Frobenius-Schur Theorem for Association Schemes

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Let G be a finite group. For $\chi \in \mathrm{Irr}(G)$, put

$$\nu_2(\chi) = \frac{1}{|G|} \sum_{g \in G} \chi(g^2)$$

and call this the *Frobenius-Schur indicator* of χ. Put

$$\theta(g) = \sharp\{h \in G \mid h^2 = g\}.$$

Frobenius-Schur Theorem.

(1) $\nu_2(\chi) \in \{-1, 0, 1\}$.

(2) $\nu_2(\chi) = 0$ if and only if $\overline{\chi} \neq \chi$.

(3) $\nu_2(\chi) = 1$ if and only if χ is afforded by a real representation.

(4) $\sum_{\chi \in \mathrm{Irr}(G)} \nu_2(\chi)\chi = \theta$.
Especially $\sum_{\chi \in \mathrm{Irr}(G)} \nu_2(\chi)\chi(1) = \sharp\{h \in G \mid h^2 = 1\}$.

Let (X, G) be an association scheme. For $\chi \in \mathrm{Irr}(G)$, put

$$\nu_2(\chi) = \frac{m_\chi}{n_G \chi(1)} \sum_{g \in G} \frac{1}{n_g} \chi({\sigma_g}^2)$$

and call this the *Frobenius-Schur indicator* of χ.

Frobenius-Schur Theorem for Association Schemes.

(1) $\nu_2(\chi) \in \{-1, 0, 1\}$.

(2) $\nu_2(\chi) = 0$ if and only if $\overline{\chi} \neq \chi$.

(3) If χ is afforded by a real representation, then $\nu_2(\chi) = 1$.

(4) $\sum_{\chi \in \mathrm{Irr}(G)} \nu_2(\chi)\chi(1) = \sharp\{h \in G \mid h^* = h\}$.

These are easy by a result on Hopf algebras in [Linchenko-Montgomery 2000].

Question. Can we say that $\nu_2(\chi) = 1$ implies that χ is afforded by a real representation ?

Put

$$\theta(\sigma_g) = \sum_{h \in G} \frac{n_g}{n_h} p^g_{hh}.$$

We do not know whether θ is a linear combination of $\mathrm{Irr}(G)$. The following is an easy fact by orthogonality relations.

Proposition. If θ is a linear combination of $\mathrm{Irr}(G)$, then $\theta = \sum_{\chi \in \mathrm{Irr}(G)} \nu_2(\chi)\chi$.

An Application.

Put

$$I(G) = \{g \in G \mid g^* = g \neq 1\}.$$

Proposition. Suppose $|G| > 1$. Then

$$|\mathrm{Irr}(G)| = \dim_{\mathbb{C}} Z(\mathbb{C}G) \geq \frac{|I(G)|^2}{|G| - 1} + 1.$$

Roughly speaking, we can say that

If G has many symmetric relations, then G is near commutative.