Clifford Theory for Commutative Association Schemes

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(Clifford Theorem for Finite Groups)
Let G be a finite group and $H \lhd G$. For $\varphi \in \mathrm{Irr}(H)$, put

$$T = T_\varphi := \{g \in G \mid \varphi^g = \varphi\},$$

where $\varphi^g(x) = \varphi(gxg^{-1})$. Then

(1) For $\chi \in \mathrm{Irr}(G \mid \varphi)$,

$$\chi_H = e \sum_{t \in T \backslash G} \varphi^t.$$

(2) The correspondence $\mathrm{Irr}(T \mid \varphi) \to \mathrm{Irr}(G \mid \varphi)$ defined by $\eta \mapsto \eta^G$ is a bijection.

(3) If there exists $\chi \in \mathrm{Irr}(G \mid \varphi)$ such that $\chi_H = \varphi$, then

$$\mathrm{Irr}(G \mid \varphi) = \{\chi\xi \mid \xi \in \mathrm{Irr}(G/H)\}.$$

(X, G) : an association scheme
(not necessary commutative)
H : a normal closed subset of G
($gH = Hg$ for any $g \in G$)

Example (as12-40)

	g_0	g_1	g_2	g_3	g_4	g_5	g_6	m_i
χ_1	1	1	2	2	2	2	2	1
χ_2	1	1	2	-1	-1	-1	-1	2
χ_3	1	-1	0	-1	-1	1	1	3
χ_4	2	0	-2	1	1	-1	-1	3

$$H := \{g_0, g_1, g_2\} \lhd G$$

	g_0	g_1	g_2	m_i
φ_1	1	1	2	1
φ_2	1	1	-2	1
φ_3	1	-1	0	2

$$(\chi_1)_H = \varphi_1, \quad (\chi_2)_H = \varphi_1,$$
$$(\chi_3)_H = \varphi_3, \quad (\chi_4)_H = \varphi_2 + \varphi_3.$$

Clifford theory dose not hold.

We will assume that H is strongly normal.

Group-Graded Algebras

Let F be a field. Suppose all algebras and modules over F are finite dimensional, and modules will be right modules.

Definition. Let S be a finite group, and let A be an F-algebra. Suppose A is a direct sum of F-subspaces A_s, $s \in S$. The algebra A is called S-**graded** (group-graded) if

(1) $A_s A_t \subseteq A_{st}$ for $s, t \in S$.

For an S-graded algebra A, A_1 is a subalgebra of A. Furthermore, if

(2) $A_s A_t = A_{st}$ for $s, t \in S$,

we say that A is **strongly** S-**graded**.

If an S-graded algebra A satisfies

(3) for every $s \in S$, A_s contains a unit a_s in A,

then the condition (2) holds, and in this case, it is known that A is a **crossed product** of S over A_1, and that Clifford theory holds.

(We do not define crossed products. You may consider the condition (3) is the definition of them.)

M : a right A-module

restriction : M_{A_1}

L : a right A_1-module

induced module : $L^A := L \otimes_{A_1} A$

$$L^A = \bigoplus_{s \in S} L \otimes A_1 a_s = \bigoplus_{s \in S} L \otimes a_s$$

For $s \in S$, $L \otimes a_s$ is an A_1-submodule of $(L^A)_{A_1}$ (**conjugate** of L).

If a_s and a'_s are units in A_s, then

$$L \otimes a_s \cong L \otimes a'_s \quad \text{(as } A_1\text{-modules)}.$$

$$T := \{t \in S \mid L \otimes a_t \cong L \otimes 1\}$$

is a subgroup of S.

Theorem (Clifford Theorem).

Let A be an S-graded F-algebra with the property (3) above, M a simple A-module, and L a simple A_1-submodule of M_{A_1}. Put

$$T := \{t \in S \mid L \otimes a_t \cong L \otimes 1\}$$

Then the followings hold.

(1) M_{A_1} is semisimple and

$$M_{A_1} = e\left(\bigoplus_{t \in T\backslash G} L \otimes a_t\right).$$

(2) Put $B := \sum_{t\in T} A_t$. Then

$$\text{Irr}(B \mid L) \to \text{Irr}(A \mid L) \quad (N \mapsto N^A)$$

is a bijection.

(X, G) : an association scheme
(not necessary commutative)
H : a strongly normal closed subset of G
($gHg^* = H$ for any $g \in G$)

The factor scheme $G//H$ is thin, so we can consider $G//H$ is a finite group.

$$\mathbb{C}G = \bigoplus_{g^H \in G//H} \mathbb{C}(HgH).$$

$\mathbb{C}G$ is $G//H$-graded. Let L be a simple $\mathbb{C}H$-module. Then

$$L^G = L \otimes_{\mathbb{C}H} \mathbb{C}G = \bigoplus_{g^H \in G//H} L \otimes \mathbb{C}(HgH),$$

and $L \otimes \mathbb{C}(HgH)$ is a $\mathbb{C}H$-submodule of $(L^G)_H$ (this can be a zero module). We write L^g for $L \otimes \mathbb{C}(HgH)$.

Conjecture (Clifford Theorem for Association Schemes).

Let (X, G) be an association scheme, H a strongly normal closed subset of G, and let L be a simple $\mathbb{C}H$-module. Put

$$T//H := \{g^H \in G//H \mid L^g \cong L\}.$$

Then

(1) For $M \in \mathrm{Irr}(G \mid L)$,

$$M_H = e \bigoplus_{t \in T \backslash G} L^t.$$

(2) The correspondence $\mathrm{Irr}(T \mid L) \to \mathrm{Irr}(G \mid L)$ defined by $N \mapsto N^G$ is a bijection.

If $\mathbb{C}G$ is a crossed product of $G//H$ over $\mathbb{C}H$, then Conjecture is true.

Example.

(1) Let (X, G) be an association scheme, and let Θ be a finite group acting on (X, G). The semidirect product $(X, G)\Theta$ (defined in Zieschang's Lecture Note) is a crossed product.

(2)

$$\left(\begin{array}{ccccccc|ccccccc}
0 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 3 & 3 & 3 & 3 \\
1 & 0 & 1 & 1 & 1 & 1 & 1 & 3 & 3 & 2 & 2 & 2 & 3 & 3 \\
1 & 1 & 0 & 1 & 1 & 1 & 1 & 2 & 3 & 3 & 3 & 2 & 2 & 3 \\
1 & 1 & 1 & 0 & 1 & 1 & 1 & 2 & 3 & 3 & 2 & 3 & 3 & 2 \\
1 & 1 & 1 & 1 & 0 & 1 & 1 & 3 & 2 & 3 & 3 & 2 & 3 & 2 \\
1 & 1 & 1 & 1 & 1 & 0 & 1 & 3 & 3 & 2 & 3 & 3 & 2 & 2 \\
1 & 1 & 1 & 1 & 1 & 1 & 0 & 3 & 2 & 3 & 2 & 3 & 2 & 3 \\
\hline
2 & 3 & 2 & 2 & 3 & 3 & 3 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\
2 & 3 & 3 & 3 & 2 & 3 & 2 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\
2 & 2 & 3 & 3 & 3 & 2 & 3 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\
3 & 2 & 3 & 2 & 3 & 3 & 2 & 1 & 1 & 1 & 0 & 1 & 1 & 1 \\
3 & 2 & 2 & 3 & 2 & 3 & 3 & 1 & 1 & 1 & 1 & 0 & 1 & 1 \\
3 & 3 & 2 & 3 & 3 & 2 & 2 & 1 & 1 & 1 & 1 & 1 & 0 & 1 \\
3 & 3 & 3 & 2 & 2 & 2 & 3 & 1 & 1 & 1 & 1 & 1 & 1 & 0
\end{array}\right)$$

(defined by the symmetric $(7, 3, 1)$-design, $PG(2, 2)$).

Clifford Theory for Commutative Schemes

Example (Wreath product)

$$\left(\begin{array}{cc|cc|cc} 0 & 1 & 2 & 2 & 3 & 3 \\ 1 & 0 & 2 & 2 & 3 & 3 \\ \hline 3 & 3 & 0 & 1 & 2 & 2 \\ 3 & 3 & 1 & 0 & 2 & 2 \\ \hline 2 & 2 & 3 & 3 & 0 & 1 \\ 2 & 2 & 3 & 3 & 1 & 0 \end{array}\right)$$

					m_i
χ_1	1	1	2	2	1
χ_2	1	1	2ω	$2\omega^2$	1
χ_3	1	1	$2\omega^2$	2ω	1
χ_4	1	-1	0	0	3

$\omega^3 = 1$

Example (as06-5 $\subset$ as12-39 $\subset$ as24-360)

										m_i
χ_1	1	1	2	2	2	2	2	6	6	1
χ_2	1	1	2	2	-2	-2	-2	-6	6	1
χ_3	1	1	2	2	-2	-2	-2	6	-6	1
χ_4	1	1	2	2	2	2	2	-6	-6	1
χ_5	1	1	-1	-1	-2	1	1	0	0	4
χ_6	1	1	-1	-1	2	-1	-1	0	0	4
χ_7	1	-1	-1	1	0	$\omega-\omega^2$	$-\omega+\omega^2$	0	0	4
χ_8	1	-1	-1	1	0	$-\omega+\omega^2$	$\omega-\omega^2$	0	0	4
χ_9	1	-1	2	-2	0	0	0	0	0	4

$\omega^3 = 1$

Let (X, G) be a commutative scheme, and H a strongly normal closed subset. For $\varphi \in \text{Irr}(H)$, e_φ is a central idempotent of $\mathbb{C}G$. We consider the decompositions

$$\mathbb{C}H = \bigoplus_{\varphi \in \text{Irr}(H)} e_\varphi \mathbb{C}H,$$

$$\mathbb{C}G = \bigoplus_{\varphi \in \text{Irr}(H)} e_\varphi \mathbb{C}G.$$

Now $e_\varphi \mathbb{C}H$ and $e_\varphi \mathbb{C}G$ are $\mathbb{C}$-algebras with the identity e_φ. We will consider Clifford theory for $e_\varphi \mathbb{C}H$ and $e_\varphi \mathbb{C}G$. Note that $\text{Irr}(e_\varphi \mathbb{C}H) = \{\varphi\}$ and $\text{Irr}(e_\varphi \mathbb{C}G) = \text{Irr}(G \mid \varphi)$.

Consider the decomposition

$$e_\varphi \mathbb{C}G = \bigoplus_{g^H \in G//H} e_\varphi \mathbb{C}(HgH),$$

then $e_\varphi \mathbb{C}G$ is $G//H$-graded.

Put

$$Z//H := \{g^H \in G//H \mid e_\varphi \mathbb{C}(HgH) \neq 0\}.$$

Lemma. For $g \in G$, $e_\varphi \mathbb{C}(HgH) \neq 0$ if and only if $e_\varphi \mathbb{C}(HgH)$ contains a unit in $e_\varphi \mathbb{C}G$.

Proposition. Let (X, G) be a (not necessary commutative) association scheme, and let H be a strongly normal closed subset of G. For a character χ of G and a character τ of $G//H$, define $\chi\tau(\sigma_g) = \chi(\sigma_g)\tau(\sigma_{g^H})$. Then $\chi\tau$ is a character of G. Moreover, if $\chi \in \mathrm{Irr}(G)$ and $\tau(1) = 1$, then $\chi\tau \in \mathrm{Irr}(G)$ and $m_{\chi\tau} = m_\chi$.

If $G//H$ is an abelian group, then so is $\mathrm{Irr}(G//H)$, and $\mathrm{Irr}(G//H)$ acts on $\mathrm{Irr}(G)$.

Proof of Lemma. By Frobenius reciprocity, we have

$$\varphi = \sum_{\chi \in \mathrm{Irr}(G|\varphi)} \chi.$$

So, for $g \in G$,

$e_\varphi \sigma_g \neq 0 \iff \chi(\sigma_g) \neq 0$ for some $\chi \in \mathrm{Irr}(G \mid \varphi)$,

$e_\varphi \sigma_g$ is a unit $\iff \chi(\sigma_g) \neq 0$ for all $\chi \in \mathrm{Irr}(G \mid \varphi)$.

We will show that $e_\varphi \sigma_g \neq 0$ implies that $e_\varphi \sigma_g$ is a unit.

Suppose $\chi(\sigma_g) \neq 0$. Note that $\text{Irr}(G//H)$ has a structure of abelian group. Now $\text{Irr}(G//H)$ acts on $\text{Irr}(G \mid \varphi)$ preserving the multiplicities. Put

$$U := \{\chi\tau \mid \tau \in \text{Irr}(G//H)\},$$

$$\text{Stab}_\chi := \{\tau \in \text{Irr}(G//H) \mid \chi\tau = \chi\}.$$

$$\begin{aligned} e_U &:= \sum_{\eta \in U} e_\eta \\ &= \frac{1}{|\text{Stab}_\chi|} \sum_{\tau \in \text{Irr}(G//H)} \frac{m_\chi}{n_G} \sum_{f \in G} \frac{1}{n_f} \overline{\chi\tau(\sigma_f)} \sigma_f \\ &= \frac{1}{|\text{Stab}_\chi|} \sum_{\tau \in \text{Irr}(G//H)} \frac{m_\chi}{n_G} \sum_{f \in G} \frac{1}{n_f} \overline{\chi(\sigma_f)\tau(\sigma_{f^H})} \sigma_f \\ &= \frac{m_\chi}{n_G |\text{Stab}_\chi|} \sum_{f \in G} \frac{1}{n_f} \overline{\chi(\sigma_f)} \left(\sum_{\tau \in \text{Irr}(Z//H)} \overline{\tau(\sigma_{f^H})} \right) \sigma_f \end{aligned}$$

If $f \notin H$ ($\Leftrightarrow f^H \neq 1^H$), then the coefficient of σ_f is 0. So $e_U \in \mathbb{C}H$. But e_φ is primitive in $\mathbb{C}H$, so $U = \text{Irr}(G \mid \varphi)$. Now

$$\chi\tau(\sigma_g) = \chi(\sigma_g)\tau(\sigma_{g^H}) \neq 0$$

for any $\tau \in \text{Irr}(G//H)$, and $\varphi\sigma_g$ is a unit in $\varphi\mathbb{C}G$. (q.e.d.)

Proposition. $Z//H$ is a subgroup of $G//H$ (Z is a closed subset of G), and $e_\varphi \mathbb{C}G$ is a crossed product.

Theorem. Let (X, G) be a commutative association scheme, H a strongly normal closed closed subset of G, and $\varphi \in \mathrm{Irr}(H)$. Put

$$Z//H := \{g^H \in G//H \mid e_\varphi \mathbb{C}(HgH) \neq 0\}.$$

Then we have the followings.

(1) Take $\xi \in \mathrm{Irr}(Z \mid \varphi)$ and fix it. Then

$$\mathrm{Irr}(Z \mid \varphi) = \{\xi\tau \mid \tau \in \mathrm{Irr}(Z//H)\}.$$

(2) The map

$$\mathrm{Irr}(Z \mid \varphi) \to \mathrm{Irr}(G \mid \varphi), \quad (\eta \mapsto \eta^G)$$

is a bijection. Here $\eta^G(\sigma_g) = \eta(\sigma_g)$ for $g \in Z$, and 0 otherwise.

(3) For $\chi \in \mathrm{Irr}(G \mid \varphi)$,

$$m_\chi = \frac{n_G}{n_Z} m_\varphi.$$

For commutative schemes, my conjecture is true.

Corollary. Let (X, G) be a commutative association scheme, and H a strongly normal closed closed subset of G. Then

$$|H| + |G//H| - 1 \leq |G| \leq |H| \cdot |G//H|.$$

Moreover

$$\text{wreath product} \iff |G| = |H| + |G//H| - 1,$$

$$\text{crossed product} \iff |G| = |H| \cdot |G//H|.$$

END.