Block Theory of Association Schemes

Akihide Hanaki
Shinshu University

We consider **representation theory of** (not necessary commutative) **association schemes** (homogeneous coherent configurations). **Representation** means a linear representation of the adjacency algebra (Bose-Mesner algebra). The adjacency algebra is defined over an arbitrary commutative ring with 1. We say an **ordinary representation** for it over a field of characteristic 0, and a **modular representation** for it over a field of positive characteristic. To simplify our argument, we assume that the coefficient field is algebraically closed, in this talk.

It is well known that the adjacency algebra over a field of characteristic 0 is a semisimple algebra. So representation theory is almostly the same as character theory, and many facts are known, in this case. (But I think that it is not enough, especially for non-commutative case.)

For modular representations, we know a few facts.

General Results:

- [Arad-Fisman-Muzychuk 1999, H. 2000]
  Semisimplicity of adjacency algebras
  (a generalization of Maschke's Theorem)
- [H. 2002] If the order is a p-power,
  then the adjacency algebra is a local algebra.

Standard modules:

- [Brouwer-van Eijl 1989]
  p-ranks of adjacency matrices for SRG
- [Peeters 2002]
  p-ranks of adjacency matrices for DRG
- [Yoshikawa-H. preprint] Structure of
  standard modules for small schemes

Special Schemes:

- [Yoshikawa preprint] Structure of adjacency
  algebras of the Hamming schemes
- [Shimabukuro preprint] Block decomposition of
  irreducible characters of the Johnson schemes

Other:

- [Arad-Erez-Muzychuk 2003]
  On even generalized table algebras

We want to consider general theory for modular representations. We need good tools.

It is natural to consider generalization of modular representation theory of finite groups. In group representation theory, **block theory** is very important and the **defect group**, a p-subgroup, plays a crucial role.

A block of an algebra is an indecomposable direct summand of the algebra as a two-sided ideal. We can also consider blocks for adjacency algebras. But we can not consider the defect group of it, since we can not consider something like Sylow p-subgroups in an association scheme. So I try to define the **defect number** for a block of adjacency algebras. But, I can not define it, at present.

I will define some invariants. I believe that it is closely related to the defect numbers.

p : a prime
(K, R, F) : a p-modular system

R : a complete discrete valuation ring with the maximal ideal (π),
K : the quotient field of R, $\mathrm{char} K = 0$,
F : the residue field $R/(\pi)$, $\mathrm{char} F = p$.

We assume K is algebraically closed, then so is F. The image of $a \in R$ by the natural epimorphism $R \to F$ is denoted by a^*.

Let ν be the valuation of R with $\nu(p) = 1$. For a rational integer $n = p^e m$, $p \nmid m$, we have $\nu(n) = e$.

Note that

$$R = \{a \in K \mid \nu(a) \geq 0\},$$

$\nu(ab) = \nu(a) + \nu(b)$, $\nu(a + b) \geq \min(\nu(a), \nu(b))$, and $\nu(0) = \infty$.

Let (X, G) be an association scheme. We denote the adjacency matrix of $g \in G$ by σ_g. Put

$$\mathbb{Z}G := \bigoplus_{g \in G} \mathbb{Z}\sigma_g,$$

and

$$\mathcal{O}G := \mathbb{Z}G \otimes_{\mathbb{Z}} \mathcal{O},$$

for $\mathcal{O} \in \{K, R, F\}$. We call $\mathcal{O}G$ the adjacency algebra of (X, G) over $\mathcal{O}$.

Remark. KG is essentially the same as $\mathbb{C}G$, and it is semisimple.

The image of $a = \sum_{g \in G} a_g \sigma_g \in RG$ by the natural epimorphism $RG \to RG/\pi RG \cong FG$ is also denoted by a^*. Namely,

$$\left(\sum_{g \in G} a_g \sigma_g \right)^* = \sum_{g \in G} {a_g}^* {\sigma_g}^*.$$

There are natural bijections between the following sets.

(1) The set of central primitive idempotents of RG : e_B.

(2) The set of central primitive idempotents of FG : ${e_B}^*$.

(3) The set of indecomposable direct summands of RG as two-sided ideals : $B = e_B RG$.

(4) The set of indecomposable direct summands of FG as two-sided ideals : $B^* = {e_B}^* FG$.

We say that B is a *block* (or a p-block) of the association scheme (X, G), and e_B is the block idempotent. Of course, B is an algebra with the identity e_B.

Let $\mathrm{Bl}(G)$ be the set of blocks of G. For $\chi \in \mathrm{Irr}(G)$ and $B \in \mathrm{Bl}(G)$, we say that χ *belongs to* B if $\chi(e_B) \neq 0$, and put $\mathrm{Irr}(B) := \{\chi \in \mathrm{Irr}(G) \mid \chi(e_B) \neq 0\}$. Then

$$\mathrm{Irr}(G) = \bigcup_{B \in \mathrm{Bl}(G)} \mathrm{Irr}(B)$$

is a partition of $\mathrm{Irr}(G)$. If $\chi \in \mathrm{Irr}(B)$, then $\chi(e_B) = \chi(1)$. Also we have

$$e_B = \sum_{\chi \in \mathrm{Irr}(B)} e_\chi,$$

where e_χ is the central primitive idempotent in KG corresponding to $\chi \in \mathrm{Irr}(G)$. Actually, $\mathrm{Irr}(B)$ is a minimal subset of $\mathrm{Irr}(G)$ such that $\sum_{\chi \in \mathrm{Irr}(B)} e_\chi$ is in RG.

The *trivial character* 1_G is the map $\sigma_g \mapsto n_g$. The *principal block* is the block B with $1_G \in \mathrm{Irr}(B)$, and we denote it by $B_0(G)$ or B_0.

For $\chi \in \mathrm{Irr}(G)$ and $z \in Z(KG)$, we put

$$\omega_\chi(z) := \frac{\chi(z)}{\chi(1)}.$$

Then $\chi \neq \varphi$ implies $\omega_\chi \neq \omega_\varphi$, and moreover, $\{\omega_\chi \mid \chi \in \mathrm{Irr}(G)\}$ is the set of all irreducible characters of $Z(KG)$. If (X, G) is commutative, then $\omega_\chi = \chi$.

For $\chi, \varphi \in \mathrm{Irr}(G)$, we can see that they are in the same block if and only if

$$\omega_\chi(z)^* = \omega_\varphi(z)^*$$

for any $z \in Z(RG)$.

So in the case $\chi \in \mathrm{Irr}(B)$, we write ${\omega_B}^* := {\omega_\chi}^*$.

For a commutative scheme (X, G), $\chi, \varphi \in \mathrm{Irr}(G)$ are in the same block if and only if

$$\chi(\sigma_g)^* = \varphi(\sigma_g)^*, \quad \text{for all } g \in G.$$

Some invariants for commutative schemes

In this section, we always assume that the association scheme (X, G) is commutative.

Let $B \in \mathrm{Bl}(G)$. Since FG is a splitting commutative algebra, its indecomposable direct summand B^* is a local algebra. So ${\omega_B}^*$ is the unique irreducible representation of B^*. We put

$$s(B) := \max\{\nu(n_g) \mid {\omega_B}^*({\sigma_g}^*) \neq 0\}.$$

This is our first invariant of B. Since ${\omega_B}^*({e_B}^*) = 1$, there exists $g \in G$ such that ${\omega_B}^*({\sigma_g}^*) \neq 0$. So this is well-defined. The next proposition is clear.

Proposition. $s(B_0) = 0$.

Remark. For noncommutative schemes, we can also define $s(B)$ by the same definition. But it is not a good definition, I think.

Let ℓ be a nonnegative integer. We define

$$I_\ell := \bigoplus_{\nu(n_g)\geq\ell} F{\sigma_g}^*.$$

Then I_ℓ is an ideal of FG. For $B \in \mathrm{Bl}(G)$, we put

$$s'(B) := \max\{\ell \mid {e_B}^* \in I_\ell\}.$$

This is our second invariant of B, but we have the following.

Proposition. $s(B) = s'(B)$.

Proof. Firstly, we will show that, if $\nu(n_g) > s'(B)$, then ${\omega_B}^*({\sigma_g}^*) = 0$. Since B^* is a local algebra, B^* has the unique maximal ideal, the Jacobson radical $\mathrm{rad}B^*$. Now ${e_B}^*{\sigma_g}^*$ is in a proper ideal ${e_B}^*I_{s'(B)+1}$, so it is in $\mathrm{rad}B^*$. The Jacobson radical annihilates any simple module, so ${\omega_B}^*({\sigma_g}^*) = {\omega_B}^*({e_B}^*{\sigma_g}^*) = 0$. This means $s(B) \leq s'(B)$.

Now ${\omega_B}^*(I_{s'(B)}) \ni {\omega_B}^*({e_B}^*) \neq 0$, so $s(B) \geq s'(B)$. $\square$

We write

$$e_B = \sum_{g \in G} \beta_B(g)\sigma_g \qquad (\beta_B(g) \in R).$$

Clearly we have

$$s(B) = \min\{\nu(n_g) \mid \beta_B(g)^* \neq 0\}.$$

We put $\mathrm{Bl}_\ell(G) := \{B \in \mathrm{Bl}(G) \mid s(B) = \ell\}$ and $G_\ell := \{g \in G \mid \nu(n_g) = \ell\}$.

Proposition. Let $B, B' \in \mathrm{Bl}_\ell(G)$, $\chi \in \mathrm{Irr}(B)$, and $\chi' \in \mathrm{Irr}(B')$. Then $B = B'$ if and only if

$$\chi(\sigma_g)^* = \chi'(\sigma_g)^*,$$

for any $g \in G_\ell$. Moreover $\{(\omega_B{}^* |_{G_\ell}) \mid B \in \mathrm{Bl}_\ell(G)\}$ is linearly independent over F, so we have $|\mathrm{Bl}_\ell(G)| \leq |G_\ell|$.

Proof. Suppose $\chi(\sigma_g)^* = \chi'(\sigma_g)^*$ for any $g \in G_\ell$. Then we have

$$\begin{aligned} \omega_B{}^*(e_{B'}{}^*) &= \sum_{g \in G_\ell} \beta_{B'}(g)^* \omega_B{}^*(\sigma_g) \\ &= \sum_{g \in G_\ell} \beta_B(g)^* \omega_B{}^*(\sigma_g) = \omega_B{}^*(e_B{}^*) = 1, \end{aligned}$$

and so $B = B'$. The converse is clear.

Suppose $\sum_{B\in \mathrm{Bl}_\ell(G)} \alpha_B(\omega_B{}^* \mid_{G_\ell}) = 0$. Then, for $B' \in \mathrm{Bl}_\ell(G)$,

$$0 = \sum_B \alpha_B(\omega_B{}^* \mid_{G_\ell})(e_{B'}{}^*) = \sum_B \alpha_B \omega_B{}^*(e_{B'}{}^*) = \alpha_{B'}$$

so $\{(\omega_B{}^* \mid_{G_\ell}) \mid B \in \mathrm{Bl}_\ell(G)\}$ is linearly independent. $\square$

Now we define the third invariant $t(B)$. Let FX be the (right) standard module of (X, G) over F. We consider the dimension of $FXe_B{}^*$. Then we have

$$\dim_F FXe_B{}^* = \sum_{\chi\in \mathrm{Irr}(B)} m_\chi \chi(1),$$

where m_χ is the multiplicity. Of course, $\chi(1) = 1$ since (X, G) is commutative, but this relation is also true for non-commutative schemes.

Lemma. $\nu\,(\dim_F FXe_B{}^*) \geq \nu(n_G)$. (This is true for arbitrary schemes.)

Proof. We have $\beta_B(1) = \left(\sum_{\chi\in \mathrm{Irr}(B)} m_\chi \chi(1)\right)/n_G$, and this is in R. $\square$

We put

$$t(B) = \nu\left(\dim_F FXe_B{}^*\right) - \nu(n_G).$$

Lemma says that $t(B) \geq 0$. There are many examples such that $s(B) = t(B)$ but also many examples such that $s(B) \neq t(B)$.

Conjecture 1. $s(B) \leq t(B)$.

Conjecture 2. $\nu(\beta_B(g)) \geq s(B) - \nu(n_g)$.

If Conjecture 2 is true, then

$$t(B) = \nu(\beta_B(1)) \geq s(B) - \nu(n_1) = s(B)$$

and so Conjecture 1 is true.

Conjecture 2 is true for
- ∘ group association schemes (we will see later)
- ∘ $G = J(v, k)$, $1 \leq v \leq 40$, FG is not semisimple (by computer calculations)

Remark. There exist examples such that $t(B_0) > 0$.

Example. Let (X, G) be a commutative scheme. Assume that FG is semisimple.
(i.e. $p \nmid n_G$ and $\sum_{g \in G} \nu(n_g) = \sum_{\chi \in \mathrm{Irr}(G)} \nu(m_\chi)$.)
Then $|\mathrm{Bl}_\ell(G)| = |G_\ell|$ for any ℓ, and $s(B) = t(B)$ for any $B \in \mathrm{Bl}(G)$. So Conjecture 1 is true in this case.

(We can restrict possibilities of the character table.)

Defect Number for a Block of Group Algebras

Let Θ be a finite group, and $(\Theta, \widehat{\Theta})$ be the group association scheme constructed by Θ. Now there is a natural bijection between $\mathrm{Bl}(\Theta)$ and $\mathrm{Bl}(\widehat{\Theta})$ since $Z(F\Theta) = F\widehat{\Theta}$. Let $\widehat{B} \in \mathrm{Bl}(\widehat{\Theta})$ and $B \in \mathrm{Bl}(\Theta)$ be corresponding blocks. In group representation theory, the *defect* $d(B)$ of B is defined by

$$d(B) := \min\{\nu(|\Theta|/\chi(1)) \mid \chi \in \mathrm{Irr}(B)\}.$$

Now it is known that

$$s(\widehat{B}) = t(\widehat{B}) = \nu(|\Theta|) - d(B).$$

Conjecture 1 and 2 are also true in this case.

Problem. Consider the similar argument as above for noncommutative schemes.

Problem. Consider a reasonable definition of defect numbers for blocks of association schemes.

Other Problem 1.

We want to know when $|\mathrm{Irr}(B)| = 1$**.**

Of cource, $|\mathrm{Irr}(B)| = 1$ if and only if $e_B \in RG$. But it is not so easy to check this condition.

If G is thin (FG is a group algebra), then $|\mathrm{Irr}(B)| = 1$ if and only if B^* is a simple algebra. But this is not true for association schemes. There is an example such that $|\mathrm{Irr}(B)| = 1$ but B^* is not simple.

Question. Is it true that $|\mathrm{Irr}(B)| = 1$ if and only if $\dim_F Z(B^*) = 1$?

If G is thin, then every B^* is a symmetric algebra. If B^* is a symmetric algebra, then $\dim_F Z(B^*) = 1$ implies that B^* is simple.

Question. Can we characterize $|\mathrm{Irr}(B)| = 1$ by $s(B)$, $t(B)$ and some other invariants ?

If G is thin, then $|\mathrm{Irr}(B)| = 1$ if and only if the defect number $d(B) = 0$.

Fact. $|\mathrm{Irr}(B_0)| = 1$ if and only if $p \nmid n_G$.

Example. Let G be the unique noncommutative scheme of order $n_G = 15$, and let $p = 2$. Then

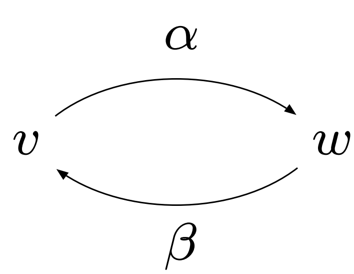
$$FG = {B_0}^* \oplus {B_1}^* \oplus {B_2}^*, \quad {B_0}^* \cong {B_1}^* \cong F.$$

(Note that $2 = p \nmid n_G = 15$. The principal block is simple.) The block ${B_2}^*$ is not simple, but $|\mathrm{Irr}(B_2)| = 1$ (only one character of degree 2). The structure of ${B_2}^*$ is as follows.

basis: $\{v, w, \alpha, \beta\}$
multiplication:

	v	w	α	β
v	v	0	α	0
w	0	w	0	β
α	0	α	0	0
β	β	0	0	0

$(1 = v + w)$



The algebra is not symmetric, and the dimension of the center is one.

Other Problem 2.

Let H be a normal closed subset of G. Define

$$\tau : \mathbb{Z}G \to \mathbb{Z}(G//H), \quad (\sigma_g \mapsto \frac{n_g}{n_{g^H}}\ \sigma_{g^H}).$$

Then τ is an algebra homomorphism. Since $n_g/n_{g^H} \in \mathbb{Z}$, we can define $\tau_{\mathcal{O}} : \mathcal{O}G \to \mathcal{O}(G//H)$ for any commutative ring $\mathcal{O}$ with 1.

Let $T : \mathcal{O}(G//H) \to M_d(\mathcal{O})$ be a representation of $G//H$. Then $T \circ \tau_{\mathcal{O}} : \mathcal{O}G \to M_d(\mathcal{O})$ is a representation of G.

If $\mathcal{O}$ is an algebraically closed field and $\tau_{\mathcal{O}}$ is an epimorphism, then the followings hold.

- ∘ If T is irreducible, then so is $T \circ \tau_{\mathcal{O}}$.
- ∘ If $T \not\sim T'$, then $T \circ \tau_{\mathcal{O}} \not\sim T' \circ \tau_{\mathcal{O}}$.
- ∘ Irr($\mathcal{O}(G//H)$) is embedded into Irr($\mathcal{O}G$).

If $\mathcal{O}$ is a field of characteristic 0, then $\tau_{\mathcal{O}}$ is an epimorphism. But it is not true, in general.

Example. Let H be a scheme such that n_H is a p-power, and let G be any scheme. Consider the wreath product $H \wr G$. In this case, $F(H \wr G)$ is a local algebra. So, $1 = |\mathrm{Irr}(F(H \wr G))| \leq |\mathrm{Irr}(FG)|$. If T is a non-linear representation of G, then $T \circ \tau_F$ is reducible.

If G is a finite group and H is a normal p-subgroup of G, then H is in the kernel of every irreducible F-representation of G. I want to generalize this to association scheme.

Problem. Let (X, G) be an association scheme, and let H be a normal closed subset such that n_H is p-power. Is any irreducible representation of FG given by a representation of $F(G//H)$?

Example.

$$\left(\begin{array}{ccccccc|ccccccc}
0 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 3 & 3 & 3 & 3 \\
1 & 0 & 1 & 1 & 1 & 1 & 1 & 3 & 3 & 2 & 2 & 2 & 3 & 3 \\
1 & 1 & 0 & 1 & 1 & 1 & 1 & 2 & 3 & 3 & 3 & 2 & 2 & 3 \\
1 & 1 & 1 & 0 & 1 & 1 & 1 & 2 & 3 & 3 & 2 & 3 & 3 & 2 \\
1 & 1 & 1 & 1 & 0 & 1 & 1 & 3 & 2 & 3 & 3 & 2 & 3 & 2 \\
1 & 1 & 1 & 1 & 1 & 0 & 1 & 3 & 3 & 2 & 3 & 3 & 2 & 2 \\
1 & 1 & 1 & 1 & 1 & 1 & 0 & 3 & 2 & 3 & 2 & 3 & 2 & 3 \\
\hline
2 & 3 & 2 & 2 & 3 & 3 & 3 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\
2 & 3 & 3 & 3 & 2 & 3 & 2 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\
2 & 2 & 3 & 3 & 3 & 2 & 3 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\
3 & 2 & 3 & 2 & 3 & 3 & 2 & 1 & 1 & 1 & 0 & 1 & 1 & 1 \\
3 & 2 & 2 & 3 & 2 & 3 & 3 & 1 & 1 & 1 & 1 & 0 & 1 & 1 \\
3 & 3 & 2 & 3 & 3 & 2 & 2 & 1 & 1 & 1 & 1 & 1 & 0 & 1 \\
3 & 3 & 3 & 2 & 2 & 2 & 3 & 1 & 1 & 1 & 1 & 1 & 1 & 0
\end{array}\right)$$

	g_0	g_1	g_2	g_3	m_i
χ_1	1	6	3	4	1
χ_2	1	6	-3	-4	1
χ_3	1	-1	$\sqrt{2}$	$-\sqrt{2}$	6
χ_4	1	-1	$-\sqrt{2}$	$\sqrt{2}$	6

$p = 7$, $\chi_1^* = \chi_a^*$, $\chi_2^* = \chi_b^*$

($\{a, b\} = \{3, 4\}$. The choice of $\{a, b\}$ depends on the p-modular system.)