A NOTE ON COMPLEX MATRIX REPRESENTATIONS OF ASSOCIATION SCHEMES

AKIHIDE HANAKI

Abstract. I write this note for a supplement of Higman's paper [1]. Especially, we prove Frobenius-Schur Theorem for association schemes. This note contains no new result.

1. Preliminaries

We denote by $\overline{\alpha}$ the complex conjugate of a complex number or a complex matrix α. The transposition of a matrix M will be denoted by $M^\top$. The (i,j)-entry of a matrix M will be denoted by M_{ij}. For a complex matrix M, we denote by M^* the Hermitian adjoint of M, namely $M^* = \overline{M}^\top$. By E, we denote the identity matrix. Vectors in $\mathbb{C}^n$ are row vectors. The standard inner product of $\mathbb{C}^n$ will be denoted by $\langle x, y\rangle = xy^*$.

Always (X,S) is an association scheme. For a representation $\Phi : \mathbb{C}S \to M_n(\mathbb{C})$ of (X,S) and a non-singular matrix P, we denote by $P^{-1}\Phi P$ the representation of (X,S) defined by $\sigma_s \mapsto P^{-1}\Phi(\sigma_s)P$.

2. $*$-Representations

We often consider the condition

$$\Phi(\sigma_{s^*}) = \Phi(\sigma_s)^* \quad \text{for all } s \in S.$$

We call this condition $*$-*condition* and a representation with this condition a $*$-*representation*.

Proposition 2.1. *Let $\Phi : \mathbb{C}S \to M_n(\mathbb{C})$ be an irreducible representation of an association scheme (X,S). Then there is a $*$-representation Ψ of (X,S) which is similar to Φ.*

To prove Proposition 2.1, we put

$$A = \sum_{s\in S} \frac{1}{n_s}\Phi(\sigma_s)\Phi(\sigma_s)^*.$$

Lemma 2.2. *The matrix A is a positive definite Hermitian matrix.*

Proof. By definition, it is clear that A is a positive semidefinite Hermitian matrix. Suppose $xAx^* = 0$ for $x \in \mathbb{C}^n$. Since $\Phi(\sigma_s)\Phi(\sigma_s)^*$ $(s \in S)$ are positive semidefinite, we have $x\Phi(\sigma_s) = 0$ for all $s \in S$. Since Φ is irreducible, $\Phi(\sigma_s)$ $(s \in S)$ span $M_n(\mathbb{C})$. Thus we have $x = 0$. Now A is positive definite. □

Date: September 27, 2016, revised June 1, 2017.

Lemma 2.3. *For every $t \in S$, $\Phi(\sigma_{t^*})A = A\Phi(\sigma_t)^*$ holds.*

Proof. We have

$$\begin{aligned}
\Phi(\sigma_{t^*})A &= \sum_{s\in S}\frac{1}{n_s}\Phi(\sigma_{t^*})\Phi(\sigma_s)\Phi(\sigma_s)^* = \sum_{s\in S}\sum_{u\in S}\frac{1}{n_s}p_{t^*s}^u\Phi(\sigma_u)\Phi(\sigma_s)^*,\\
A\Phi(\sigma_t)^* &= \sum_{s\in S}\frac{1}{n_s}\Phi(\sigma_s)\Phi(\sigma_s)^*\Phi(\sigma_t)^* = \sum_{s\in S}\frac{1}{n_s}\Phi(\sigma_s)\Phi(\sigma_t\sigma_s)^*\\
&= \sum_{s\in S}\sum_{u\in S}\frac{1}{n_s}p_{ts}^u\Phi(\sigma_s)\Phi(\sigma_u)^* = \sum_{s\in S}\sum_{u\in S}\frac{1}{n_u}p_{u^*t}^{s^*}\Phi(\sigma_s)\Phi(\sigma_u)^*\\
&= \sum_{s\in S}\sum_{u\in S}\frac{1}{n_u}p_{t^*u}^{s}\Phi(\sigma_s)\Phi(\sigma_u)^*.
\end{aligned}$$

The equation holds. □

Proof of Proposition 2.1. By Lemma 2.2, there is a non-singular square matrix B such that $A = BB^*$. By Lemma 2.3, we have $\Phi(\sigma_{t^*})BB^* = BB^*\Phi(\sigma_t)^*$, and so $B^{-1}\Phi(\sigma_{t^*})B = B^*\Phi(\sigma_t)^*(B^*)^{-1} = (B^{-1}\Phi(\sigma_t)B)^*$ for all $t \in S$. Then $\Psi = B^{-1}\Phi B$ satisfies the required condition. □

Remark. For an irreducible representation $\Phi : \mathbb{C}S \to M_n(\mathbb{C})$, we suppose that $\Phi(\sigma_s) \in M_n(\mathbb{R})$ for all $s \in S$. Then there is a $*$-representation Ψ which is similar to Φ and $\Psi(\sigma_s) \in M_n(\mathbb{R})$ for all $s \in S$.

Corollary 2.4. *Let $\Phi : \mathbb{C}S \to M_n(\mathbb{C})$ be a representation of an association scheme (X, S). Then there is a $*$-representation Ψ of (X, S) which is similar to Φ.*

Proof. Consider an irreducible decomposition of Φ and apply Proposition 2.1. □

Proposition 2.5. *Let Φ and Ψ be similar $*$-representations of an association scheme (X, S). Then there exists a unitary matrix U such that $\Psi = U^{-1}\Phi U$.*

The proof will be done by some steps.

Lemma 2.6. *Let Φ and Ψ be similar irreducible $*$-representations of an association scheme (X, S). Then there exists a unitary matrix U such that $\Psi = U^{-1}\Phi U$.*

Proof. Since Φ and Ψ are similar, there is a non-singular matrix P such that $\Psi = P^{-1}\Phi P$. Now $\Psi(\sigma_s) = P^{-1}\Phi(\sigma_s)P$ and so $\Psi(\sigma_{s^*}) = \Psi(\sigma_s)^* = P^*\Phi(\sigma_{s^*})(P^{-1})^* = P^*\Phi(\sigma_{s^*})(P^*)^{-1}$. Also we have $\Psi(\sigma_{s^*}) = P^{-1}\Phi(\sigma_{s^*})P$. Therefore $(PP^*)^{-1}\Phi(\sigma_{s^*})(PP^*) = \Phi(\sigma_{s^*})$ for all $s \in S$. Since $\Phi(\sigma_{s^*})$ $(s \in S)$ span the full matrix algebra, $PP^* = \alpha E$ for some $\alpha \in \mathbb{C}$. Since PP^* is positive definite, α must be a positive real number. Put $U = \frac{1}{\sqrt{\alpha}}P$, then U is unitary and $\Psi = U^{-1}\Phi U$. □

Lemma 2.7. *Let Φ be a reducible $*$-representation of an association scheme (X, S). Then there exists a unitary matrix U such that*

$$U^{-1}\Phi(\sigma_s)U = \begin{pmatrix} \Phi_1(\sigma_s) & \\ & \Phi_2(\sigma_s) \end{pmatrix},$$

for all $s \in S$. In this case, Φ_i $(i = 1, 2)$ are also $*$-representations.

Proof. Let $\mathbb{C}^n$ be the representation space of Φ, and let W be a $\mathbb{C}S$-invariant subspace of $\mathbb{C}^n$. We consider $W^\perp = \{x \in \mathbb{C}^n \mid \langle W, x\rangle = 0\}$, where $\langle \bullet, \bullet \rangle$ is the standard inner product. We show that $W^\perp$ is also $\mathbb{C}S$-invariant. Suppose $x \in W^\perp$. Then, for any $w \in W$ and $s \in S$, we have

$$\langle w, x\Phi(\sigma_s)\rangle = w\Phi(\sigma_s)^* x^* = w\Phi(\sigma_{s^*})x^* = \langle w\Phi(\sigma_{s^*}), x\rangle = 0$$

since $w\Phi(\sigma_{s^*}) \in W$. This means that $W^\perp$ is also $\mathbb{C}S$-invariant.

Now we choose orthonormal bases of W and $W^\perp$ and combine them to get an orthonormal basis of $\mathbb{C}^n$. We make a unitary matrix U by the orthonormal basis. Then we have a decomposition of Φ. It is clear that Φ_i $(i = 1, 2)$ satisfy $*$-condition. □

Proof of Proposition 2.5. By Lemma 2.7, we have irreducible decompositions of Φ and Ψ by unitary matrices. Since they are similar, the assertion holds by Lemma 2.6. □

3. Frobenius-Schur Theorem

For $\chi \in \mathrm{Irr}(S)$, we define

$$\nu_2(\chi) = \frac{m_\chi}{|X|\chi(1)} \sum_{s\in S} \frac{1}{n_s}\chi(\sigma_s^2)$$

and call this number the *Frobenius-Schur indicator* of χ.

For $\chi \in \mathrm{Irr}(S)$, we say that

- χ is of the *first kind* if χ is afforded by a real representation,
- χ is of the *second kind* if χ is real and not afforded by a real representation, and
- χ is of the *third kind* if χ is not real,

where we say that χ is *real* if $\chi(\sigma_s) \in \mathbb{R}$ for all $s \in S$, and a representation Φ is *real* if $\Phi(\sigma_s)$ are matrices over $\mathbb{R}$ for all $s \in S$. Obviously, a real representation affords a real character.

We will prove Frobenius-Schur Theorem for association schemes.

Theorem 3.1 (Frobenius-Schur Theorem [1, (7.5)])**.** *Let χ be an irreducible character of an association scheme (X, S). Then $\nu_2(\chi) \in \{-1, 0, 1\}$ and*

- *$\nu_2(\chi) = 1$ if and only if χ if of the first kind,*
- *$\nu_2(\chi) = -1$ if and only if χ if of the second kind, and*
- *$\nu_2(\chi) = 0$ if and only if χ if of the third kind.*

We will prove Theorem 3.1 by the following way :

- If χ is of the first kind, then $\nu_2(\chi) = 1$ (Lemma 3.2).
- If χ is of the third kind, then $\nu_2(\chi) = 0$ (Lemma 3.2).
- If χ is of the first or second kind, then $\nu_2(\chi) \in \{-1, 1\}$ (Lemma 3.3).
- If $\nu_2(\chi) = 1$, then χ is of the first kind (Lemma 3.4).

Combining these facts, we can prove Theorem 3.1.

Lemma 3.2. *If χ is of the third kind, then $\nu_2(\chi) = 0$. If χ is of the first kind, then $\nu_2(\chi) = 1$.*

Proof. Let Φ be a representation affording χ. We may assume that Φ satisfies $*$-condition. We have

$$\begin{aligned}
\nu_2(\chi) &= \frac{m_\chi}{|X|\chi(1)}\sum_{s\in S}\frac{1}{n_s}\chi(\sigma_s^2) = \frac{m_\chi}{|X|\chi(1)}\sum_{s\in S}\frac{1}{n_s}\mathrm{trace}(\Phi(\sigma_s)\Phi(\sigma_s)) \\
&= \frac{m_\chi}{|X|\chi(1)}\sum_{s\in S}\frac{1}{n_s}\sum_{i=1}^{\chi(1)}\sum_{j=1}^{\chi(1)}\Phi(\sigma_s)_{ij}\Phi(\sigma_s)_{ji} \\
&= \frac{m_\chi}{|X|\chi(1)}\sum_{s\in S}\frac{1}{n_s}\sum_{i=1}^{\chi(1)}\sum_{j=1}^{\chi(1)}\overline{\Phi(\sigma_{s^*})_{ji}}\Phi(\sigma_s)_{ji}.
\end{aligned}$$

Suppose that χ is of the third kind. Then Φ and $\overline{\Phi}$ are non-similar. Thus $\nu_2(\chi) = 0$ by Schur relations ([1, (3.8)] or [3, Theorem 4.2.4]).

Suppose that χ is of the first kind. By Remark after the proof of Proposition 2.1, χ is afforded by a real $*$-representation. Hence we assume that Φ is a real $*$-representation. Then, again by Schur relations, we have

$$\begin{aligned}
\nu_2(\chi) &= \frac{m_\chi}{|X|\chi(1)}\sum_{s\in S}\frac{1}{n_s}\sum_{i=1}^{\chi(1)}\sum_{j=1}^{\chi(1)}\overline{\Phi(\sigma_{s^*})_{ji}}\Phi(\sigma_s)_{ji} \\
&= \frac{m_\chi}{|X|\chi(1)}\sum_{s\in S}\frac{1}{n_s}\sum_{i=1}^{\chi(1)}\sum_{j=1}^{\chi(1)}\Phi(\sigma_{s^*})_{ji}\Phi(\sigma_s)_{ji} \\
&= \frac{m_\chi}{|X|\chi(1)}\sum_{s\in S}\frac{1}{n_s}\sum_{i=1}^{\chi(1)}\Phi(\sigma_{s^*})_{ii}\Phi(\sigma_s)_{ii} \\
&= \frac{m_\chi}{|X|\chi(1)}\chi(1)\frac{|X|}{m_\chi} = 1.
\end{aligned}$$

Thus the lemma holds. □

Lemma 3.3. *Suppose that χ is of the first or second kind. Let Φ be a $*$-representation affording χ. Then there is a unitary matrix U such that $\overline{\Phi} = U^{-1}\Phi U$. In this case, $U^\top = \nu_2(\chi)U$ and $\nu_2(\chi) \in \{-1, 1\}$.*

Proof. By assumption, $\overline{\Phi}$ and Φ are similar. Thus there is a unitary matrix U such that $\overline{\Phi} = U^{-1}\Phi U$. By $\overline{\Phi(\sigma_{s^*})} = U^{-1}\Phi(\sigma_{s^*})U$, we have $\Phi(\sigma_s) = \overline{U}^{-1}\overline{\Phi(\sigma_s)}\overline{U}$. Thus

$$\overline{\Phi(\sigma_s)} = U^{-1}\overline{U}^{-1}\overline{\Phi(\sigma_s)}\overline{U}U = (\overline{U}U)^{-1}\overline{\Phi(\sigma_s)}(\overline{U}U).$$

This equation holds for all $s \in S$ and $\Phi(\sigma_s)$ $(s \in S)$ span $M_{\chi(1)}(\mathbb{C})$, we can write $\overline{U}U = \alpha E$ for some $\alpha \in \mathbb{C}$. Since U is unitary, we have $U = \alpha U^\top = \alpha^2 U$. Therefore $\alpha^2 = 1$ and $\alpha = \pm 1$.

We will show that $\alpha = \nu_2(\chi)$. We put $U = (u_{ij})$. By definition, $u_{ji} = \alpha u_{ij}$. By Schur relations, we have

$$\begin{aligned}\nu_2(\chi) &= \frac{m_\chi}{|X|\chi(1)} \sum_{s\in S} \frac{1}{n_s} \sum_{i=1}^{\chi(1)} \sum_{j=1}^{\chi(1)} \overline{\Phi(\sigma_{s^*})_{ji}} \Phi(\sigma_s)_{ji} \\ &= \frac{m_\chi}{|X|\chi(1)} \sum_{s\in S} \frac{1}{n_s} \sum_{i=1}^{\chi(1)} \sum_{j=1}^{\chi(1)} \sum_{k=1}^{\chi(1)} \sum_{\ell=1}^{\chi(1)} \overline{u_{kj}} \Phi(\sigma_{s^*})_{k\ell} u_{\ell i} \Phi(\sigma_s)_{ji} \\ &= \frac{m_\chi}{|X|\chi(1)} \sum_{i=1}^{\chi(1)} \sum_{j=1}^{\chi(1)} \overline{u_{ij}} u_{ji} \sum_{s\in S} \frac{1}{n_s} \Phi(\sigma_{s^*})_{ij} \Phi(\sigma_s)_{ji} \\ &= \frac{1}{\chi(1)} \sum_{i=1}^{\chi(1)} \sum_{j=1}^{\chi(1)} \overline{u_{ij}} u_{ji} = \frac{\alpha}{\chi(1)} \sum_{i=1}^{\chi(1)} \sum_{j=1}^{\chi(1)} \overline{u_{ij}} u_{ij} \\ &= \frac{\alpha}{\chi(1)} \mathrm{trace}(\overline{U} U^\top) = \frac{\alpha}{\chi(1)} \mathrm{trace}(E) = \alpha.\end{aligned}$$

Thus $\alpha = \nu_2(\chi)$ holds. □

Lemma 3.4. *If $\nu_2(\chi) = 1$, then χ is of the first kind.*

Proof. Let Φ be a $*$-representation affording χ. Suppose $\nu_2(\chi) = 1$. There is a unitary matrix U such that $\overline{\Phi} = U^{-1}\Phi U$ and $U^\top = U$ by Lemma 3.3. By [2, Lemma 4.18], there is a non-singular matrix V such that $U = V\overline{V}^{-1}$. Now $\overline{\Phi(\sigma_s)} = \overline{V} V^{-1} \Phi(\sigma_s) V \overline{V}^{-1}$ and so $\overline{V}^{-1} \overline{\Phi(\sigma_s)} \overline{V} = V^{-1}\Phi(\sigma_s) V$ for all $s \in S$. This means that the representation $V^{-1}\Phi V$ is a real representation. □

Now Theorem 3.1 was proved.

Corollary 3.5. *We have $\sum_{\chi\in \mathrm{Irr}(S)} \nu_2(\chi)\chi(1) = \sharp\{s \in S \mid s = s^*\}$.*

Proof. We denote by γ the standard character of (X, S). Then

$$\sum_{s\in S} \frac{1}{n_s} \gamma(\sigma_s^2) = \sum_{s\in S} \frac{1}{n_s} \sum_{\chi\in \mathrm{Irr}(S)} m_\chi \chi(\sigma_s^2) = \sum_{\chi\in \mathrm{Irr}(S)} m_\chi \nu_2(\chi) \frac{|X|\chi(1)}{m_\chi}$$

and

$$\sum_{s\in S} \frac{1}{n_s} \gamma(\sigma_s^2) = \sum_{s\in S} \frac{1}{n_s} p_{ss}^1 |X| = \sum_{s=s^*} |X| = |X| \cdot \sharp\{s \in S \mid s = s^*\}.$$

Thus the statement holds. □

Example 3.6. Let (X, S) be a noncommutative association scheme with $|S| = 6$. Then we can set $\mathrm{Irr}(S) = \{\chi_1, \chi_2, \chi_3\}$, $\chi_1(1) = \chi_2(1) = 1$ and $\chi_3(1) = 2$. Since there is a trivial character, we can see that $\nu_2(\chi_1) = \nu_2(\chi_2) = 1$. Also we have $\overline{\chi_3} = \chi_3$. Suppose $\nu_2(\chi_3) = -1$. Then $\sharp\{s \in S \mid s = s^*\} = 1 + 1 - 2 = 0$, but this is impossible since $1 = 1^*$. Hence $\nu_2(\chi_3) = 1$. Now we can see that $\sharp\{s \in S \mid s = s^*\} = 1 + 1 + 2 = 4$ and χ_3 is afforded by a real representation.

References

[1] D. G. Higman. Coherent configurations. I. Ordinary representation theory. *Geometriae Dedicata*, 4(1):1–32, 1975.

[2] I. M. Isaacs. *Character theory of finite groups*. Academic Press [Harcourt Brace Jovanovich Publishers], New York, 1976.

[3] P.-H. Zieschang. *An algebraic approach to association schemes*, volume 1628 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1996.

E-mail address: hanaki@shinshu-u.ac.jp (Akihide Hanaki)