A REMARK ON OUR PAPER "ALGEBRAIC STRUCTURE OF ASSOCIATION SCHEMES OF PRIME ORDER"

AKIHIDE HANAKI AND KATSUHIRO UNO

In the paper [2], the following theorem is proved.

Theorem 1 ([2, Theorem 3.3]). *Let (X, G) be an association scheme. If $|X|$ is a prime number, then (X, G) is commutative. Moreover, all nontrivial irreducible characters are algebraically conjugate, and all valencies of nontrivial relations and all multiplicities of nontrivial irreducible characters are constant.*

To prove the theorem, we used [1, Theorem 1.2] by Arad et al., but we have found a simpler proof of it which does not use their result. Moreover, it strengthens [2, Lemma 3.2]. Notations are the same as in [2].

Proposition 2. *If all nontrivial irreducible characters of G have the same multiplicities, then G is commutative and all nontrivial relations have the same valencies.*

Proof. Note that $|X| = \sum_{\chi\in\mathrm{Irr}(G)} m_\chi \chi(1)$. Suppose $m_\chi = m$ for every nontrivial irreducible character χ of G. Then $|X| = 1 + m\sum_{\chi\neq 1_G} \chi(1)$. Hence m is prime to $|X|$. Also note that $\sum_{\chi\neq 1_G} \chi(1) \leq \sum_{\chi\neq 1_G} \chi(1)^2 = |G| - 1$ and the equality holds if and only if G is commutative.

The Frame number is

$$\mathcal{F}(G) = |X|^{|G|} \frac{\prod_{g\in G} n_g}{\prod_{\chi\in\mathrm{Irr}(G)} {m_\chi}^{(\chi(1)^2)}} = |X|^{|G|} \frac{\prod_{g\neq 1} n_g}{m^{|G|-1}} \in \mathbb{Z}.$$

Since m is prime to $|X|$, we have

$$\frac{\prod_{g\neq 1} n_g}{m^{|G|-1}} \in \mathbb{Z}$$

and especially $\prod_{g\neq 1} n_g \geq m^{|G|-1}$.

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On the other hand, since the geometric mean is at most equal to the arithmetic mean, we have

$$\left(\prod_{g\neq 1} n_g\right)^{\frac{1}{|G|-1}} \leq \frac{\sum_{g\neq 1} n_g}{|G|-1} \leq \frac{\sum_{g\neq 1} n_g}{\sum_{\chi\neq 1_G}\chi(1)} = \frac{|X|-1}{\sum_{\chi\neq 1_G}\chi(1)} = m.$$

So the equality holds in the above inequality. Especially, $|G|-1 = \sum_{\chi\neq 1_G}\chi(1)$ and this means that G is commutative. Also we have $n_g = m$ for every $1 \neq g \in G$. □

By [2, Lemma 3.1] and the above proposition, Theorem 1 holds.

References

1. Z. Arad, E. Fisman, and M. Muzychuk, *Generalized table algebras*, Israel J. Math. **114** (1999), 29–60.
2. A. Hanaki and K. Uno, *Algebraic structure of association schemes of prime order*, J. Algebraic Combin. **23** (2006), no. 2, 189–195.

Faculty of Science, Shinshu University, Matsumoto, 390-8621, Japan
E-mail address: hanaki@math.shinshu-u.ac.jp

Department of Mathematical Sciences, Osaka Kyoiku University, Kashiwara, Osaka, 582-8582, Japan
E-mail address: uno@cc.osaka-kyoiku.ac.jp