Representations of p'-Valenced Schemes

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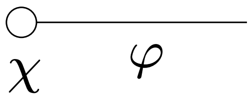
Blocks of group algebras of defect 1 (a cyclic defect group) [R. Brauer, E. C. Dade]

Brauer tree

vertex	$\longleftrightarrow$	p-conjugate class of irreducible ordinary characters
edge	$\longleftrightarrow$	irreducible modular character



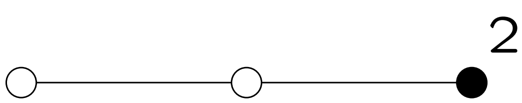
$\iff \chi|_{G_{p'}}$ contains φ.

At most one p-conjugate class has the size greater than one (the **exceptional vertex**).

Examples

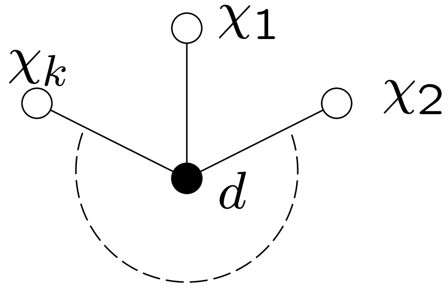
$SL(2,5)$, $(p=5)$

(principal block)



$$D_B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}$$

$C_p \rtimes C_k$, $p = 1 + kd$

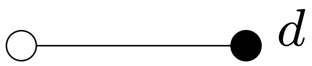


$$D_B = \left(\begin{array}{ccc} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \\ \hline 1 & \cdots & 1 \\ & \cdots & \\ 1 & \cdots & 1 \end{array}\right)$$

Remark. (1) If G is solvable, then the shape of the graph is a "star" by Fong-Swan's Theorem.
(2) If the block is local, then the graph has the only one edge.

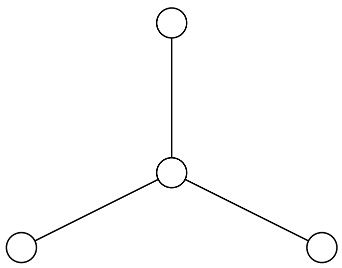
Examples of blocks of association schemes

$Cyc(p, d)$



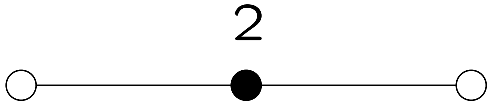
$$D_B = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$$

as21-29 ($p = 7$)



$$D_B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

as26-27 ($p = 13$)



$$D_B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix}$$

Questions. Let (X, G) be a p'-valenced scheme, B a block of (X, G) with "defect" 1.

(1) Is it true that $d_{\chi\varphi} = 0$ or 1 ?

(2) For an irreducible modular character φ, is it true that

$$\sharp\{\chi \in \mathrm{Irr}(B) \mid d_{\chi\varphi} \geq 1\}/(\mathfrak{P}\text{-conjugate}) = 2 \quad ?$$

(3) If (2) is true, then we can define a graph by decomposition numbers. Is the graph a tree ?

(4) Is it true that there exists at most one exceptional vertex ?

(5) Does B^* have finite representation type ? Is it a Brauer tree algebra ?

Note that we have not defined the "defect" for a block of an association scheme. So the statements above is incomplete.

Definitions.

Let X be a finite set, G a collection of non-empty subsets of $X \times X$. For $g \in G$, we define the **adjacency matrix** $\sigma_g \in Mat_X(\mathbb{Z})$ by $(\sigma_g)_{xy} = 1$ if $(x, y) \in g$, and 0 otherwise.

(X, G) is called an **association scheme** if

(1) $X \times X = \bigcup_{g \in G} g$ (disjoint),
(2) $1 := \{(x, x) \mid x \in X\} \in G$,
(3) if $g \in G$, then $g^* := \{(y, x) \mid (x, y) \in g\} \in G$,
(4) and $\sigma_f \sigma_g = \sum_{h \in G} p_{fg}^h \sigma_h$ for some $p_{fg}^h \in \mathbb{Z}$.

Then every row (column) of σ_g contains exactly $n_g := p_{gg^*}^1$ ones. We call n_g the **valency** of $g \in G$.

(X, G) is said to be p'-**valenced** if every valency is a p'-number.

Define

$$\mathbb{Z}G = \bigoplus_{g \in G} \mathbb{Z}\sigma_g \subset Mat_X(\mathbb{Z}),$$

then $\mathbb{Z}G$ is a $\mathbb{Z}$-algebra. For a commutative ring R with unity, we define

$$RG = R \otimes_{\mathbb{Z}} \mathbb{Z}G$$

and call this the **adjacency algebra** of (X, G) over R.

We say that (X, G) is **commutative** if $\mathbb{Z}G$ is a commutative ring.

The followings are known.

(1) If K is a field of characteristic zero, then KG is separable (semisimple).

(2) If F is a field of characteristic $p > 0$ and (X, G) is p'-valenced, then FG is a symmetric algebra. (Note that Brauer tree algebras are symmetric algebras.)

We say that a field K is a **splitting field** of (X, G) if K is a splitting field of $\mathbb{Q}G$, namely char$K = 0$ and KG is isomorphic to a direct sum of full matrix algebras over K.

For an association scheme (X, G), there exists a finite Galois extension K of $\mathbb{Q}$ which is a splitting field of (X, G). We fix such K and denote the ring of integers in K by $\mathcal{O}$. Let p be a (rational) prime number, $\mathfrak{P}$ a prime ideal of $\mathcal{O}$ lying above $p\mathbb{Z}$. The **inertia group** T of $\mathfrak{P}$ is defined by

$$T = \{\tau \in Gal(K/\mathbb{Q}) \mid a - a^\tau \in \mathfrak{P} \quad \forall a \in \mathcal{O}\}.$$

We call the corresponding subfield of K the **inertia field** of $\mathfrak{P}$ and denote it by L. We denote $\mathcal{O}_L$ for the ring of integers in L, and $\mathfrak{p}$ for the unique prime ideal of $\mathcal{O}_L$ lying below $\mathfrak{P}$. It is known that $\mathfrak{p}$ is unramified in $L/\mathbb{Q}$, namely $p \notin \mathfrak{p}^2$.

Let $\mathcal{O}_{\mathfrak{P}}$ be the localization of $\mathcal{O}$ by $\mathfrak{P}$. Put $F = \mathcal{O}_{\mathfrak{P}}/\mathfrak{P}\mathcal{O}_{\mathfrak{P}} \cong \mathcal{O}/\mathfrak{P}$, a field of characteristic p. We also suppose F is large enough. For $\alpha \in \mathcal{O}_{\mathfrak{P}}$, we denote $\alpha^* \in F$ for the image of the natural epimorphism $\mathcal{O}_{\mathfrak{P}} \to F$.

We denote the set of all irreducible characters of KG and FG by Irr(G) and IBr(G), respectively.

Let γ be the standard character, namely the character of the representation $\sigma_g \mapsto \sigma_g$. For $\chi \in$ Irr(G), we denote m_χ for the multiplicity of χ in γ and call it the **multiplicity** of χ.

An indecomposable direct summand B of $\mathcal{O}_{\mathfrak{P}}G$ as a two-sided ideal is called a $\mathfrak{P}$-**block** of (X, G). Then there exists a central primitive idempotent e_B of $\mathcal{O}_{\mathfrak{P}}G$ such that $e_B\mathcal{O}_{\mathfrak{P}}G = B$.

We say $\chi \in$ Irr(G) belongs to a $\mathfrak{P}$-block B if $\chi(e_B) \neq 0$, and denote Irr(B) for the set of irreducible ordinary characters belonging to B.

It is known that

$$e_B = \sum_{\chi \in \mathrm{Irr}(B)} e_\chi,$$

where $e_\chi = \frac{m_\chi}{n_G} \sum_{g \in G} \frac{1}{n_g} \chi(\sigma_{g^*}) \sigma_g$. Also $\mathrm{Irr}(B)$ is a minimal subset S of $\mathrm{Irr}(G)$ such that $\sum_{\chi \in S} e_\chi \in \mathcal{O}_{\mathfrak{P}} G$.

Let Ψ be a matrix representation affording $\chi \in \mathrm{Irr}(G)$. We can suppose $\Psi(\sigma_g) \in Mat_{\chi(1)}(\mathcal{O}_{\mathfrak{P}})$ for every $g \in G$. Then we obtain a representation Ψ^* of FG. Consider the irreducible constituents of Ψ^* and denote the multiplicity of an irreducible modular character φ in Ψ^* by $d_{\chi\varphi}$. We call $d_{\chi\varphi}$ the **decomposition number** and the matrix $D = (d_{\chi\varphi})$ the **decomposition matrix**.

We say that $\varphi \in \mathrm{IBr}(G)$ belongs to a block B if there exists $\chi \in \mathrm{Irr}(B)$ such that $d_{\chi\varphi} \neq 0$. Then φ belongs to the only one block. We denote $\mathrm{IBr}(B)$ for the set of modular irreducible characters belonging to B.

If $\chi \in \mathrm{Irr}(B)$, $\varphi \in \mathrm{IBr}(B')$, and $B \neq B'$, then $d_{\chi\varphi} = 0$. So we can consider the decomposition matrix D_B of a block B.

Let Ψ be a matrix representation affording $\chi \in \mathrm{Irr}(G)$ such that $\Psi(\sigma_g) \in Mat_{\chi(1)}(\mathcal{O}_{\mathfrak{P}})$ for every $g \in G$ as before. For $\tau \in Gal(K/\mathbb{Q})$, we can define a representation Ψ^τ by $\Psi^\tau(\sigma_g) = \Psi(\sigma_g)^\tau$ (entry-wise action), and denote its character by χ^τ.

In general, χ and χ^τ may belong to different blocks. But if $\tau \in Gal(K/L)$, L is the inertia field of $\mathfrak{P}$, then they belong to the same block.

We say that two irreducible ordinary characters are $\mathfrak{P}$-**conjugate** if they are conjugate by the action of the inertia group $Gal(K/L)$.

Now $\mathrm{Irr}(B)$ is a disjoint union of some $\mathfrak{P}$-conjugate classes. We denote the size of the $\mathfrak{P}$-conjugate class containing χ by r_χ.

We denote ν_p for the $\mathfrak{P}$-valuation on K such that $\nu_p(p) = 1$. Namely, if $p\mathcal{O}_{\mathfrak{P}} = \mathfrak{P}^e\mathcal{O}_{\mathfrak{P}}$ and $\alpha\mathcal{O}_{\mathfrak{P}} = \mathfrak{P}^f\mathcal{O}_{\mathfrak{P}}$, then $\nu_p(\alpha) = f/e$.

Block of "defect 0"

In group representation theory, "defect 0" means the block over a field of characteristic p is a simple algebra.

In the following, we suppose B is a block of an association scheme (X, G) and $\chi \in \mathrm{Irr}(B)$.

Proposition. Let (X, G) be a p'-valenced scheme. If $\nu_p(m_\chi) \geq \nu_p(|X|)$, then $\nu_p(m_\chi) = \nu_p(|X|)$, $\mathrm{Irr}(B) = \{\chi\}$, χ^* is irreducible, and $\mathrm{IBr}(B) = \{\chi^*\}$.

Proposition. Let (X, G) be a p'-valenced scheme. Suppose $\nu_p(\chi(1)) = 0$. Then the following conditions are equivalent.
(1) $\nu_p(m_\chi) \geq \nu_p(|X|)$.
(2) $\nu_p(m_\chi) = \nu_p(|X|)$.
(3) $\mathrm{Irr}(B) = \{\chi\}$.

Remark. If (X, G) is not p'-valenced, then this is not true.

Proposition. Let (X, G) be a commutative scheme. If $\nu_p(m_\chi) < \nu_p(|X|)$, then $|\mathrm{Irr}(B)| \geq 2$.

Block of “defect 1”

In group representation theory, the structure of a block of defect 1 is almost determined by the Brauer tree.

For a p'-valenced scheme, we consider a block B with a character χ such that $\nu_p(m_\chi)+1 = \nu_p(|X|)$.

Proposition. Let (X, G) be a p'-valenced scheme. If $\nu_p(m_\chi) + 1 = \nu_p(|X|)$ and $\nu_p(r_\chi) > 0$, then $\mathrm{Irr}(B) = \{\chi^\tau \mid \tau \in Gal(K/L)\}$.

For a block satisfying the property in the above Proposition, we cannot define the Brauer tree, since it has only one vertex. But I do not know such an example.

We denote K^G for the set of K-valued functions on $\{\sigma_g \mid g \in G\}$. For $\alpha, \beta \in K^G$, we define

$$[\alpha, \beta] = \sum_{g \in G} \frac{1}{n_g} \alpha(\sigma_{g^*}) \beta(\sigma_g).$$

Let Φ be a matrix representatation of KG. We denote $\Phi_{ij} \in K^G$ for the (i,j)-entries of Φ, namely $\Phi_{ij}(\sigma_g) = \Phi(\sigma_g)_{ij}$.

Proposition (Schur relation).

(1) If Φ is an irreducible representation affording χ, then $[\Phi_{ij}, \Phi_{k\ell}] = \delta_{i\ell}\delta_{jk}|X|/m_\chi$. ($\delta$ is the Kronecker's delta.)

(2) If Φ and Ψ have no common irreducible constituent, then $[\Phi_{ij}, \Psi_{k\ell}] = 0$.

Let Ψ_i, $i = 1, 2, 3$, be irreducible representations of KG affording ψ_i, respectively. We may assume that all $\Psi_i(\sigma_g)$, $g \in G$ are matrices over $\mathcal{O}_{\mathfrak{P}}$, and then, we can consider representations ${\Psi_i}^*$ of FG. Suppose ${\Psi_i}^*$, $i = 1, 2, 3$, have a common irreducible constituent S. We may assume

$$\Psi_i = \begin{pmatrix} S_i & * \\ * & * \end{pmatrix},$$

where ${S_i}^* = S$.

We define $u, v \in K^G$ by $u = (\Psi_1)_{11} - (\Psi_2)_{11}$ and $v = (\Psi_1)_{11} - (\Psi_3)_{11}$. Then $u(\sigma_g), v(\sigma_g) \in \mathfrak{P}\mathcal{O}_{\mathfrak{P}}$ for every $g \in G$.

By Schur relation, we have

$$[(\Psi_1)_{11}, (\Psi_1)_{11}] = \frac{|X|}{m_{\psi_1}}.$$

Then

$$\begin{aligned} 0 &= [(\Psi_1)_{11}, (\Psi_2)_{11}] \\ &= [(\Psi_1)_{11}, (\Psi_1)_{11}] - [(\Psi_1)_{11}, u]. \end{aligned}$$

So we have

$$[(\Psi_1)_{11}, u] = [(\Psi_1)_{11}, (\Psi_1)_{11}].$$

Similarly

$$[(\Psi_1)_{11}, v] = [(\Psi_1)_{11}, (\Psi_1)_{11}].$$

Now

$$\begin{aligned} 0 &= [(\Psi_2)_{11}, (\Psi_3)_{11}] \\ &= [(\Psi_1)_{11}, (\Psi_1)_{11}] \\ &\quad -[u, (\Psi_1)_{11}] - [(\Psi_1)_{11}, v] + [u, v] \\ &= -[(\Psi_1)_{11}, (\Psi_1)_{11}] + [u, v]. \end{aligned}$$

This means

$$\frac{|X|}{m_{\psi_1}} = [u, v].$$

Consider the traces over K/L of u and v, then we have

$$\frac{|X| \cdot |K : L|^2}{m_{\psi_1}} = \sum_{g \in G} \frac{1}{n_g} \mathrm{Tr}_{K/L}(u(\sigma_{g^*}))\mathrm{Tr}_{K/L}(v(\sigma_g)).$$

Suppose (X, G) is p'-valenced, $\nu_p(m_{\psi_1})+1 = \nu_p(|X|)$, and ψ_i, $i = 1, 2, 3$, are not $\mathfrak{P}$-conjugate to each other. Then we have $\nu_p(r_{\psi_i}) = 0$, $i = 1, 2, 3$.

<u>Case 1. K is cyclotomic (abelian).</u>

In this case, we can prove that

$$\nu_p(\mathrm{Tr}_{K/L}(u(\sigma_{g^*}))) \geq \nu_p(|K : L|) + 1$$

and

$$\nu_p(\mathrm{Tr}_{K/L}(v(\sigma_g))) \geq \nu_p(|K : L|) + 1.$$

This is a contradiction.

<u>Case 2. $\nu_p(|K : L|) = 0$.</u>

In this case, we can prove that

$$\nu_p(\mathrm{Tr}_{K/L}(u(\sigma_{g^*}))) \geq 1$$

and this is a contradiction.

(This condition is equivalent to that p is tamely ramified in $K/\mathbb{Q}$.)

Proposition. Let (X, G) be a p'-valenced scheme, B a block of G, and $\varphi \in \mathrm{IBr}(B)$. Assume there exists $\chi \in \mathrm{Irr}(B)$ with $\nu_p(m_\chi) + 1 = \nu_p(|X|)$. Suppose that the minimal splitting field K of G is abelian or $\nu_p(|K : L|) = 0$ (p is tamely ramified in $K/\mathbb{Q}$). Then the number of $\mathfrak{P}$-conjugate classes of $\mathrm{Irr}(B)$ such that their modular characters contain φ is at most two.
For $\psi \in \mathrm{Irr}(B)$ such that $d_{\psi\varphi} \geq 0$, we suppose $\nu_p(r_\psi \psi(1)) = 0$. Then the number is exactly two.

Remark. If $\nu_p(r_\psi \psi(1)) = 0$ for all $\psi \in \mathrm{Irr}(B)$, then we may assume $\nu_p(|K : L|) = 0$.

If all the numbers above are two, then we can draw a graph. Its vartex is a $\mathfrak{P}$-conjugate class, and its edge is an irreducible modular character.

By a similar argument, we can show that the following.

Proposition. Let (X, G) be a commutative p'-valenced scheme, B a block of G, and $\chi \in \mathrm{Irr}(B)$. Suppose $\nu_p(m_\chi) + 1 = \nu_p(|X|)$ and $\nu_p(r_\chi) = 0$. Then $\nu_p(m_\psi) + 1 = \nu_p(|X|)$ for all $\psi \in \mathrm{Irr}(B)$ and the number of $\mathfrak{P}$-conjugate classes of $\mathrm{Irr}(B)$ is exactly two.

Corollary. Let (X, G) be a commutative p'-valenced scheme with $\nu_p(|X|) = 1$. Then all non-trivial irreducible ordinary characters in the principal block are $\mathfrak{P}$-conjugate.

Proposition. If $|X| = p$, then all non-trivial irreducible ordinary characters are $\mathfrak{P}$-conjugate.

Using this fact, we can prove that (X, G) is commutative, if $|X| = p$.

Proposition. Let (X, G) be a p'-valenced scheme, $\psi \in \mathrm{Irr}(G)$. Suppose $\nu_p(m_\chi) + 1 = \nu_p(|X|)$. If the Schur index $m_L(\chi) = 1$, $\nu_p(r_\chi) = 0$, and $p \neq 2$, then $d_{\chi\varphi} \leq 1$ for every $\varphi \in \mathrm{IBr}(G)$.
(The assumption on Schur indices holds if there exists an L-representation of G affording χ.)

Remark. (1) If $p \neq 2$, then the Schur index $m_L(\chi)$ equals to one for a group character χ. (Note that the base field is not $\mathbb{Q}$.) The assumption $\nu_p(r_\chi) = 0$ can be replaced by that $L(\chi(\sigma_g) \mid g \in G)$ is a Galois extension of L.

(2) If we can define a graph, $d_{\chi\varphi} \leq 1$ holds for $\chi \in \mathrm{Irr}(B)$ and $\varphi \in \mathrm{IBr}(B)$, and $p \neq 2$, then the graph is bipartite. (Of cource, a tree is bipartite.)

(3) Almost all results in this talk are not true for non p'-valenced schemes.

(4) For commutative p'-valenced scheme, it is reasonable to define the "defect" of a block by $\max\{\nu_p(|X|) - \nu_p(m_\chi) \mid \chi \in \mathrm{Irr}(B)\}$. But, in general, it is still difficult.

Let (X, G) be a p'-valenced scheme. Suppose $\nu_p(m_\chi)+1 = \nu_p(|X|)$, $d_{\chi\varphi} \leq 1$ for all $\chi \in \mathrm{Irr}(B)$ and all $\varphi \in \mathrm{IBr}(B)$, and a graph is defined. Then the graph is a tree if and only if $\mathrm{rank} D_B = |\mathrm{IBr}(B)|$. Especially, if the Cartan matrix C_B is invertible, then the graph is a tree.

Question. For a p'-valenced scheme, is the Cartan matrix invertible ?

END.