

Convergence rate of lattice approximations for reflected SPDEs driven by Gaussian space-time white noise

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Abstract

In this paper, we study the convergence rate of a lattice approximation scheme for a reflected stochastic partial differential equation (SPDE) driven by Gaussian space-time white noise, which can be viewed as an infinite-dimensional Skorohod problem. We establish the $L^p(\Omega)$ -convergence rate for the lattice approximation in the uniform topology under Lipschitz assumptions. The proof is based on a penalization method and a detailed analysis of deterministic parabolic obstacle problems. Our result shows that the convergence rate is logarithmic, reflecting the additional difficulty caused by the reflection term. To the best of our knowledge, this is the first explicit convergence rate for approximations of reflected SPDEs.

Keywords: Reflected SPDE, obstacle problem, Skorohod type equation, lattice approximation, penalization, convergence rate.

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1 Introduction

In this paper, we investigate the convergence rate of the lattice approximation for the reflected stochastic partial differential equation (SPDE) on $[0, 1]$ driven by multiplicative Gaussian space-time white noise

$$\left\{ \begin{array}{l} \frac{\partial u}{\partial t}(t, x) = \frac{\partial^2 u}{\partial x^2}(t, x) + f(t, x, u(t, x)) \\ \quad + \sigma(t, x, u(t, x))\dot{W}(t, x) + \eta(dt, dx), \quad t > 0, \quad x \in (0, 1), \\ u(t, 0) = u(t, 1) = 0, \quad t \geq 0, \\ u(0, x) = u_0(x), \quad x \in [0, 1], \\ u(t, x) \geq 0, \quad t > 0, \quad x \in [0, 1] \text{ a.s.}, \end{array} \right. \quad (1.1)$$

where the initial value u_0 is a non-negative random function defined on the interval $[0, 1]$ with $u_0(0) = u_0(1) = 0$, the coefficients $f, \sigma : \mathbb{R}_+ \times [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$ are measurable mappings, $\{\dot{W}(t, x) : t \geq 0, x \in [0, 1]\}$ denotes the space-time white noise on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration $(\mathcal{F}_t)_{t \geq 0}$, and η denotes a positive random measure on $[0, \infty) \times [0, 1]$ supported on the zero set of u . We assume that initial values are $\mathcal{F}_0$ -measurable and $\mathcal{F}_0$ is independent of

$\{\dot{W}(t, x) : t \geq 0, x \in [0, 1]\}$. As usual, let $(\mathcal{F}_t)_{t \geq 0}$ be the filtration generated by $\mathcal{F}_0 \vee \sigma(W(s, x) : s \in [0, t], x \in [0, 1]) \vee \mathcal{N}$, where $\mathcal{N}$ denotes the family of $\mathbb{P}$ -null sets.

Such equations can be interpreted as infinite-dimensional Skorohod problems and arise naturally in various applications, including stochastic interface models [13] and financial markets [19]. The reflected SPDE (1.1) was first introduced in [27] in 1992 for additive noise (i.e., $\sigma \equiv 1$). It was later extended to the case of multiplicative noise in [10] in 1993, where the existence of solutions was established, while the uniqueness remained open until it was resolved in [33]. Due to the presence of the reflection constraint $u(t, x) \geq 0$ and the associated random measure η , the analysis of the reflected SPDE (1.1) is significantly more involved than that of standard SPDEs, so it has attracted the interest of many researchers. Various important properties have been studied, including reversible measures [13], [34], the integration by parts formulae [35], hitting properties [8], occupation densities [36], the large deviation principle of the Freidlin–Wentzell type [33], strong Feller property [38], Hölder continuity [9], hypercontractivity [31], log-Harnack inequality [32] and so on. T. Funaki and S. Olla [13] in 2001 proved that the fluctuation of a $\nabla\phi$ -interface model on a hard wall weakly converges to the stationary solution of the reflected SPDE driven by additive noise. We also refer the reader to the monograph [37] for a comprehensive account.

On the numerical side, lattice approximations (also called finite difference schemes) for SPDEs without reflection have been extensively studied. Lattice approximations reduce the infinite dimensional problem to finite dimensional stochastic differential equations (SDEs), allowing the application of finite dimensional stochastic analysis. This approach was first introduced for SPDEs by Funaki [12] with application to the study of invariant measures. After that many results have been published, such as SPDEs with one-sided Lipschitz continuity [28], stochastic elliptic type [26] and stochastic Burgers equation [1]. Recently, it has also been widely used in the study of singular SPDEs after the groundbreaking work by Hairer [20], for example, weak universality of the 2d parabolic Anderson model [25] and the dynamical Φ_3^4 model [40]. On the other hand, the convergence rate of numerical approximations to various SPDEs is very important both in theory and applications, and it has attracted considerable attention. The convergence rate of spatial lattice approximations of SPDE without reflection was first studied by Gyöngy [17], and the implicit and explicit schemes were studied in [18]. It also was shown in [11] that the optimal rate of uniform convergence in this case is $\frac{1}{2}$ – as the spatial discretisation tends to zero. In addition, convergence rates of numerical approximations have also been studied for non-linear SPDEs in various schemes, including 2D Navier-Stokes equations [5], stochastic Landau-Lifshitz-Bloch equation [16], stochastic Burgers-type equations [21], stochastic Allen-Cahn-type equations [15], stochastic nonlinear Schrödinger equations [6].

However, the presence of reflection in the reflected SPDE (1.1) introduces additional difficulties, and much less is known about its numerical approximation. The lattice approximation for (1.1) was studied by Zhang in [39] in 2016, where the convergence of this scheme was established using tightness arguments, but no explicit convergence rate was established. In particular, the author pointed out that “Because of the appearance of the random measure (local time) term it seems hard to get the precise rate of the convergence”, [39, Remark 2.1]. This is the main motivation of the present work.

The primary objective of this paper is to establish an explicit convergence rate for lattice approximations of the reflected SPDE (1.1); see (2.3) below for the associated lattice scheme (also called a Skorohod-type system). Following the approach in [39], the core of our analysis lies in studying deterministic parabolic obstacle problems and their lattice approximations. The strategy

is based on a penalization method: First, we establish convergence rates for the penalized equations of (3.1) and (3.2) respectively. Those allow us to quantify the penalization error. Second, we derive error estimates between the parabolic penalized equation and its lattice scheme. Finally, by balancing the penalization and its discretization errors, we obtain a convergence rate for (3.1), which yields, as an application, the desired convergence rate for (1.1).

Now let us state our assumptions and introduce some notations. We assume the following are satisfied throughout this paper.

(A.0) The initial value u_0 is $\mathcal{F}_0$ -measurable, and $u_0 \in C^\alpha([0, 1])$, $\alpha \in (0, 1)$ with $u_0(0) = u_0(1) = 0$ such that for any $p \geq 1$

$$\mathbb{E} \left[\|u_0\|_{C^\alpha}^p \right] < \infty.$$

(A.1) (Lipschitz condition) There exists a constant $C > 0$ such that for any $x, y \in [0, 1]$, $t \in [0, T]$, $u, v \in \mathbb{R}$,

$$|f(t, x, u) - f(t, y, v)| + |\sigma(t, x, u) - \sigma(t, y, v)| \leq C(|x - y| + |u - v|).$$

(A.2) (Linear growth) There exists a constant $C > 0$ such that for any $x \in [0, 1]$, $t \in [0, T]$, $u \in \mathbb{R}$,

$$|f(t, x, u)| + |\sigma(t, x, u)| \leq C(1 + |u|).$$

Let us prepare some notations. Throughout the paper, let $T > 0$ be fixed and let $C > 0$ denote a generic constant, independent of $n \in \mathbb{N}$, ε and λ , whose value may change from line to line. Let $C([0, T] \times [0, 1])$ be the space of continuous functions $u(t, x)$ defined on $[0, T] \times [0, 1]$ such that

$$\|u\|_{T, \infty} := \sup_{t \in [0, T], x \in [0, 1]} |u(t, x)| < \infty.$$

Similarly, $C([0, T]; \mathbb{R}^n)$ denotes the class of all $\mathbb{R}^n$ -valued continuous functions U defined on $[0, T]$ such that

$$\|U\|_{T, \infty} := \sup_{0 \leq t \leq T} \max_{1 \leq k \leq n} |U_k(t)| < \infty. \quad (1.2)$$

Although the same symbol is also used to denote the norm of $C([0, T] \times [0, 1])$ above, the context ensures that no ambiguity will arise. For brevity, we also sometimes write $\mathcal{Q}_T := [0, T] \times [0, 1]$ and use $\rho((t, x), (s, y)) := |t - s|^{\frac{1}{2}} + |x - y|$ for the parabolic metric.

Under the above assumptions, we prove that the lattice approximation u^n defined by (2.5) below converges to the solution u of the reflected SPDE (1.1) in the uniform topology $\|\cdot\|_{T, \infty}$ with the explicit logarithmic rate $(\log n)^{-\kappa p/2}$, $0 < \kappa < \gamma := \frac{1}{2} \wedge \alpha$, see Theorem 2.2 below. The logarithmic rate arises from the use of Lipschitz estimates for the penalized system, which leads to exponential dependence on the penalization parameter. Although the penalized term is in fact monotone, exploiting this structure at the supremum norm level remains a challenging problem. It remains an interesting open question whether the logarithmic rate can be improved to a polynomial one.

The remainder of this paper is organized as follows. In Section 2, we introduce the lattice approximation framework and state the main result. In Section 3, we analyze deterministic obstacle problems and establish convergence rates for their lattice approximations via penalization methods. In Section 4, we apply these results to the reflected SPDE (1.1) and complete the proof of the main theorem.

2 Preliminaries, Lattice framework and main result

We begin by recalling the definition of the solution to the reflected SPDE (1.1), following [10]; see also [39].

Definition 2.1. Let $u_0 \in C([0, 1])$ be a non-negative function with $u_0(0) = u_0(1) = 0$. A pair (u, η) is called a solution of the reflected SPDE (1.1) if the following are satisfied:

- (i) $u = \{u(t, x)\}_{t \geq 0, x \in [0, 1]}$ is an $\mathcal{F}_t$ -adapted continuous random field, and $u(t, x) \geq 0$ a.s.
- (ii) η is a random measure on $\mathbb{R}_+ \times (0, 1)$ such that
 - (a) $\eta(\{t\} \times (0, 1)) = 0$ for every $t \geq 0$;
 - (b)

$$\int_0^t \int_0^1 x(1-x) \eta(ds, dx) < \infty, \quad t \geq 0;$$

- (c) for every measurable function $\psi : [0, \infty) \times [0, 1] \rightarrow \mathbb{R}_+$,

$$\int_0^t \int_0^1 \psi(s, x) \eta(ds, dx)$$

is $\mathcal{F}_t$ -measurable.

- (iii) For every $\phi \in C^2([0, 1])$ with $\phi(0) = \phi(1) = 0$,

$$\begin{aligned} \langle u(t), \phi \rangle - \int_0^t \langle u(s), \phi'' \rangle ds - \int_0^t \langle f(s, \cdot, u(s)), \phi \rangle ds \\ = \langle u_0, \phi \rangle + \int_0^t \int_0^1 \phi(x) \sigma(s, x, u(s, x)) W(ds, dx) + \int_0^t \int_0^1 \phi(x) \eta(ds, dx). \end{aligned}$$

- (iv) (Reflection condition)

$$\int_0^\infty \int_0^1 u(t, x) \eta(dt, dx) = 0.$$

Remark (i) on p.19 [10] tells us that the weak form (iii) above is equivalent to the mild solution

$$\begin{aligned} u(t, x) = \int_0^1 G(t, x, y) u_0(y) dy + \int_0^t \int_0^1 G(t-s, x, y) f(s, y, u(s, y)) dy ds \\ + \int_0^t \int_0^1 G(t-s, x, y) \sigma(s, y, u(s, y)) W(ds, dy) \\ + \int_0^t \int_0^1 G(t-s, x, y) \eta(ds, dy), \end{aligned} \tag{2.1}$$

where $G(t, x, y)$ is the heat kernel of the Laplacian $\frac{\partial^2}{\partial x^2}$ on the interval $[0, 1]$ with Dirichlet boundary conditions, i.e.,

$$G(t, x, y) = \sum_{j=1}^{\infty} \exp(-j^2 \pi^2 t) \phi_j(x) \phi_j(y),$$

and $\phi_j(x) := \sqrt{2} \sin(jx\pi)$. In this paper, instead of the weak form, we mainly use the mild form as [39].

We will recall some important properties including the well-posedness and regularity results at the end of this section.

Next, let us introduce the lattice approximation for the reflected SPDE (1.1) according to [39], see also [17] and [12] for the case of SPDEs without reflection. For each $n \in \mathbb{N}$, let $x_k := k/n$, $k = 0, \dots, n$. We define the $(n-1)$ -dimensional stochastic process $u^n(t, x_k)$, $k = 1, \dots, n-1$ as the unique solution of $(n-1)$ -dimensional reflected SDE (also called the Skorohod-type SDE)

$$\begin{cases} du^n(t, x_k) = n^2(u^n(t, x_{k+1}) - 2u^n(t, x_k) + u^n(t, x_{k-1})) dt \\ \quad + n \sigma(t, x_k, u^n(t, x_k)) d(W(t, x_{k+1}) - W(t, x_k)) \\ \quad + f(t, x_k, u^n(t, x_k)) dt + d\eta_k^n(t), \quad k = 1, \dots, n-1, \\ u^n(t, x_k) \geq 0, \\ u^n(0, x_k) = u_0(x_k) \end{cases} \quad (2.2)$$

with the constraints

$$u^n(t, 0) = u^n(t, 1) = 0, \quad t \geq 0.$$

Since the definition of the reflected SDE (2.2) is standard and similar to that of the reflected SPDE (1.1), we omit the details here, see [39, Definition 2.1] for example. We just point out that $\eta_k^n(t)$, $t \geq 0$ is an $\mathcal{F}_t$ -adapted, continuous, increasing process with $\eta_k^n(0) = 0$ such that $\int_0^t u^n(s, x_k) \eta_k^n(ds) = 0$ for all $t \geq 0$ and $k = 1, 2, \dots, n-1$.

Let us here introduce some notations to write (2.2) in its abstract form. Set

$$u_k^n(t) := u^n(t, x_k) \text{ and } W_k^n(t) := \sqrt{n} (W(t, x_{k+1}) - W(t, x_k)).$$

By the properties of $\dot{W}$, one knows that $W^n = (W_1^n(t), \dots, W_{n-1}^n(t))$ is an $(n-1)$ -dimensional Brownian motion. Let $A^n = (A_{ki}^n)$ denote the $(n-1) \times (n-1)$ matrix with elements

$$A_{kk}^n = -2, \quad A_{ki}^n = 1 \text{ for } |k-i| = 1, \quad A_{ki}^n = 0 \text{ for } |k-i| > 1.$$

For $t \geq 0$ and each $y = (y_1, y_2, \dots, y_{n-1}) \in \mathbb{R}^{n-1}$, let $F^n(t, y) = (f(t, x_1, y_1), \dots, f(t, x_{n-1}, y_{n-1}))$ and let $\Sigma^n(t, y)$ be the $(n-1) \times (n-1)$ diagonal matrix with $\Sigma_{kk}^n(t, y) = \sigma(t, x_k, y_k)$. Then using those notations, as explained in [39], the system (2.2) can be reformulated as the reflected SDE on the domain $\mathcal{Q}_n := \{z \in \mathbb{R}^{n-1} : z_i \geq 0, i = 1, 2, \dots, n-1\}$

$$du^n(t) = n^2 A^n u^n(t) dt + F^n(t, u^n(t)) dt + \sqrt{n} \Sigma^n(t, u^n(t)) dW^n(t) + d\eta^n(t) \quad (2.3)$$

with the initial value $u^n(0) = (u_0(x_1), u_0(x_2), \dots, u_0(x_{n-1}))$. Since the domain $\mathcal{Q}_n$ is convex, the well-posedness of the system (2.3) is well-known, we refer to [24] and [29]. It is known that

$$e_j = \left(\sqrt{\frac{2}{n}} \sin(jx_1\pi), \dots, \sqrt{\frac{2}{n}} \sin(jx_{k-1}\pi), \dots, \sqrt{\frac{2}{n}} \sin(jx_{n-1}\pi) \right) \in \mathbb{R}^{n-1}, j = 1, 2, \dots, n-1$$

are the eigenvectors of $n^2 A^n$ with corresponding eigenvalues

$$\lambda_j^n := -4n^2 \sin^2 \left(\frac{j}{2n} \pi \right),$$

which form an orthonormal basis of $\mathbb{R}^{n-1}$. It is known that the semigroup

$$G^n(t) := \exp(n^2 A^n t)$$

generated by $n^2 A^n$ on $\mathbb{R}^{n-1}$ admits the representation

$$G^n(t)x = \sum_{j=1}^{n-1} e^{\lambda_j^n t} \langle x, e_j \rangle e_j, \quad x \in \mathbb{R}^{n-1}. \quad (2.4)$$

Using the semigroup $G^n(t)$ and Itô's formula, the solution of (2.3) admits the mild representation

$$\begin{aligned} u^n(t) = & G^n(t)u^n(0) + \int_0^t G^n(t-s)F^n(s, u^n(s))ds \\ & + \sqrt{n} \int_0^t G^n(t-s)\Sigma^n(s, u^n(s))dW^n(s) + \int_0^t G^n(t-s)d\eta^n(s). \end{aligned}$$

For $n \in \mathbb{N}$, we extend the solution $u^n(t) \in \mathbb{R}^{n-1}$ to a continuous random field on $[0, T] \times [0, 1]$ by the polygonal approximation. More precisely, for $x \in [x_k, x_{k+1})$, $k = 0, \dots, n-1$, define

$$u^n(t, x) := u^n(t, x_k) + (nx - k)(u^n(t, x_{k+1}) - u^n(t, x_k)) \quad (2.5)$$

for $x \in [x_k, x_{k+1})$, $k = 0, \dots, n-1$ with $u^n(t, 0) = u^n(t, 1) := 0$.

The main purpose of this paper is to study the convergence rate of the interpolated lattice approximation $u^n(t, x)$ to the solution $u(t, x)$ of (1.1). To simplify the notation, for $a = (a_0, \dots, a_n) \in \mathbb{R}^{n+1}$ with $a_0 = a_n = 0$, we denote by $\mathcal{I}_n a(x)$, $x \in [0, 1]$ its piecewise linear interpolation, namely

$$(\mathcal{I}_n a)(x) := a_k + (nx - k)(a_{k+1} - a_k), \quad x \in [x_k, x_{k+1})$$

with $\mathcal{I}_n a(0) = \mathcal{I}_n a(1) := 0$. Using this convention, we have

$$u^n(t, x) = \mathcal{I}_n u^n(t)(x), \quad t \in [0, T], x \in [0, 1]. \quad (2.6)$$

We now state the main convergence theorem, which gives the explicit convergence rate of $u^n(t, x)$ to the solution $u(t, x)$ of the reflected SPDE (1.1).

Theorem 2.2. *Assume (A.0)–(A.2) are satisfied. Then for every $0 < \kappa < \gamma$ and $p \geq 1$, there exists a constant $C > 0$ such that*

$$\mathbb{E}[\|u^n - u\|_{T, \infty}^p] \leq C(\log n)^{-\frac{\kappa p}{2}}, \quad n \geq 2. \quad (2.7)$$

Here and in the sequel, we set $\gamma := \frac{1}{2} \wedge \alpha$.

In particular, the lattice approximation converges to the solution in $L^p(\Omega; C([0, T] \times [0, 1]))$ with an explicit logarithmic rate.

Remark 2.3. (i) For SPDEs without reflection, it is well known that the optimal strong convergence rate of lattice approximations is polynomial of order $n^{-1/2-}$, see [17, 11].

(ii) In contrast, the convergence rate obtained in Theorem 2.2 is logarithmic. This slower rate reflects the additional analytical difficulty caused by the reflection term. As discussed in Section 1, the use of penalization introduces exponential dependence on the penalization parameter, which leads to a logarithmic rate after balancing errors. Whether this rate can be improved remains an open problem.

As preparation, let us summarize some important properties about solutions of (1.1) and (2.3), which will be used later. The first one is about the moment estimates of solutions. By Theorem 2.1 [33], Proposition 5.1 [39] and [29], we have the next theorem.

Theorem 2.4. *Under the assumptions in Theorem 2.2, the existence and uniqueness of solutions hold for both (1.1) and (2.3). Moreover, for each $p \geq 1$, there exists a constant $C > 0$ such that*

$$\mathbb{E}[\|u\|_{T,\infty}^p] + \sup_{n \in \mathbb{N}} \mathbb{E}[\|u^n\|_{T,\infty}^p] \leq C.$$

It was well known that Hölder continuity of solutions to SPDEs is important to study the convergence rate of lattice approximation. However, it is not trivial for reflected SPDEs. It was first proved by Dalang and Zhang in 2013 for the reflected SPDE (1.1) with coefficients f and σ independent of t and x , see [9, Theorem 3.3]. On the other hand, in [9], the authors assume that the initial value $u_0(x) \equiv 0$. So, [9, Theorem 3.3] can not be applied to our case directly. In the following, we generalize the main result in [9] to the reflected SPDE (1.1) under the assumptions (A.0)–(A.2), which will be used in the proof of Theorem 2.2 in Section 4.

Theorem 2.5. *Assume (A.0)–(A.2) are satisfied and let $\alpha_* := (2\alpha) \wedge 1$. Then for any*

$$0 < \bar{\alpha} < \frac{\alpha_*}{2},$$

there exists a nonnegative random variable $\mathcal{M}$ such that

$$|u(t, x) - u(s, y)| \leq \mathcal{M} (|t - s|^{\frac{1}{2}} + |x - y|)^{\bar{\alpha}}, \quad s, t \in [0, T], \quad x, y \in [0, 1] \text{ a.s.}, \quad (2.8)$$

and for any $p \geq 1$,

$$\mathbb{E}[\mathcal{M}^p] < \infty. \quad (2.9)$$

In particular, for each $p \geq 1$, there exists a constant $C > 0$ such that

$$\mathbb{E} \left[\sup_{(t,x) \in \mathcal{Q}_T} |u(t, k_n(x)) - u(t, x)|^p \right] \leq C n^{-p\bar{\alpha}}, \quad (2.10)$$

where $k_n(x) := \frac{[nx]}{n}$, $x \in [0, 1]$.

Proof. This can be proved essentially by mimicking the proof of Theorem 3.3 [9] by Dalang and Zhang. Because our assumptions differ slightly from those in [9], for convenience of readers, we give some explanation. As in [9, (2.1)], for each $\varepsilon > 0$, set

$$g_\varepsilon(r) := \frac{\arctan([r \wedge 0]^2)}{\varepsilon}, \quad r \in \mathbb{R},$$

and consider the penalized SPDE

$$\begin{cases} \frac{\partial u^\varepsilon(t, x)}{\partial t} = \frac{\partial^2 u^\varepsilon(t, x)}{\partial x^2} + f(t, x, u^\varepsilon(t, x)) + g_\varepsilon(u^\varepsilon(t, x)) + \sigma(t, x, u^\varepsilon(t, x)) \dot{W}(t, x), \\ u^\varepsilon(0, x) = u_0(x), \\ u^\varepsilon(t, 0) = u^\varepsilon(t, 1) = 0. \end{cases} \quad (2.11)$$

Although, unlike those in [9], the coefficients f and σ depend on t and x , the assumption **(A.1)** implies the Lipschitz condition uniformly on $t \in [0, T]$ and $x \in [0, 1]$, that is, there exists a constant $C > 0$ such that for any $x \in [0, 1], t \in [0, T], u, v \in \mathbb{R}$,

$$|f(t, x, u) - f(t, x, v)| + |\sigma(t, x, u) - \sigma(t, x, v)| \leq C|u - v|. \quad (2.12)$$

Then, the penalization result in [33] or [10] gives, for every $p \geq 1$,

$$\lim_{\varepsilon \downarrow 0} \mathbb{E}[\|u^\varepsilon - u\|_{T,\infty}^p] = 0, \quad (2.13)$$

and in particular,

$$\sup_{\varepsilon > 0} \mathbb{E}[\|u^\varepsilon\|_{T,\infty}^p] < \infty. \quad (2.14)$$

Set

$$S_t u_0(x) = \int_0^1 G(t, x, y) u_0(y) dy, \quad t \geq 0, x \in [0, 1]. \quad (2.15)$$

We now modify [9, (2.5)] by including the initial-value term $S_t u_0(x)$, and define

$$\begin{aligned} N^\varepsilon(t, x) := & S_t u_0(x) + \int_0^t \int_0^1 G(t-r, x, y) f(r, y, u^\varepsilon(r, y)) dy dr \\ & + \int_0^t \int_0^1 G(t-r, x, y) \sigma(r, y, u^\varepsilon(r, y)) W(dr, dy), \quad t \geq 0, x \in [0, 1]. \end{aligned} \quad (2.16)$$

Let $v^\varepsilon(t, x) := u^\varepsilon(t, x) - N^\varepsilon(t, x)$. Then, as explained in [9], by (2.11) and (2.16), we know that v^ε satisfies the random PDE

$$\begin{cases} \frac{\partial v^\varepsilon(t, x)}{\partial t} = \frac{\partial^2 v^\varepsilon(t, x)}{\partial x^2} + g_\varepsilon(v^\varepsilon(t, x) + N^\varepsilon(t, x)), \\ v^\varepsilon(0, x) = 0, \\ v^\varepsilon(t, 0) = v^\varepsilon(t, 1) = 0. \end{cases} \quad (2.17)$$

This is exactly equation (2.6) of [9]. The only difference is that the stochastic process N^ε now contains $S_t u_0$. We can obtain that for every $\alpha' \in (0, \frac{\alpha_*}{2})$ and $\varepsilon > 0$, there exists a nonnegative random variable $\mathcal{M}_\varepsilon$ such that

$$|N^\varepsilon(t, x) - N^\varepsilon(s, y)| \leq \mathcal{M}_\varepsilon \rho((t, x), (s, y))^{\alpha'}, \quad (t, x), (s, y) \in \mathcal{Q}_T \text{ a.s.}, \quad (2.18)$$

and for any $p \geq 1$,

$$\sup_{\varepsilon > 0} \mathbb{E}[|\mathcal{M}_\varepsilon|^p] < \infty. \quad (2.19)$$

This is the analogue of [9, Lemma 2.1]. Let us now give its brief proof. Write

$$N^\varepsilon(t, x) := S_t u_0(x) + I_1^\varepsilon(t, x) + I_2^\varepsilon(t, x), \quad (t, x) \in \mathcal{Q}_T.$$

Under the assumption **(A.0)**, we easily have that for every $p \geq 1$, there exists a constant $C > 0$ such that

$$\mathbb{E}[|S_t u_0(x) - S_s u_0(y)|^p] \leq C \rho((t, x), (s, y))^{\alpha p}, \quad (t, x), (s, y) \in \mathcal{Q}_T. \quad (2.20)$$

Indeed, noting that the heat semigroup S_t preserves Hölder regularity,

$$\|S_t u_0\|_{C^\alpha} \leq C \|u_0\|_{C^\alpha},$$

we have

$$|S_t u_0(x) - S_t u_0(y)| \leq C \|u_0\|_{C^\alpha} |x - y|^\alpha, \quad x, y \in [0, 1].$$

By the standard semigroup estimate, we also have

$$\sup_{x \in [0, 1]} |S_t u_0(x) - S_s u_0(x)| \leq C \|u_0\|_{C^\alpha} |t - s|^{\alpha/2}, \quad s, t \in [0, T].$$

Combining the two estimates, we have

$$|S_t u_0(x) - S_s u_0(y)| \leq C \|u_0\|_{C^\alpha} \rho((t, x), (s, y))^\alpha, \quad (t, x), (s, y) \in \mathcal{Q}_T,$$

which yields (2.20) by taking the p -th power and then expectations together with **(A.0)**.

On the other hand, for I_1^ε and I_2^ε , the proof of [9, Lemma 2.1] or [17, Lemma 3.5] can be applied to our case, thanks to the uniform p -th moment estimate (2.14) and the Lipschitz continuity (2.12). More precisely, by (2.14), (2.12), **(A.2)**, the heat-kernel estimates [9, (2.8)–(2.10)], the Burkholder-Davis-Gundy inequality and the integrated Minkowski inequality, for every $p \geq 2$,

$$\sup_{\varepsilon > 0} \mathbb{E}[|I_i^\varepsilon(t, x) - I_i^\varepsilon(s, y)|^p] \leq C \rho((t, x), (s, y))^{\frac{p}{2}}, \quad i = 1, 2, \quad (t, x), (s, y) \in \mathcal{Q}_T. \quad (2.21)$$

Combining this with (2.20), we obtain that

$$\sup_{\varepsilon > 0} \mathbb{E}[|N^\varepsilon(t, x) - N^\varepsilon(s, y)|^p] \leq C \rho((t, x), (s, y))^{\frac{\alpha_* p}{2}}, \quad (t, x), (s, y) \in \mathcal{Q}_T. \quad (2.22)$$

For every $\alpha' \in (0, \frac{\alpha_*}{2})$, if $p > \frac{6}{\alpha_* - 2\alpha'}$, then we have

$$\alpha' < \frac{\alpha_*}{2} - \frac{3}{p}.$$

Applying the anisotropic Kolmogorov continuity theorem ([7, Corollary A.3] and its proof) together with (2.22), we can easily obtain the desired result (2.18) and (2.19); note that p can be chosen to be arbitrarily large.

Now noting (2.17), we can repeat the proof of [9, Theorem 3.3] and especially the estimate (3.18) [9] to complete the proof of the first part of this theorem. In fact, by the same argument together with the estimate (2.18) on N^ε and (2.19), we can obtain

$$\mathbb{E}[|u^\varepsilon(t, x) - u^\varepsilon(s, y)|^p] \leq C \mathbb{E}[|\mathcal{M}_\varepsilon|^p] \rho((t, x), (s, y))^{\alpha' p}. \quad (2.23)$$

Then letting $\varepsilon \downarrow 0$ together with (2.13) and Fatou's lemma, we have

$$\mathbb{E}[|u(t, x) - u(s, y)|^p] \leq C \rho((t, x), (s, y))^{\alpha' p}. \quad (2.24)$$

Hence, similarly, applying the anisotropic Kolmogorov continuity theorem again, we know that there exists a nonnegative random variable $\mathcal{M}$ satisfying (2.9) such that (2.8) holds for any $0 < \bar{\alpha} < \frac{\alpha_*}{2}$; note that $\alpha' \in (0, \frac{\alpha_*}{2})$ is arbitrary.

Finally, we prove the second part of this theorem, i.e., (2.10). By (2.8) and $k_n(x) = \frac{[nx]}{n}$, we get

$$|u(t, x) - u(t, k_n(x))| \leq \mathcal{M} n^{-\bar{\alpha}}, \quad x \in [0, 1] \text{ a.s.}$$

Taking the supremum over $(t, x) \in \mathcal{Q}_T$, the p -th power and then taking expectation, we obtain (2.10). Consequently, the proof of this theorem is complete. $\square$

Remark 2.6. In [9], for simplicity, the authors only gave the proof for reflected SPDE with the initial value $u_0 \equiv 0$. For the general case, the authors pointed out that $u(t, x) - S_t u_0(x)$ will solve a similar reflected SPDE with initial data 0, see the second paragraph on P. 135 [9]. But, by the change of variables $\mathbf{u}(t, x) := u(t, x) - S_t u_0(x)$, the obstacle condition is changed to $\mathbf{u}(t, x) \geq -S_t u_0(x)$, which is a space-time dependent case. Therefore, it seems that [9, Theorem 3.3] cannot be applied directly for the study on the Hölder continuity of the reflected SPDE with non-zero initial values. Noting this point, in the proof of Theorem 2.5, instead of the change of variables proposed by Dalang and Zhang, we put the initial value $S_t u_0$ into $N^\varepsilon(t, x)$, see (2.16). This permits us to prove the result by repeating the proof of [9, Theorem 3.3].

3 Convergence rate of lattice approximation for deterministic parabolic obstacle problems

Following the main idea in [27] and [39], we first introduce the deterministic parabolic obstacle problem together with its lattice approximation, and then we establish the explicit convergence rate of this scheme.

Let u_0 be deterministic such that $u_0 \in C^\alpha([0, 1])$, $\alpha \in (0, 1)$ with $u_0(0) = u_0(1) = 0$, and let $V \in C(\mathbb{R}_+ \times [0, 1])$ with $V(0, x) = u_0(x) \geq 0$ and satisfying Dirichlet boundary conditions $V(t, 0) = V(t, 1) = 0, t \geq 0$.

Consider the deterministic parabolic obstacle problem

$$\begin{cases} \frac{\partial Z(t, x)}{\partial t} - \frac{\partial^2 Z(t, x)}{\partial x^2} = \dot{\eta}(t, x), & x \in [0, 1], \\ Z(t, x) \geq -V(t, x), \\ \int_0^t \int_0^1 (Z(s, x) + V(s, x)) \eta(ds, dx) = 0. \end{cases} \quad (3.1)$$

As in [39], we also consider its lattice approximation system by using the following deterministic obstacle problem (called also the deterministic Skorohod-type problem) in $\mathbb{R}^{n-1}$:

$$\begin{cases} dZ^n(t) = n^2 A^n Z^n(t) dt + d\eta^n(t), \\ Z^n(t) \geq -V^n(t), \\ \int_0^T \langle Z^n(t) + V^n(t), d\eta^n(t) \rangle = 0, \quad T > 0, \end{cases} \quad (3.2)$$

where the obstacle $V^n = (V_1^n, \dots, V_{n-1}^n) \in C([0, \infty); \mathbb{R}^{n-1})$ is given and $\langle \cdot, \cdot \rangle$ denotes the inner product of $\mathbb{R}^{n-1}$. Recall that A^n is the matrix defined in Section 2.

We first recall the definitions of solutions to (3.1) and its discretization scheme (3.2), and then summarize several important facts.

Definition 3.1. A pair (Z, η) is called a solution of (3.1) if the following are satisfied:

(i) $Z \in C(\mathbb{R}_+ \times [0, 1])$ and

$$Z(0, x) = 0, \quad Z(t, 0) = Z(t, 1) = 0, \quad Z(t, x) \geq -V(t, x).$$

(ii) η is a deterministic measure on $\mathbb{R}_+ \times (0, 1)$ such that for all $\varepsilon > 0, T > 0$

$$\eta([0, T] \times (\varepsilon, (1 - \varepsilon))) < \infty.$$

(iii) For any test function $\phi \in C^2([0, 1])$ with $\phi(0) = \phi(1) = 0$,

$$(Z(t), \phi) - \int_0^t (Z(s), \phi'') ds = \int_0^t \int_0^1 \phi(x) \eta(ds, dx), \quad t \geq 0.$$

(iv) For any $t \geq 0$, $\int_0^t \int_0^1 (Z(s, x) + V(s, x)) \eta(ds, dx) = 0$.

Definition 3.2. A pair (Z^n, η^n) is called a solution of (3.2) if the following are satisfied:

(i) $Z^n = (Z_1^n, \dots, Z_{n-1}^n) \in C([0, \infty); \mathbb{R}^{n-1})$ and $Z^n(t) \geq -V^n(t)$.

(ii) $\eta^n = (\eta_1^n, \dots, \eta_{n-1}^n) \in C([0, \infty); \mathbb{R}^{n-1})$ and for each i , $\eta_i^n(t)$ is an increasing continuous function with $\eta_i^n(0) = 0$.

(iii) For all $t \geq 0$,

$$Z^n(t) = \int_0^t n^2 A^n Z^n(s) ds + \eta^n(t).$$

(iv) For $t \geq 0$, $\sum_{i=1}^{n-1} \int_0^t (Z_i^n(s) + V_i^n(s)) \eta_i^n(ds) = 0$.

In this paper, we call the function V in (3.1) and $V^n \in \mathbb{R}^{n-1}$ in (3.2) obstacles. Furthermore, to emphasize the relation between the solution $Z(t, x)$ of (3.1) and its obstacle $V(t, x)$, we sometimes write $Z(t, x) = Z(V)(t, x)$ according to our purpose. Similarly, we also write $Z^n(t) = Z^n(V^n)(t)$ for the solution of (3.2).

Lemma 3.3. (i) *The continuous obstacle problem (3.1) admits a unique solution. Moreover, the solution map $V \mapsto Z(V)$ is 1-Lipschitz continuous, that is, for any two obstacles V^1 and V^2 ,*

$$\|Z(V^1) - Z(V^2)\|_{T, \infty} \leq \|V^1 - V^2\|_{T, \infty}.$$

(ii) *The lattice approximation system (3.2) has a unique solution and the solution map $V^n \mapsto Z^n(V^n)$ is also 1-Lipschitz continuous, that is, for any two obstacles $V^{n,1}$ and $V^{n,2}$,*

$$\|Z^n(V^{n,1}) - Z^n(V^{n,2})\|_{T, \infty} \leq \|V^{n,1} - V^{n,2}\|_{T, \infty},$$

where the norm $\|\cdot\|_{T, \infty}$ is defined by (1.2).

For the first part, see [27, Theorem 1.2 and Section 2] and for the second part, see [39, Lemma 3.2 and Theorem 3.1].

The main purpose of this section is to specify the convergence rate of the solution Z^n of (3.2) to the solution $Z(t, x)$ of (3.1).

Throughout this section, we assume the obstacle $V(t, x)$ in (3.1) satisfies the condition **(V)**:
Let $\alpha_1, \alpha_2 \in (0, 1)$. There exists a constant $\mathcal{K} > 0$ such that

$$|V(t, x) - V(s, y)| \leq \mathcal{K}(|t - s|^{\alpha_1} + |x - y|^{\alpha_2}), \quad s, t \in [0, T], x, y \in [0, 1]. \quad (3.3)$$

For a given obstacle $V(t, x)$ in (3.1) and any $n \in \mathbb{N}$, let $V^n(t)$ be the restriction of $V(t, x)$ on the mesh point $x_k, k = 1, 2, \dots, n-1$, i.e.,

$$V^n(t) = (V(t, x_1), V(t, x_2), \dots, V(t, x_{n-1})). \quad (3.4)$$

Now for $n \geq 1$, let $Z^n(t, x) := \mathcal{I}_n(Z^n(t))(x), t \in [0, T], x \in [0, 1]$ denote the linear interpolation of the solution $Z^n(t)$ of (3.2) with the obstacle $V^n(t)$ defined by (3.4), that is,

$$Z^n(t, x) := Z_k^n(t) + (nx - k)(Z_{k+1}^n(t) - Z_k^n(t)) \quad (3.5)$$

for $x \in [x_k, x_{k+1}), k = 0, \dots, n-1$, where $Z_0^n(t) = Z_n^n(t) = 0$. Then the main result of this section is the following.

Theorem 3.4. *Assume **(V)** holds. Let Z be the solution to (3.1) with the obstacle $V(t, x)$ and Z^n be the continuous function defined by (3.5). Then there exists a constant $C > 0$ such that*

$$\|Z^n - Z\|_{T, \infty} \leq C\mathcal{K}(\log n)^{-\kappa/2}, \quad n \geq 2, \quad (3.6)$$

where $\mathcal{K}$ is the constant used in (3.3) and

$$\kappa := \min\{2\alpha_1, \alpha_2\}. \quad (3.7)$$

The proof will be given in Subsection 3.4.

Remark 3.5. In [39, Theorem 4.1], it is proved that $|Z^n(t, x) - Z(t, x)|$ converges to 0 in the uniform topology as $n \rightarrow \infty$ by the compactness argument without an explicit convergence rate. Theorem 3.4 gives an explicit convergence rate. As we pointed out, determining the optimal rate remains challenging.

Finally, we give a smooth approximation of the obstacle $V(t, x)$ satisfying (3.3) in Lemma 3.6 below, which will be frequently used. Since we will need explicit ε -dependence later, we state the following quantitative smoothing lemma, together with a brief proof. A general parabolic mollification estimate can be found, for example, in [23, Chapter 4] or [22, Chapter 3].

Lemma 3.6. *Let $V(t, x)$ be the obstacle satisfying (3.3) and let $\kappa := \min\{2\alpha_1, \alpha_2\}$ as before. Then for each $\varepsilon > 0$, there exists a smooth function V^ε on $[0, T] \times [0, 1] \rightarrow \mathbb{R}$ such that $V^\varepsilon(t, 0) = V^\varepsilon(t, 1) = 0, t \in [0, T]$, and*

$$\|V^\varepsilon - V\|_{T, \infty} \leq C\mathcal{K}\varepsilon^\kappa, \quad (3.8)$$

$$\left\| \frac{\partial V^\varepsilon}{\partial t} \right\|_{T, \infty} + \left\| \frac{\partial^2 V^\varepsilon}{\partial x^2} \right\|_{T, \infty} \leq C\mathcal{K}\varepsilon^{\kappa-2}, \quad (3.9)$$

where the constant $C = C(T, \kappa) > 0$ is independent of $\mathcal{K}$.

Proof. The form CK of the constant appearing in the estimates (3.8) and (3.9) is vital for us to apply to SPDEs in Section 4. So we give a short proof of (3.9) to specify the constant CK as an example (and omit the proof of (3.8)). Let $\rho \in C_c^\infty(\mathbb{R}^2)$ be a standard parabolic mollifier satisfying

$$\rho \geq 0 \text{ and } \int_{\mathbb{R}^2} \rho(t, x) dt dx = 1,$$

and define

$$\rho_\varepsilon(t, x) = \varepsilon^{-3} \rho\left(\frac{t}{\varepsilon^2}, \frac{x}{\varepsilon}\right), \quad \varepsilon \in (0, 1).$$

Extending V outside $[0, T] \times [0, 1]$ in a standard way preserving the Hölder bound, we define

$$V^\varepsilon := \rho_\varepsilon * V.$$

Then we claim that V^ε is the desired function. As we said, we only show (3.9). Since $\rho_\varepsilon \in C_c^\infty(\mathbb{R}^2)$, we have $\int \int_{\mathbb{R}^2} \frac{\partial \rho_\varepsilon(t, x)}{\partial t} dt dx = 0$, and then we write

$$\frac{\partial V^\varepsilon(t, x)}{\partial t} = \iint_{\mathbb{R}^2} \partial_t \rho_\varepsilon(s, y) (V(t - s, x - y) - V(t, x)) ds dy.$$

Hence, by the assumption **(V)**, we have

$$\left| \frac{\partial V^\varepsilon(t, x)}{\partial t} \right| \leq K \iint |\partial_t \rho_\varepsilon(s, y)| (|s|^{\alpha_1} + |y|^{\alpha_2}) ds dy.$$

Noting that

$$\partial_t \rho_\varepsilon(s, y) = \varepsilon^{-5} (\partial_t \rho)\left(\frac{s}{\varepsilon^2}, \frac{y}{\varepsilon}\right),$$

change of variables gives that

$$\left| \frac{\partial V^\varepsilon(t, x)}{\partial t} \right| \leq K (\varepsilon^{2\alpha_1-2} + \varepsilon^{\alpha_2-2}) \leq CK \varepsilon^{\kappa-2}.$$

Hence, we know $\|\partial_t V^\varepsilon\|_{T, \infty}$ satisfies the estimate (3.9). Similar argument gives

$$\left\| \frac{\partial^2 V^\varepsilon}{\partial x^2} \right\|_{T, \infty} \leq CK \varepsilon^{\kappa-2}.$$

Thus the proof of (3.9) is complete. $\square$

3.1 Convergence rate of solutions to deterministic parabolic obstacle problems via penalization

In this part, following the construction of the solution (3.1) in [27], we study the convergence rate of the penalization system to that of (3.1). For a given $V(t, x)$ satisfying (3.3), we consider the deterministic parabolic penalized equation

$$\begin{cases} \frac{\partial Z^\lambda(t, x)}{\partial t} - \frac{\partial^2 Z^\lambda(t, x)}{\partial x^2} = \lambda (Z^\lambda(t, x) + V(t, x))^- , \\ Z^\lambda(t, 0) = Z^\lambda(t, 1) = 0, \\ Z^\lambda(0, x) = 0, \end{cases} \quad (3.10)$$

where $\lambda \geq 1$. In [27, Theorem 1.4], it is proved that Z^λ converges to the solution $Z(t, x)$ of (3.1) uniformly on $\mathcal{Q}_T$ as $\lambda \rightarrow \infty$. We aim to give its convergence rate in λ .

Theorem 3.7. *Let (Z, η) be the solution of the deterministic obstacle problem (3.1) and let Z^λ solve the penalized equation (3.10). Then there exists a constant $C > 0$ such that*

$$\|Z - Z^\lambda\|_{T,\infty} \leq CK \lambda^{-\kappa/2}, \quad \lambda \geq 1, \quad (3.11)$$

where K and κ are the same as in Theorem 3.4.

Proof. Let us first recall two standard facts concerning the penalization equation (3.10). By [4, Corollary 2.3] and [27, Section 2.2], we know that

- (i) For each $V \in C([0, T] \times [0, 1])$, (3.10) has a unique strong solution $Z^\lambda \in L^2([0, T]; H^1(0, 1)) \cap C([0, T] \times [0, 1])$ and it is increasing in λ .
- (ii) The solution map $V \mapsto Z^\lambda := Z^\lambda(V)$ of (3.10) is 1-Lipschitz continuous, that is,

$$\|Z^\lambda(V^1) - Z^\lambda(V^2)\|_{T,\infty} \leq \|V^1 - V^2\|_{T,\infty}$$

for any two continuous obstacles V^1 and V^2 .

Fix $\varepsilon > 0$, which will be specified later. Let V^ε satisfy (3.8) and (3.9) in Lemma 3.6 hereafter. Let $(Z^\varepsilon, \eta^\varepsilon)$ denote the solution of the obstacle problem (3.1) with the smooth obstacle V^ε , and let $Z^{\lambda,\varepsilon}$ be the solution of the penalized equation (3.10) with the same V^ε , that is,

$$\begin{cases} \frac{\partial Z^{\lambda,\varepsilon}(t, x)}{\partial t} - \frac{\partial^2 Z^{\lambda,\varepsilon}(t, x)}{\partial x^2} = \lambda(Z^{\lambda,\varepsilon}(t, x) + V^\varepsilon(t, x))^- , \\ Z^{\lambda,\varepsilon}(t, 0) = Z^{\lambda,\varepsilon}(t, 1) = 0, \\ Z^{\lambda,\varepsilon}(0, x) = 0. \end{cases} \quad (3.12)$$

We first point out that by (ii) and (3.8), one has

$$\|Z^\lambda(V) - Z^{\lambda,\varepsilon}(V^\varepsilon)\|_{T,\infty} \leq \|V - V^\varepsilon\|_{T,\infty} \leq CK \varepsilon^\kappa. \quad (3.13)$$

Here and in the sequel, we use $Z^{\lambda,\varepsilon} = Z^{\lambda,\varepsilon}(V^\varepsilon) := Z^\lambda(V^\varepsilon)$ to denote the unique solution of the penalized equation (3.10) (or equivalently (3.12)) with V^ε replaced by V .

Noting that V^ε is smooth and satisfies (3.9), we set

$$G_\varepsilon := \|(\partial_t - \partial_{xx})V^\varepsilon\|_{T,\infty}.$$

We claim that

$$\|Z^\varepsilon - Z^{\lambda,\varepsilon}\|_{T,\infty} \leq \frac{G_\varepsilon}{\lambda} \quad (3.14)$$

holds for each $\lambda \geq 1$ and $\varepsilon > 0$, where $Z^\varepsilon = Z(V^\varepsilon)$ denotes the solution of (3.1) with the obstacle V replaced by V^ε . Recall that for the smooth obstacle $Z(V^\varepsilon)$, (3.1) and the penalized equation (3.12) has a unique strong solution, see Corollary 2.3 [4]. By (i) above and [27, Step 2], one knows that

$$Z^{\lambda,\varepsilon}(t, x) \leq Z^\varepsilon(t, x).$$

Thus, in order to show the claim (3.14), it is enough to show for all $t \in [0, T]$ and $x \in [0, 1]$

$$Z^\varepsilon(t, x) \leq Z^{\lambda,\varepsilon}(t, x) + \frac{G_\varepsilon}{\lambda}. \quad (3.15)$$

Set

$$Y^{\lambda,\varepsilon} := Z^{\lambda,\varepsilon} + V^\varepsilon.$$

By (3.12), we know that $Y^{\lambda,\varepsilon}$ solves

$$\partial_t Y^{\lambda,\varepsilon} - \partial_{xx} Y^{\lambda,\varepsilon} = (\partial_t - \partial_{xx})V^\varepsilon + \lambda(Y^{\lambda,\varepsilon})^-. \quad (3.16)$$

Consider the constant function

$$\underline{y} := -\frac{G_\varepsilon}{\lambda}.$$

Since $|(\partial_t - \partial_{xx})V^\varepsilon| \leq G_\varepsilon$,

$$\partial_t \underline{y} - \partial_{xx} \underline{y} = 0 \leq (\partial_t - \partial_{xx})V^\varepsilon + \lambda(\underline{y})^-.$$

Thus $\underline{y}$ is a subsolution of (3.16) by noting that

$$\underline{y}(0, x) = -\frac{G_\varepsilon}{\lambda} \leq V^\varepsilon(0, x) = Y^{\lambda,\varepsilon}(0, x), \quad \underline{y}(t, 0) = \underline{y}(t, 1) = -\frac{G_\varepsilon}{\lambda} \leq 0.$$

By the parabolic comparison principle for semilinear equations, see [23, Theorem 2.9] for instance, we have

$$Y^{\lambda,\varepsilon}(t, x) \geq -\frac{G_\varepsilon}{\lambda},$$

which gives that

$$Z^{\lambda,\varepsilon} + V^\varepsilon + \frac{G_\varepsilon}{\lambda} \geq 0. \quad (3.17)$$

Let now $\widehat{Z}^{\lambda,\varepsilon} := Z^{\lambda,\varepsilon} + \frac{G_\varepsilon}{\lambda}$. By (3.17), we know that $\widehat{Z}^{\lambda,\varepsilon} + V^\varepsilon \geq 0$. Set

$$\widehat{w} = (Z^\varepsilon - \widehat{Z}^{\lambda,\varepsilon})^+.$$

Subtracting (3.1) with V replaced by the smooth obstacle V^ε from (3.12) gives that

$$\partial_t (Z^\varepsilon - \widehat{Z}^{\lambda,\varepsilon}) - \partial_{xx} (Z^\varepsilon - \widehat{Z}^{\lambda,\varepsilon}) = \eta^\varepsilon - \lambda(Z^{\lambda,\varepsilon} + V^\varepsilon)^-. \quad (3.18)$$

Let us multiply (3.18) by $\widehat{w}$ and integrate over $[0, 1]$. Then using standard arguments for positive parts (see [4, Lemma 6.1] or [27] for example), we obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\widehat{w}(t)\|_{L^2}^2 + \|\partial_x \widehat{w}(t)\|_{L^2}^2 &= \int_0^1 \widehat{w}(t, x) \eta^\varepsilon(dx, dt) \\ &\quad - \lambda \int_0^1 \widehat{w}(t, x) (Z^{\lambda,\varepsilon}(t, x) + V^\varepsilon(t, x))^- dx. \end{aligned} \quad (3.19)$$

On the set $\{x : \widehat{w}(t, x) > 0\}$, we have $Z^\varepsilon > \widehat{Z}^{\lambda,\varepsilon} = Z^{\lambda,\varepsilon} + \frac{G_\varepsilon}{\lambda}$. Hence $Z^\varepsilon + V^\varepsilon > Z^{\lambda,\varepsilon} + V^\varepsilon + \frac{G_\varepsilon}{\lambda}$. By (3.17), we have

$$Z^\varepsilon + V^\varepsilon > 0.$$

Therefore, the first term on the right hand side of (3.19) equals to 0, see Definition 3.1. On the other hand, by the positivity of $\widehat{w}$, it is clear that the second term on the right hand side of (3.19) is positive. Therefore, noting that $\lambda \geq 1$, we obtain

$$\frac{1}{2} \frac{d}{dt} \|\widehat{w}(t)\|_{L^2}^2 + \|\partial_x \widehat{w}(t)\|_{L^2}^2 \leq 0.$$

Since $\widehat{w}(0) = 0$, it follows that $\widehat{w} \equiv 0$, and hence (3.15) holds. Therefore, the claim (3.14) is proved. Combining (3.14) with (3.9) in Lemma 3.6, we have

$$\|Z^\varepsilon - Z^{\lambda,\varepsilon}\|_{T,\infty} \leq C\mathcal{K}\lambda^{-1}\varepsilon^{\kappa-2}. \quad (3.20)$$

Using Lemma 3.3, (3.13) and (3.20), we obtain

$$\begin{aligned} & \|Z(V) - Z^\lambda(V)\|_{T,\infty} \\ & \leq \|Z(V) - Z^\varepsilon(V^\varepsilon)\|_{T,\infty} + \|Z^\varepsilon(V^\varepsilon) - Z^{\lambda,\varepsilon}(V^\varepsilon)\|_{T,\infty} + \|Z^{\lambda,\varepsilon}(V^\varepsilon) - Z^\lambda(V)\|_{T,\infty} \\ & \leq C\mathcal{K}(\varepsilon^\kappa + \lambda^{-1}\varepsilon^{\kappa-2}). \end{aligned}$$

Now choosing

$$\varepsilon = \lambda^{-1/2},$$

we have

$$\|Z - Z^\lambda\|_{T,\infty} \leq C\mathcal{K}\lambda^{-\kappa/2},$$

which completes the proof of (3.11). $\square$

Remark 3.8. Using Theorem 3.7 together with the construction of solutions to the reflected SPDE (1.1) as in [10], one can also derive the convergence rate for the corresponding stochastic penalized system to (1.1), see also Remark 4.12 for more information.

3.2 Convergence rate of solutions to deterministic Skorohod-type equations via penalization

This part aims to study the convergence rate of solutions of the penalization equations to that of the deterministic Skorohod-type equation (3.2). More precisely, let $(Z^n(t), \eta^n(t))$ be the solution of (3.2) with the obstacle $V^n(t)$ and let $Z^{n,\lambda}$ solve the finite dimensional penalized ODE

$$\begin{cases} \frac{d}{dt}Z^{n,\lambda}(t) = n^2 A^n Z^{n,\lambda}(t) + \lambda(Z^{n,\lambda}(t) + V^n(t))^- , \\ Z^{n,\lambda}(0) = 0. \end{cases} \quad (3.21)$$

By [39, Theorems 3.1 and 4.1], it is known that

$$\lim_{\lambda \rightarrow \infty} \|Z^n - Z^{n,\lambda}\|_{T,\infty} = 0.$$

Recall that for $U \in C([0, T]; \mathbb{R}^{n-1})$,

$$\|U\|_{T,\infty} = \sup_{0 \leq t \leq T} \max_{1 \leq k \leq n-1} |U_k(t)|.$$

The goal is to give the convergence rate in λ . The detail is stated in the next theorem.

Theorem 3.9. *Let $V(t, x)$ satisfy (3.3) and let $V_k^n(t)$ be defined by (3.4), which is determined by $V(t, x)$. Further, let $(Z^n(t), \eta^n(t))$ be the solution of (3.2) and let $Z^{n,\lambda}$ be the solution of (3.21) with the same obstacle $V^n(t)$. Then there exists a constant $C > 0$ such that for all $\lambda \geq 1$,*

$$\sup_{n \in \mathbb{N}} \|Z^n - Z^{n,\lambda}\|_{T,\infty} \leq C\mathcal{K}\lambda^{-\kappa/2}; \quad (3.22)$$

recall that $\mathcal{K}$ is the constant in (3.3) and κ is defined by (3.7).

Let us first give some preparation and then give the proof of Theorem 3.9 in the last part of this subsection. The first one is the one-Lipschitz of the penalized ODE (3.21) on its obstacle $V^n(t)$.

Lemma 3.10. *Let*

$$V^n = (V_1^n, \dots, V_{n-1}^n) \text{ and } \tilde{V}^n = (\tilde{V}_1^n, \dots, \tilde{V}_{n-1}^n)$$

belong to $C([0, T]; \mathbb{R}^{n-1})$. Let $Z^{n, \lambda}$ and $\tilde{Z}^{n, \lambda}$ be solutions of the penalized system (3.21) with obstacles $V^n(t)$ and $\tilde{V}^n(t)$ respectively. Then we have

$$\|Z^{n, \lambda} - \tilde{Z}^{n, \lambda}\|_{T, \infty} \leq \|V^n - \tilde{V}^n\|_{T, \infty}.$$

Proof. This can be shown essentially as in [39, Lemma 3.2], in which a similar result is established for the deterministic Skorohod-type problem (3.2). For the reader's convenience, we state the proof here. Set

$$\delta := \|V^n - \tilde{V}^n\|_{T, \infty}.$$

It suffices to prove the one-sided estimate

$$Z_k^{n, \lambda}(t) \leq \tilde{Z}_k^{n, \lambda}(t) + \delta, \quad 0 \leq t \leq T, \quad 1 \leq k \leq n-1. \quad (3.23)$$

The reverse inequality then follows by symmetry, after exchanging $(V^n, Z^{n, \lambda})$ and $(\tilde{V}^n, \tilde{Z}^{n, \lambda})$.

Define

$$b_k(t) := Z_k^{n, \lambda}(t) - \tilde{Z}_k^{n, \lambda}(t) - \delta, \quad b(t) := (b_1(t), \dots, b_{n-1}(t)).$$

Noting that $Z_k^{n, \lambda}(0) = \tilde{Z}_k^{n, \lambda}(0) = 0$, we know

$$b_k(0) = -\delta \leq 0.$$

By the definitions of $Z^{n, \lambda}$ and $\tilde{Z}^{n, \lambda}$, we easily obtain

$$\frac{d}{dt} b_k(t) = n^2 (A^n b(t))_k + n^2 (A^n \delta \mathbf{1})_k + \lambda \left[(Z_k^{n, \lambda}(t) + V_k^n(t))^- - (\tilde{Z}_k^{n, \lambda}(t) + \tilde{V}_k^n(t))^- \right],$$

where $\mathbf{1} := (1, \dots, 1)^\tau \in \mathbb{R}^{n-1}$. Set

$$b_k^+(t) := \max\{b_k(t), 0\}, \quad b^+(t) := (b_1^+(t), \dots, b_{n-1}^+(t)).$$

Applying the chain rule to $\sum_{k=1}^{n-1} (b_k^+(t))^2$, we get

$$\begin{aligned} \frac{d}{dt} \sum_{k=1}^{n-1} (b_k^+(t))^2 &= 2n^2 \sum_{k=1}^{n-1} b_k^+(t) ((A^n b(t))_k + (A^n \delta \mathbf{1})_k) \\ &\quad + 2\lambda \sum_{k=1}^{n-1} b_k^+(t) \left[(Z_k^{n, \lambda}(t) + V_k^n(t))^- - (\tilde{Z}_k^{n, \lambda}(t) + \tilde{V}_k^n(t))^- \right]. \end{aligned} \quad (3.24)$$

By [39, Lemma 3.1], we know that

$$\langle b^+, A^n b \rangle \leq 0.$$

By the definition of A^n , we also have $\sum_{k=1}^{n-1} b_k^+(t) (A^n \delta \mathbf{1})_k = -\delta (b_1^+(t) + b_{n-1}^+(t))$. Combining those, we know that the first term on the right hand side of (3.24) is non-positive.

We can show that the second term on the right hand side of (3.24) is also non-positive. Indeed, whenever $b_k^+(t) > 0$, then $b_k(t) > 0$, that is, $Z_k^{n,\lambda}(t) > \tilde{Z}_k^{n,\lambda}(t) + \delta$. Since $|V_k^n(t) - \tilde{V}_k^n(t)| \leq \delta$, we have

$$Z_k^{n,\lambda}(t) + V_k^n(t) > \tilde{Z}_k^{n,\lambda}(t) + \delta + \tilde{V}_k^n(t) - \delta = \tilde{Z}_k^{n,\lambda}(t) + \tilde{V}_k^n(t).$$

Since the map $r \mapsto r^-$ is nonincreasing,

$$(Z_k^{n,\lambda}(t) + V_k^n(t))^- \leq (\tilde{Z}_k^{n,\lambda}(t) + \tilde{V}_k^n(t))^-.$$

Thus, whenever $b_k^+(t) > 0$,

$$b_k^+(t) \left[(Z_k^{n,\lambda}(t) + V_k^n(t))^- - (\tilde{Z}_k^{n,\lambda}(t) + \tilde{V}_k^n(t))^- \right] \leq 0.$$

Therefore, combining the above observations with (3.24), we obtain

$$\frac{d}{dt} \sum_{k=1}^{n-1} (b_k^+(t))^2 \leq 0.$$

Since $b_k^+(0) = 0$ for all k , it follows that

$$b_k^+(t) \equiv 0 \quad \text{for all } 0 \leq t \leq T, \quad 1 \leq k \leq n-1.$$

Hence, we obtain

$$Z_k^{n,\lambda}(t) - \tilde{Z}_k^{n,\lambda}(t) - \delta \leq 0,$$

that is, (3.23) holds. Therefore, the proof of this lemma is complete. $\square$

Lemma 3.11. *Assume the conditions in Theorem 3.9 hold. We further assume*

$$V \in C^{1,2}([0, T] \times [0, 1]) \text{ with } V(t, 0) = V(t, 1) = 0,$$

and $V^n(t)$ is determined by $V(t, x)$ as before. Then

$$0 \leq Z_k^n(t) - Z_k^{n,\lambda}(t) \leq \frac{G_n}{\lambda}, \quad 0 \leq t \leq T, \quad 1 \leq k \leq n-1, \quad (3.25)$$

where

$$G_n := \sup_{0 \leq t \leq T, \quad 1 \leq k \leq n-1} \left| \frac{dV(t, x_k)}{dt} - (n^2 A^n V^n(t))_k \right|.$$

In particular, we have

$$\|Z^n - Z^{n,\lambda}\|_{T,\infty} \leq \lambda^{-1} \left(\left\| \frac{\partial V}{\partial t} \right\|_{T,\infty} + \left\| \frac{\partial^2 V}{\partial x^2} \right\|_{T,\infty} \right). \quad (3.26)$$

Proof. Set

$$Y_k^n(t) := Z_k^n(t) + V_k^n(t), \quad Y_k^{n,\lambda}(t) := Z_k^{n,\lambda}(t) + V_k^n(t), \quad k = 1, \dots, n-1.$$

Then by the definition of the solution to (3.2), we have $Y_k^n(t) \geq 0$ for all k, t . We also know $Y^{n,\lambda}$ solves

$$\frac{d}{dt} Y^{n,\lambda}(t) = n^2 A^n Y^{n,\lambda}(t) + \lambda (Y^{n,\lambda}(t))^- + g^n(t), \quad (3.27)$$

where

$$g_k^n(t) := \partial_t V(t, x_k) - (n^2 A^n V^n(t))_k.$$

We claim that

$$Y_k^{n,\lambda}(t) \geq -\frac{G_n}{\lambda}, \quad 0 \leq t \leq T, \quad 1 \leq k \leq n-1. \quad (3.28)$$

Define the constant vector

$$\underline{y} = \underline{y}(t) := \left(-\frac{G_n}{\lambda}, \dots, -\frac{G_n}{\lambda} \right) \in \mathbb{R}^{n-1}.$$

Then one easily knows that $\underline{y}$ is a *subsolution* of (3.27). In fact, since $\underline{y}$ is constant in time, we have

$$\frac{d}{dt} \underline{y}(t) = 0.$$

Moreover, by the definition of $\underline{y}$, we see $\underline{y} \leq 0$ and then $(\underline{y})^- = \frac{G_n}{\lambda} \mathbf{1}$; Recall that $\mathbf{1} := (1, \dots, 1)^\tau \in \mathbb{R}^{n-1}$. Therefore, the following holds componentwisely:

$$\frac{d}{dt} \underline{y} - n^2 A^n \underline{y} - \lambda(\underline{y})^- = -n^2 (A^n \underline{y}) - G_n \mathbf{1}.$$

By the definition of A^n , it easily follows that

$$(A^n \underline{y})_k \geq 0, \quad k = 1, 2, \dots, n-1.$$

Hence

$$\frac{d}{dt} \underline{y} - n^2 A^n \underline{y} - \lambda(\underline{y})^- \leq -G_n \mathbf{1}.$$

By the definition of G_n , we have

$$-G_n \leq g_k^n(t) \leq G_n, \quad 0 \leq t \leq T, \quad 1 \leq k \leq n-1,$$

so we conclude

$$\frac{d}{dt} \underline{y} \leq n^2 A^n \underline{y} + \lambda(\underline{y})^- + g^n(t).$$

Thus $\underline{y}$ is a subsolution of (3.27).

Let us define the vector field

$$F(t, y) := n^2 A^n y + \lambda y^- + g^n(t).$$

Then, the system

$$\dot{y} = F(t, y)$$

is cooperative. Indeed, for $i \neq j$,

$$\frac{\partial}{\partial y_j} (n^2 A^n y)_i = n^2 (A^n)_{ij} \geq 0,$$

and the map $y \mapsto y^-$ acts componentwisely, so it does not create any coupling between different components. Hence

$$\frac{\partial F_i}{\partial y_j}(t, y) \geq 0 \quad \text{for all } i \neq j,$$

and the system is cooperative. Since

$$\underline{y}(0) = -\frac{G_n}{\lambda} \mathbf{1} \leq V^n(0) = Y^{n,\lambda}(0),$$

and $\underline{y}$ is a subsolution while $Y^{n,\lambda}$ is a solution of (3.27), the comparison principle for cooperative finite-dimensional ODE systems (see, e.g., [2, Chapter 3]) yields

$$\underline{y}(t) \leq Y^{n,\lambda}(t) \quad \text{componentwise for all } 0 \leq t \leq T,$$

which gives the claim (3.28).

Next, using (3.28), we show that (3.25). It suffices to show the second inequality $Z^n \leq Z^{n,\lambda} + G_n/\lambda$ of (3.25), because the first part is well-known. Set

$$b_k(t) := Z_k^n(t) - Z_k^{n,\lambda}(t) - \frac{G_n}{\lambda}, \quad w_k(t) := (b_k(t))^+, \quad k = 1, \dots, n-1.$$

We shall prove that $w \equiv 0$, which is equivalent to

$$Z_k^n(t) \leq Z_k^{n,\lambda}(t) + \frac{G_n}{\lambda}, \quad 0 \leq t \leq T, \quad 1 \leq k \leq n-1. \quad (3.29)$$

Recall that

$$Y_k^{n,\lambda}(t) := Z_k^{n,\lambda}(t) + V_k^n(t).$$

Subtracting the equation (3.21) from the reflected system (3.2), we obtain

$$d(Z_k^n(t) - Z_k^{n,\lambda}(t)) = n^2 (A^n(Z^n - Z^{n,\lambda})(t))_k dt + d\eta_k^n(t) - \lambda(Y_k^{n,\lambda}(t))^- dt.$$

Since

$$Z^n - Z^{n,\lambda} = b + \frac{G_n}{\lambda} \mathbf{1},$$

it follows that

$$db_k(t) = n^2 (A^n b(t))_k dt + \frac{G_n}{\lambda} n^2 (A^n \mathbf{1})_k dt + d\eta_k^n(t) - \lambda(Y_k^{n,\lambda}(t))^- dt.$$

By the chain rule,

$$\begin{aligned} d\left(\sum_{k=1}^{n-1} (b_k^+(t))^2\right) &= 2n^2 \sum_{k=1}^{n-1} b_k^+(t) (A^n b(t))_k dt + \frac{2n^2 G_n}{\lambda} \sum_{k=1}^{n-1} b_k^+(t) (A^n \mathbf{1})_k dt \\ &\quad + 2 \sum_{k=1}^{n-1} b_k^+(t) d\eta_k^n(t) - 2\lambda \sum_{k=1}^{n-1} b_k^+(t) (Y_k^{n,\lambda}(t))^- dt \\ &=: I_1 + I_2 + I_3 + I_4. \end{aligned}$$

By [39, Lemma 3.1] and noting that $A^n \mathbf{1} = (-1, 0, \dots, 0, -1)$, we easily know that $I_1 + I_2 \leq 0$. Whenever $b_k^+(t) > 0$, we have

$$Z_k^n(t) - Z_k^{n,\lambda}(t) > \frac{G_n}{\lambda}.$$

Hence, we have

$$Y_k^n(t) = Z_k^n(t) + V_k^n(t) > Z_k^{n,\lambda}(t) + V_k^n(t) + \frac{G_n}{\lambda} = Y_k^{n,\lambda}(t) + \frac{G_n}{\lambda}.$$

Combining this with (3.28), we see that on the set $\{b_k^+(t) > 0\}$,

$$Y_k^n(t) > 0.$$

By the property of $\eta_k^n(t)$, it follows that $I_3 = 0$. On the other hand, it is clear that

$$I_4 = -2\lambda \sum_{k=1}^{n-1} b_k^+(t) (Y_k^{n,\lambda}(t))^- dt \leq 0.$$

Combining the above estimates, we obtain

$$d\left(\sum_{k=1}^{n-1} (b_k^+(t))^2\right) \leq 0.$$

Noting that $b_k(0) = Z_k^n(0) - Z_k^{n,\lambda}(0) - \frac{G_n}{\lambda} = -\frac{G_n}{\lambda} \leq 0$, we have

$$\sum_{k=1}^{n-1} (b_k^+(t))^2 = 0, \quad 0 \leq t \leq T,$$

so $b_k^+(t) \equiv 0$ for all k . Therefore, (3.29) is proved.

Finally, let us prove (3.26). It suffices to show $n^2 A^n V^n$ is uniformly bounded in n . This can be easily shown by the mean-value theorem. Indeed, for each $t \in [0, T]$ and $1 \leq k \leq n-1$, the mean-value form of the second-difference formula yields some $\xi_{k,t} \in (x_{k-1}, x_{k+1})$ such that

$$(n^2 A^n V^n(t))_k = n^2 (V(t, x_{k+1}) - 2V(t, x_k) + V(t, x_{k-1})) = \partial_{xx} V(t, \xi_{k,t}).$$

Therefore

$$\sup_{n \geq 1} \sup_{0 \leq t \leq T} \max_{1 \leq k \leq n-1} |(n^2 A^n V^n(t))_k| \leq \|\partial_{xx} V\|_{T, \infty}.$$

Hence, this together with (3.25), we obtain (3.26). $\square$

From now on, let us give the proof of Theorem 3.9.

Proof of Theorem 3.9. Let V^ε be the smooth function introduced in Lemma 3.6. Define the sampled smooth obstacle

$$V_k^{n,\varepsilon}(t) := V^\varepsilon(t, x_k), \quad k = 0, \dots, n.$$

Let $Z^{n,\varepsilon}$ and $Z^{n,\lambda,\varepsilon}$ denote the solutions of the deterministic Skorohod-type equation (3.2) with $V^n(t)$ replaced by $V^{n,\varepsilon}$ and respectively, its penalized system (3.21).

By [39, Lemma 3.2] and Lemma 3.10, we know for all $n \in \mathbb{N}$

$$\begin{aligned} & \|Z^n - Z^{n,\varepsilon}\|_{T, \infty} + \|Z^{n,\lambda} - Z^{n,\lambda,\varepsilon}\|_{T, \infty} \\ &= \|Z^n(V) - Z^n(V^\varepsilon)\|_{T, \infty} + \|Z^{n,\lambda}(V) - Z^{n,\lambda}(V^\varepsilon)\|_{T, \infty} \\ &\leq 2\|V^n - V^{n,\varepsilon}\|_{T, \infty} \\ &\leq C\mathcal{K} \varepsilon^\kappa, \end{aligned} \tag{3.30}$$

where (3.8) has been used for the last inequality.

Noting that V^ε is smooth and then using (3.26) in Lemma 3.11, we have

$$\|Z^{n,\varepsilon} - Z^{n,\lambda,\varepsilon}\|_{T,\infty} \leq \frac{\|\partial_t V^\varepsilon\|_{T,\infty} + \|\partial_{xx} V^\varepsilon\|_{T,\infty}}{\lambda}.$$

Then by (3.9), we obtain

$$\|Z^{n,\varepsilon} - Z^{n,\lambda,\varepsilon}\|_{T,\infty} \leq \frac{CK\varepsilon^{\kappa-2}}{\lambda}. \quad (3.31)$$

Combining (3.30) with (3.31), we obtain

$$\begin{aligned} \|Z^n - Z^{n,\lambda}\|_{T,\infty} &\leq \|Z^n - Z^{n,\varepsilon}\|_{T,\infty} + \|Z^{n,\varepsilon} - Z^{n,\lambda,\varepsilon}\|_{T,\infty} + \|Z^{n,\lambda,\varepsilon} - Z^{n,\lambda}\|_{T,\infty} \\ &\leq CK(\varepsilon^\kappa + \lambda^{-1}\varepsilon^{\kappa-2}). \end{aligned}$$

As in the proof of Theorem 3.7, choosing $\varepsilon := \lambda^{-1/2}$, we obtain the desired (3.22). $\square$

3.3 Rigorous error estimate between the parabolic penalized equation and its discretization scheme

The part is devoted to estimating the error between the parabolic penalized equation (3.10) and its discretization scheme (3.21). We still assume all of the conditions in Theorem 3.4 are fulfilled. Let Z^λ and $Z^{n,\lambda}$ be the solutions of (3.10) with the obstacle V and respectively, (3.21) with the obstacle V^n . Recall that for a given V , V^n is defined by (3.4).

Set

$$F_\lambda(u) = \lambda u^-, u \in \mathbb{R}$$

for simplicity. Since $Z^\lambda(0, x) = 0$, the solution $Z^\lambda(t, x)$ admits its mild representation:

$$Z^\lambda(t, x) = \int_0^t \int_0^1 G(t-s, x, y) F_\lambda(Z^\lambda(s, y) + V(s, y)) dy ds, \quad (3.32)$$

where $G(t, x, y)$ denotes the heat kernel of the Laplacian $\frac{\partial^2}{\partial x^2}$ on the interval $[0, 1]$ with Dirichlet boundary conditions as before. Similarly, by the variation of constant formula together with $Z^{n,\lambda}(0) = 0$, we also have

$$Z^{n,\lambda}(t) = \int_0^t G^n(t-s) F_\lambda(Z^{n,\lambda}(s) + V^n(s)) ds, \quad (3.33)$$

where $G^n(t) = e^{n^2 A^n t}$ is defined by (2.4). For $n \geq 1$, define the continuous function $Z^{n,\lambda}(t, x)$ as the linear interpolation $\mathcal{I}_n Z^{n,\lambda}(t)(x)$ of $Z^{n,\lambda}(t)$, that is, for $x \in [x_k, x_{k+1})$, $k = 1, 2, \dots, n-1$

$$Z^{n,\lambda}(t, x) := Z_k^{n,\lambda}(t) + (nx - k)(Z_{k+1}^{n,\lambda}(t) - Z_k^{n,\lambda}(t)), \quad x \in [x_k, x_{k+1}), k = 1, 2, \dots, n-1$$

with $Z_0^{n,\lambda}(t) = Z_n^{n,\lambda}(t) = 0$. As in [17], introduce the kernel $G^n(t, x, y)$ by

$$G^n(t, x, y) = \sum_{j=1}^{n-1} \exp(\lambda_j^n t) \phi_j^n(x) \phi_j(k_n(y)), \quad (3.34)$$

where $k_n(y) = \frac{[ny]}{n}$ as before, and for $x \in [\frac{k}{n}, \frac{k+1}{n}]$,

$$\phi_j^n(x) := \phi_j(x_k) + (nx - k)(\phi_j(x_{k+1}) - \phi_j(x_k)). \quad (3.35)$$

By (3.33) and the similar argument to [17, (2.7)] or [39, (5.8)], we have

$$Z^{n,\lambda}(t, x) = \int_0^t \int_0^1 G^n(t-s, x, y) F_\lambda(Z^{n,\lambda}(s, k_n(y)) + V(s, k_n(y))) dy ds. \quad (3.36)$$

Let us here summarize some elementary properties about the above kernels.

Lemma 3.12. *There exists a constant $C > 0$ such that the following hold:*

(i)

$$\sup_{0 < r \leq T} \sup_{x \in [0,1]} \int_0^1 G^n(r, x, y) dy \leq C.$$

(ii)

$$\int_0^\infty \int_0^1 |G(t, x, y) - G^n(t, x, y)|^2 dy dt \leq \frac{C}{n}$$

for $x \in [0, 1]$ and $n \geq 1$.

The first part (i) is the standard mass bound for the discrete heat kernel. For (ii), see [17, Lemma 3.2]. The next theorem is the main result of this subsection.

Theorem 3.13. *For any $\beta \in (0, 1)$, there exists a constant $C > 0$ such that*

$$\|Z^{n,\lambda} - Z^\lambda\|_{T,\infty} \leq C\mathcal{K} e^{C\lambda} \lambda^{2-\kappa/2} (n^{-1/2} + n^{-\alpha_2} + n^{-\beta}), \quad (3.37)$$

where α_2 is the spatial Hölder's index of $V(t, x)$, see (3.3).

Proof. Subtracting (3.32) from (3.36), we obtain

$$\begin{aligned} & |Z^{n,\lambda}(t, x) - Z^\lambda(t, x)| \\ & \leq \int_0^t \int_0^1 |G^n - G|(t-s, x, y) |F_\lambda(Z^\lambda(s, y) + V(s, y))| dy ds \\ & \quad + \int_0^t \int_0^1 G^n(t-s, x, y) |F_\lambda(Z^{n,\lambda}(s, k_n(y)) + V(s, k_n(y))) - F_\lambda(Z^\lambda(s, y) + V(s, y))| dy ds \\ & =: I_1^n(t, x) + I_2^n(t, x). \end{aligned} \quad (3.38)$$

Let Z be the solution of the parabolic obstacle problem (3.1) with the obstacle $V(t, x)$. Then, we know $Z + V \geq 0$. By the simple inequality $a^- \leq |a - b|$ for any $a \in \mathbb{R}$ and $b \geq 0$, we have

$$(Z^\lambda + V)^-(t, x) \leq |Z(t, x) - Z^\lambda(t, x)|, \quad (t, x) \in \mathcal{Q}_T.$$

Therefore, by Theorem 3.7, we have

$$\sup_{(t,x) \in \mathcal{Q}_T} (Z^\lambda(t, x) + V(t, x))^- \leq \|Z - Z^\lambda\|_{T,\infty} \leq C\mathcal{K}\lambda^{-\kappa/2},$$

which yields that

$$M_{\lambda,T} := \|F_\lambda(Z^\lambda + V)\| \leq C\mathcal{K}\lambda^{1-\kappa/2}. \quad (3.39)$$

Hence, by Lemma 3.12-(ii) and (3.39), we have for any $(t, x) \in \mathcal{Q}_T$,

$$\begin{aligned} I_1^n(t, x) &\leq M_{\lambda,T} \int_0^t \int_0^1 |G^n - G|(t-s, x, y) dy ds \\ &\leq C\mathcal{K}\lambda^{1-\kappa/2} n^{-1/2}. \end{aligned} \quad (3.40)$$

We next estimate $I_2^n(t, x)$. For this, we first show that $Z^\lambda(t, x)$ is Hölder continuous in x . Recalling the well-known estimate, see [14, Lemma 3.4.8] for example, for any $\beta \in (0, 1)$

$$\int_0^T \int_0^1 |G(s, x, y) - G(s, x', y)| dy ds \leq C_\beta |x - x'|^\beta, \quad x, x' \in [0, 1]$$

and using (3.39) again, we have that for any $t \in [0, T]$ and $x, x' \in [0, 1]$

$$\begin{aligned} |Z^\lambda(t, x) - Z^\lambda(t, x')| &\leq M_{\lambda,T} \int_0^t \int_0^1 |G(t-s, x, y) - G(t-s, x', y)| dy ds \\ &\leq C\mathcal{K}\lambda^{1-\kappa/2} |x - x'|^\beta. \end{aligned} \quad (3.41)$$

Hence, (3.41) yields that for any $s \in [0, T]$ and $y \in [0, 1]$,

$$|Z^\lambda(s, k_n(y)) - Z^\lambda(s, y)| \leq C\mathcal{K}\lambda^{1-\kappa/2} n^{-\beta}. \quad (3.42)$$

Since $F_\lambda(r) = \lambda r^-$ is λ -Lipschitz, by (3.42) and the assumption (3.3) on the obstacle V , we have

$$\begin{aligned} &|F_\lambda(Z^{n,\lambda}(s, k_n(y)) + V(s, k_n(y))) - F_\lambda(Z^\lambda(s, y) + V(s, y))| \\ &\leq \lambda \left(|Z^{n,\lambda}(s, k_n(y)) - Z^\lambda(s, k_n(y))| + |Z^\lambda(s, k_n(y)) - Z^\lambda(s, y)| + |V(s, k_n(y)) - V(s, y)| \right) \\ &\leq \lambda \left(E_n(s) + C\mathcal{K}\lambda^{1-\kappa/2} n^{-\beta} + \mathcal{K}n^{-\alpha_2} \right), \end{aligned} \quad (3.43)$$

where

$$E_n(t) := \sup_{0 \leq r \leq t, 0 \leq x \leq 1} |Z^{n,\lambda}(r, x) - Z^\lambda(r, x)|, \quad t \in [0, T].$$

Substituting this into $I_2^n(t, x)$ yields

$$\begin{aligned} I_2^n(t, x) &\leq \lambda \int_0^t \int_0^1 G^n(t-s, x, y) \left(E_n(s) + C\mathcal{K}\lambda^{1-\kappa/2} n^{-\beta} + \mathcal{K}n^{-\alpha_2} \right) dy ds \\ &\leq \lambda \int_0^t \left(E_n(s) + C\mathcal{K}\lambda^{1-\kappa/2} n^{-\beta} + \mathcal{K}n^{-\alpha_2} \right) \left(\int_0^1 G^n(t-s, x, y) dy \right) ds. \end{aligned}$$

Then, using Lemma 3.12-(i), we deduce that

$$I_2^n(t, x) \leq C\mathcal{K}\lambda(\lambda^{1-\kappa/2} n^{-\beta} + n^{-\alpha_2}) + C\lambda \int_0^t E_n(s) ds.$$

Recalling (3.38) and, then using this estimate and (3.43), we obtain

$$E_n(t) \leq C\mathcal{K}(\lambda^{1-\kappa/2}n^{-1/2} + \lambda^{2-\kappa/2}n^{-\beta} + \lambda n^{-\alpha_2}) + C\lambda \int_0^t E_n(s) ds,$$

which, by Gronwall's inequality, yields that

$$E_n(t) \leq Ce^{C\lambda}\mathcal{K}(\lambda^{1-\kappa/2}n^{-1/2} + \lambda^{2-\kappa/2}n^{-\beta} + \lambda n^{-\alpha_2}).$$

Therefore, (3.37) is proved by noting that $\kappa < 1$ and $\lambda \geq 1$. □

3.4 Proof of Theorem 3.4

In this part, we state the proof of Theorem 3.4 using the results obtained in Subsections 3.1-3.3.

Proof of Theorem 3.4. By the triangle inequality,

$$\|Z^n - Z\|_{T,\infty} \leq \|Z^n - Z^{n,\lambda}\|_{T,\infty} + \|Z^{n,\lambda} - Z^\lambda\|_{T,\infty} + \|Z^\lambda - Z\|_{T,\infty}.$$

Using Theorems 3.7, 3.9 and 3.13, we obtain

$$\|Z^n - Z\|_{T,\infty} \leq C\mathcal{K}\left(\lambda^{-\kappa/2} + e^{C\lambda}\lambda^{2-\kappa/2}(n^{-1/2} + n^{-\alpha_2} + n^{-\beta})\right); \quad (3.44)$$

recall that $\mathcal{K}$ is the constant in (3.3). Now set

$$\gamma := \min\left\{\frac{1}{2}, \alpha_2, \beta\right\}.$$

Then (3.44) becomes

$$\|Z^n - Z\|_{T,\infty} \leq C\mathcal{K}\left(\lambda^{-\kappa/2} + e^{C\lambda}\lambda^{2-\kappa/2}n^{-\gamma}\right). \quad (3.45)$$

Choose

$$\lambda_n = \delta \log n,$$

where $\delta > 0$ is small enough that

$$C\delta < \gamma.$$

Then, we see

$$e^{C\lambda_n}n^{-\gamma} = n^{C\delta-\gamma}.$$

Substituting this into (3.45), we get

$$\|Z^n - Z\|_{T,\infty} \leq C\mathcal{K}\left((\log n)^{-\kappa/2} + (\log n)^{2-\kappa/2}n^{-(\gamma-C\delta)}\right).$$

Since $\gamma - C\delta > 0$ and C is independent of n, λ , the second term is bounded by $C(\log n)^{-\kappa/2}$ for all $n \geq 2$. Hence

$$\|Z^n - Z\|_{T,\infty} \leq C\mathcal{K}(\log n)^{-\kappa/2}.$$

This proves (3.6). Therefore the proof is complete. □

Remark 3.14. From the proof, it is known that the logarithmic rate essentially comes from the error estimate obtained in Theorem 3.13. Recall that Theorem 3.13 is obtained by using the Lipschitz continuity of $F_\lambda(r) = \lambda r^-$, which leads to a Gronwall argument and produces the exponential factor $e^{C\lambda}$. On the other hand, F_λ is in fact monotone decreasing, i.e.,

$$(a - b)(F_\lambda(a) - F_\lambda(b)) \leq 0, \quad a, b \in \mathbb{R},$$

which suggests that one may instead use a monotonicity (energy) argument. Such an approach removes the exponential dependence on λ at the L^2 -level, yielding estimates without the factor $e^{C\lambda}$. Nevertheless, upgrading this improvement to the sup-norm $\|\cdot\|_{T,\infty}$ is nontrivial, due to the presence of the discretization-continuum consistency error and the lack of a direct comparison principle for the penalized equations. Consequently, the current argument yields only a logarithmic convergence rate in n after optimizing in λ .

4 Convergence rate of reflected SPDEs via lattice approximations

In this section, we devote to the proof of Theorem 2.2 by using Theorem 3.4. Let us recall the mild forms of the reflected SPDE (1.1) and its discretization scheme (2.2) (or equivalently (2.3)) based on [39]. Let $u(t, x)$ be the solution of (1.1) and recall its mild form (2.1). Let us denote by $\bar{v}(t, x)$ all of the parts except the last one $\int_0^t \int_0^1 G(t-s, x, y) \eta(ds, dy)$ on the right hand side of (2.1), i.e.,

$$\begin{aligned} \bar{v}(t, x) = & \int_0^1 G(t, x, y) u_0(y) dy + \int_0^t \int_0^1 G(t-s, x, y) f(s, y, u(s, y)) dy ds \\ & + \int_0^t \int_0^1 G(t-s, x, y) \sigma(s, y, u(s, y)) W(ds, dy). \end{aligned} \quad (4.1)$$

Lemma 4.2 below shows that $\bar{v}(t, x)$ satisfies the assumption **(V)**, see (3.3). Let

$$(\bar{z}(t, x) := u(t, x) - \bar{v}(t, x), \eta(dt, dx)).$$

Then it solves the random parabolic obstacle problem

$$\begin{cases} \frac{\partial \bar{z}(t, x)}{\partial t} - \frac{\partial^2 \bar{z}(t, x)}{\partial x^2} = \dot{\eta}(t, x), & x \in [0, 1], \\ \bar{z}(t, x) \geq -\bar{v}(t, x), \\ \int_0^t \int_0^1 (\bar{z}(s, x) + \bar{v}(s, x)) \eta(ds, dx) = 0, & t \geq 0. \end{cases} \quad (4.2)$$

Similarly, we know that the solution of (2.3) is given by

$$\begin{aligned} u^n(t) = & G^n(t) u^n(0) + \int_0^t G^n(t-s) F^n(s, u^n(s)) ds \\ & + \sqrt{n} \int_0^t G^n(t-s) \Sigma^n(s, u^n(s)) dW^n(s) + \int_0^t G^n(t-s) d\eta^n(s) \\ =: & v^n(t) + \int_0^t G^n(t-s) d\eta^n(s), \end{aligned}$$

where $G^n(t) = \exp(n^2 A^n t)$ as before. Then it is known that $(Z^n(t) := u^n(t) - v^n(t), \eta^n(t))$ solves the $(n-1)$ -dimensional random Skorohod-type system

$$\begin{cases} dZ^n(t) = n^2 A^n Z^n(t) dt + d\eta^n(t), \\ Z^n(t) \geq -v^n(t), \\ \int_0^T \langle Z^n(t) + v^n(t), d\eta^n(t) \rangle = 0. \end{cases} \quad (4.3)$$

Recall the continuous random field $u^n(t, x)$ is defined by (2.6) in Section 2. Similarly, we further introduce the random fields

$$\eta^n(t, x) = \mathcal{I}_n \eta^n(t)(x) \quad \text{and} \quad v^n(t, x) = \mathcal{I}_n v^n(t)(x), \quad t \geq 0, x \in [0, 1]$$

with $\eta_0^n(t) = \eta_n^n(t) := 0$, $v_0^n(t) = v_n^n(t) := 0$. Let the kernel $G^n(t, x, y)$ be defined by (3.34) in Section 3. Then, we have

$$\begin{aligned} u^n(t, x) &= \int_0^1 G^n(t, x, y) u^n(0, k_n(y)) dy + \int_0^t \int_0^1 G^n(t-s, x, y) f(s, k_n(y), u^n(s, k_n(y))) dy ds \\ &\quad + \int_0^t \int_0^1 G^n(t-s, x, y) \sigma(s, k_n(y), u^n(s, k_n(y))) W(ds, dy) \\ &\quad + \int_0^t \int_0^1 G^n(t-s, x, y) \eta^n(ds, k_n(y)) dy, \end{aligned}$$

and

$$v^n(t, x) := u^n(t, x) - \int_0^t \int_0^1 G^n(t-s, x, y) \eta^n(ds, k_n(y)) dy,$$

where $k_n(y) = \frac{[ny]}{n}$ as before.

However, comparing (4.2) with (4.3), it is known that Theorem 3.4 can not be applied directly. In fact, the obstacle $v^n(t)$ in (4.3) is not the restriction of $\bar{v}(t, x)$ on the gridpoints x_k . This motivates the introduction of an intermediate stochastic Skorohod-type equation with the sampled obstacle $\bar{v}^n$ below. For every $n \in \mathbb{N}$, let us denote by $\bar{v}^n(t) \in \mathbb{R}^{n-1}$ the lattice vector determined by $\bar{v}(t, x)$, that is,

$$\bar{v}^n(t) = (\bar{v}(t, x_1), \dots, \bar{v}(t, x_{n-1}));$$

see (4.1) for $\bar{v}(t, x)$. Let $(\bar{z}^n, \bar{\eta}^n)$ be the solution of the following random Skorohod-type problem with the obstacle $\bar{v}^n(t)$ in $\mathbb{R}^{n-1}$:

$$\begin{cases} d\bar{z}^n(t) = n^2 A^n \bar{z}^n(t) dt + d\bar{\eta}^n(t), \\ \bar{z}^n(t) \geq -\bar{v}^n(t), \\ \int_0^T \langle \bar{z}^n(t) + \bar{v}^n(t), d\bar{\eta}^n(t) \rangle = 0. \end{cases} \quad (4.4)$$

Introduce the continuous random field $\bar{z}^n$:

$$\bar{z}^n(t, x) := \bar{z}_k^n(t) + (nx - k) (\bar{z}_{k+1}^n(t) - \bar{z}_k^n(t)) \quad (4.5)$$

for $x \in [x_k, x_{k+1})$, $k = 0, \dots, n-1$, with $\bar{z}_0^n(t) = \bar{z}_n^n(t) := 0$. Let

$$\bar{v}^n(t, x) = \mathcal{I}_n \bar{v}^n(t)(x), \quad t \geq 0, x \in [0, 1]. \quad (4.6)$$

Then, we have

$$\begin{aligned}\bar{v}^n(t, x) &= \int_0^1 \bar{G}^n(t, x, y) u_0(y) dy + \int_0^t \int_0^1 \bar{G}^n(t-s, x, y) f(s, y, u(s, y)) dy ds \\ &\quad + \int_0^t \int_0^1 \bar{G}^n(t-s, x, y) \sigma(s, y, u(s, y)) W(ds, dy),\end{aligned}\tag{4.7}$$

where, as introduced in [39, (5.40)],

$$\begin{aligned}\bar{G}^n(t, x, y) \\ := G(t, x_k, y) + (nx - k) [G(t, x_{k+1}, y) - G(t, x_k, y)], \quad x \in [x_k, x_{k+1}), k = 0, 1, 2, \dots, n-1.\end{aligned}$$

Recalling $\phi_j(x) = \sqrt{2} \sin(j\pi x)$ and the definition of $\phi_j^n(x)$ in (3.35), one easily knows

$$\bar{G}^n(t, x, y) = \sum_{j=1}^{\infty} e^{-j^2 \pi^2 t} \phi_j^n(x) \phi_j(y).\tag{4.8}$$

Finally, set

$$\bar{u}^n(t, x) := \bar{z}^n(t, x) + \bar{v}^n(t, x), t \in [0, T], x \in [0, 1].$$

From now on, we first aim to establish the convergence rate of $\|\bar{u}^n - u\|_{T, \infty}$ in Subsection 4.1 and then complete the proof of Theorem 2.2 in Subsection 4.2.

4.1 Convergence rate of $\|\bar{u}^n - u\|_{T, \infty}$

The main goal of this subsection is to prove the logarithmic convergence rate of $\|\bar{u}^n - u\|_{T, \infty}$.

Theorem 4.1. *Under the assumptions in Theorem 2.2, for every $0 < \kappa < \gamma$ and $p \geq 1$, there exists a constant $C > 0$ such that*

$$\mathbb{E}[\|\bar{u}^n - u\|_{T, \infty}^p] \leq C(\log n)^{-\frac{\kappa p}{2}}, n \geq 2;$$

recall that $\gamma = \frac{1}{2} \wedge \alpha$.

Proof. This follows immediately from Propositions 4.3 and 4.9 below. $\square$

To prove this, we divide it into two main parts. The first one is the convergence rate of reflected systems between (4.2) and (4.5), see Proposition 4.3 below, and the second one is that of $\bar{v}^n(t, x)$ to $\bar{v}(t, x)$, see Proposition 4.9 below.

Let us start with the following lemma, which shows that $\bar{v}(t, x)$ satisfies the assumption **(V)** imposed on the obstacle in Section 3.

Lemma 4.2. *There exists a constant $C > 0$ such that for all $s, t \in [0, T], x, y \in [0, 1]$,*

$$\mathbb{E}[|\bar{v}(t, x) - \bar{v}(s, y)|^p] \leq CM_p \left(|t - s|^{1/2} + |x - y| \right)^{\gamma p},\tag{4.9}$$

where

$$M_p = 1 + \mathbb{E}[\|u_0\|_{C^\alpha([0,1])}^p] + \sup_{(r,y) \in [0,T] \times [0,1]} \mathbb{E}[|u(r, y)|^p] < \infty.$$

Moreover, for every $0 < \beta < \gamma$ and $p > \frac{3}{\gamma-\beta}$, there exists a non-negative random variable $\mathcal{K}$ such that

$$|\bar{v}(t, x) - \bar{v}(s, y)| \leq \mathcal{K} \left(|t - s|^{1/2} + |x - y| \right)^\beta, \quad s, t \in [0, T], x, y \in [0, 1] \text{ a.s.} \quad (4.10)$$

and

$$\mathbb{E}[\mathcal{K}^p] \leq CM_p. \quad (4.11)$$

Proof. Put

$$\bar{v}(t, x) := S_t u_0(x) + J_1(t, x) + J_2(t, x).$$

By (2.20) in the proof of Theorem 2.5, it is sufficient to show that there exists a constant $C > 0$ such that for all $s, t \in [0, T]$ and $x, y \in [0, 1]$

$$\mathbb{E}[|J_1(t, x) - J_1(s, y)|^p] \leq CM_p \left(|t - s|^{1/2} + |x - y| \right)^{p/2}, \quad (4.12)$$

$$\mathbb{E}[|J_2(t, x) - J_2(s, y)|^p] \leq CM_p \left(|t - s|^{1/2} + |x - y| \right)^{p/2}. \quad (4.13)$$

These can be easily shown by the usual arguments, see Lemma 3.6 [17] or [30] for instance. However, in order to see the constant M_p on the right hand sides of (4.12) and (4.13), let us give a brief explanation for (4.13) as an example. For $0 \leq s \leq t \leq T$, we write

$$J_2(t, x) - J_2(s, y) = K_1(s, t, x, y) + K_2(s, t, x), \quad x, y \in [0, 1],$$

where

$$\begin{aligned} K_1(s, t, x, y) &:= \int_0^s \int_0^1 (G(t - r, x, z) - G(s - r, y, z)) \sigma(r, z, u(r, z)) W(dr, dz), \\ K_2(s, t, x) &:= \int_s^t \int_0^1 G(t - r, x, z) \sigma(r, z, u(r, z)) W(dr, dz). \end{aligned}$$

By the Burkholder–Davis–Gundy inequality and the integrated Minkowski inequality (See Proposition 1.3 [3] for instance), we have

$$\begin{aligned} \mathbb{E}[|K_1(s, t, x, y)|^p] &\leq C \mathbb{E} \left[\left(\int_0^s \int_0^1 |G(t - r, x, z) - G(s - r, y, z)|^2 |\sigma(r, z, u(r, z))|^2 dz dr \right)^{p/2} \right] \\ &\leq C \left(\int_0^s \int_0^1 |G(t - r, x, z) - G(s - r, y, z)|^2 \mathbb{E} \left[|\sigma(r, z, u(r, z))|^p \right]^{\frac{2}{p}} dz dr \right)^{p/2} \\ &\leq \sup_{(s, z) \in \mathcal{Q}_T} \mathbb{E} \left[|\sigma(r, z, u(r, z))|^p \right] \left(\int_0^s \int_0^1 |G(t - r, x, z) - G(s - r, y, z)|^2 dz dr \right)^{p/2} \\ &\leq CM_p \left(|t - s|^{1/2} + |x - y| \right)^{p/2}, \end{aligned}$$

where we have used the fact

$$\int_0^s \int_0^1 |G(t - r, x, z) - G(s - r, y, z)|^2 dz dr \leq |t - s|^{1/2} + |x - y|,$$

see (2.8) and (2.9) [9] for instance, for the last inequality. Similarly,

$$\mathbb{E}[|K_2|^p] \leq CM_p \left(\int_s^t \int_0^1 |G(t-r, x, y)|^2 dy dr \right)^{p/2} \leq CM_p |t-s|^{p/4}.$$

Combining the above two estimates yields (4.13). Consequently, combining (2.20), (4.12) and (4.13), and noting that $p/2 \geq \gamma p$, we obtain (4.9).

Note that $0 < \beta < \gamma$ and $p > \frac{3}{\gamma-\beta}$ imply that $\beta < \gamma - \frac{3}{p}$. Hence using the anisotropic Kolmogorov continuity theorem ([7, Corollary A.3] and its proof) together with (4.9), it is easy to obtain (4.10) and (4.11). $\square$

Using Theorem 3.4 together with the above lemma, we have the explicit convergence rate of $\bar{z}^n(t, x)$ defined by (4.5) to the solution $\bar{z}(t, x)$ of (4.2) in the following sense:

Proposition 4.3. *For every $0 < \kappa < \gamma$ and $p > 1$, there exists a constant $C > 0$ such that*

$$\mathbb{E}[\|\bar{z}^n - \bar{z}\|_{T,\infty}^p] \leq C(\log n)^{-\frac{\kappa p}{2}}, \quad n \geq 2. \quad (4.14)$$

Proof. Thanks to Jensen's inequality, it is enough to prove (4.14) for large enough p . Note that $\bar{z}(t, x)$ and $\bar{z}^n(t)$ are solutions of (4.2) and (4.4) respectively. By Lemma 4.2, it is known that the obstacle $\bar{v}(t, x)$ is Hölder continuous in (t, x) and satisfies the assumption **(V)** on the obstacle, see (3.3), with $\kappa = \beta \in (0, \gamma)$ and the random Lipschitz constant $\mathcal{K}$, see (4.10). Then by Theorem 3.4, it follows that for any $t \in [0, T], x \in [0, 1]$,

$$|\bar{z}^n(t, x) - \bar{z}(t, x)| \leq C\mathcal{K}(\log n)^{-\frac{\kappa}{2}}, \quad n \geq 2 \text{ a.s.}$$

Noting that $\mathcal{K}$ satisfies (4.11), we easily obtain (4.14). $\square$

Next, we turn to the study on the convergence rate of $\bar{v}^n(t, x)$ to $\bar{v}(t, x)$. We first formulate a criterion for the convergence rate of a sequence of random fields in the uniform topology.

Proposition 4.4. *Let $U^n(t, x), (t, x) \in \mathcal{Q}_T$ be an $\mathcal{F}_t$ -adapted random field satisfying the following assumptions:*

(i) *There exists a constant $C > 0$ such that*

$$\sup_{(t,x) \in \mathcal{Q}_T} \mathbb{E}[|U_n(t, x)|^p] \leq C n^{-p/2}. \quad (4.15)$$

(ii) *There exists $C > 0$ such that*

$$\sup_{n \in \mathbb{N}} \mathbb{E}[|U_n(t, x) - U_n(s, y)|^p] \leq C \rho((t, x), (s, y))^{p/2}, \quad (t, x), (s, y) \in \mathcal{Q}_T. \quad (4.16)$$

Then for every $0 < \theta < \frac{1}{2}$ and $p > \frac{6}{1-2\theta}$, there exists $C > 0$ such that

$$\mathbb{E}[\|U_n\|_{T,\infty}^p] \leq C n^{-p\theta}.$$

Proof. Let $\delta \in (0, 1)$, which will be specified in the last step. Then one can cover $\mathcal{Q}_T$ by at most

$$N(\delta) \leq C_T \delta^{-3}$$

balls of radius δ in the metric ρ . Indeed, whenever $\rho((t, x), (s, y)) < \delta$, then $|t - s| < \delta^2$, $|x - y| < \delta$, so a grid with mesh δ^2 in time and δ in space suffices, and the number of grid points is of order δ^{-3} . Hence, there exists a finite set $\Gamma_\delta \subset \mathcal{Q}_T$ such that $\#\Gamma_\delta \leq C_T \delta^{-3}$, and for every $u \in \mathcal{Q}_T$ there exists $v \in \Gamma_\delta$ with $\rho(u, v) \leq \delta$. Noting that

$$\sup_{u \in \mathcal{Q}_T} |U_n(u)| \leq \max_{v \in \Gamma_\delta} |U_n(v)| + \sup_{\rho(u, v) \leq \delta} |U_n(u) - U_n(v)|,$$

we have that for $p \geq 1$,

$$\mathbb{E} \left[\sup_{u \in \mathcal{Q}_T} |U_n(u)|^p \right] \leq C_p \left(\mathbb{E} \left[\max_{v \in \Gamma_\delta} |U_n(v)|^p \right] + \mathbb{E} \left[\sup_{\rho(u, v) \leq \delta} |U_n(u) - U_n(v)|^p \right] \right). \quad (4.17)$$

First, the assumption (4.15) yields that

$$\mathbb{E} \left[\max_{v \in \Gamma_\delta} |U_n(v)|^p \right] \leq C \delta^{-3} n^{-p/2}. \quad (4.18)$$

On the other hand, by the assumption (4.16) and [7, Corollary A.3], for any $\beta \in (0, \frac{1}{2} - \frac{3}{p})$, there exists a non-negative random variable $\mathcal{K}$ independent of n such that

$$|U_n(u) - U_n(v)| \leq \mathcal{K} \rho(u, v)^\beta, \quad u, v \in \mathcal{Q}_T$$

and $\mathbb{E}[|\mathcal{K}|^p] < \infty$. Hence,

$$\sup_{\rho(u, v) \leq \delta} |U_n(u) - U_n(v)|^p \leq \mathcal{K}^p \delta^{\beta p}.$$

Taking expectations gives that

$$\sup_{n \in \mathbb{N}} \mathbb{E} \left[\sup_{\rho(u, v) \leq \delta} |U_n(u) - U_n(v)|^p \right] \leq C \delta^{\beta p} \quad (4.19)$$

holds for a constant $C > 0$. Combining (4.17), (4.18) and (4.19) yields that

$$\mathbb{E} \left[\sup_{u \in \mathcal{Q}_T} |U_n(u)|^p \right] \leq C \left(\delta^{-3} n^{-p/2} + \delta^{\beta p} \right).$$

Now choosing

$$\delta = n^{-\frac{p}{2(p\beta+3)}},$$

we have

$$\mathbb{E} \left[\sup_{u \in \mathcal{Q}_T} |U_n(u)|^p \right] \leq C n^{-\frac{\beta p^2}{2(p\beta+3)}}.$$

Since β can be taken arbitrarily close to $\frac{1}{2} - \frac{3}{p}$, we have $\frac{\beta p}{2(p\beta+3)}$ is arbitrarily close to $\frac{1}{2} - \frac{3}{p}$. Therefore for every

$$\theta < \frac{1}{2} - \frac{3}{p}$$

there exists a constant $C > 0$ such that $\mathbb{E} \left[\sup_{u \in \mathcal{Q}_T} |U_n(u)|^p \right] \leq C n^{-p\theta}$, which completes the proof. $\square$

Remark 4.5. Let $a_n > 0$. If we replace $Cn^{-p/2}$ in (i) by Ca_n , and C in (ii) by Ca_n , then by the similar argument, we can obtain the following:

There exists a constant $C > 0$ such that

$$\mathbb{E} \left[\|U_n\|_{T,\infty}^p \right] \leq Ca_n.$$

In particular, if $a_n = (\log n)^{-\kappa p/2}$, $\kappa \in (0, 1)$, $n \geq 2$, then we have logarithmic convergence rate

$$\mathbb{E} \left[\|U_n\|_{T,\infty}^p \right] \leq C(\log n)^{-\kappa p/2}, \quad n \geq 2.$$

This will be used in the proof of Theorem 2.2.

Recalling the representations (4.1) of $\bar{v}(t, x)$ and (4.7) of $\bar{v}^n(t, x)$, we prepare the following estimates on the kernels.

Lemma 4.6. *Let $\Delta_n(t, x, y) := G(t, x, y) - \bar{G}^n(t, x, y)$. Then there exists a constant $C > 0$ such that the following hold:*

(i)

$$\sup_{x \in [0,1]} \int_0^\infty \int_0^1 \Delta_n(t, x, y)^2 dy dt \leq \frac{C}{n}, \quad n \geq 1. \quad (4.20)$$

(ii) *For any $0 \leq s \leq t \leq T$, $x, y \in [0, 1]$, and $n \geq 1$,*

$$\int_s^t \int_0^1 |\Delta_n(t-r, x, z)|^2 dz dr \leq C |t-s|^{1/2}, \quad (4.21)$$

$$\int_0^s \int_0^1 |\Delta_n(t-r, x, z) - \Delta_n(s-r, y, z)|^2 dz dr \leq C(|t-s|^{1/2} + |x-y|). \quad (4.22)$$

Proof. By (4.8) and the series expansion

$$G(t, x, y) = \sum_{j=1}^{\infty} e^{-j^2 \pi^2 t} \phi_j(x) \phi_j(y),$$

we have

$$\Delta_n(t, x, y) = \sum_{j=1}^{\infty} e^{-j^2 \pi^2 t} (\phi_j(x) - \phi_j^n(x)) \phi_j(y).$$

Using the orthonormality of $\{\phi_j\}_{j \geq 1}$ in $L^2(0, 1)$,

$$\int_0^1 |\Delta_n(t, x, y)|^2 dy = \sum_{j=1}^{\infty} e^{-2j^2 \pi^2 t} (\phi_j(x) - \phi_j^n(x))^2. \quad (4.23)$$

For (i), integrating (4.23) in $t \in (0, \infty)$, we have

$$\int_0^\infty \int_0^1 |\Delta_n(t, x, y)|^2 dy dt = \sum_{j=1}^{\infty} \frac{(\phi_j(x) - \phi_j^n(x))^2}{2\pi^2 j^2}. \quad (4.24)$$

Inserting the elementary bound

$$\sup_{x \in [0,1]} |\phi_j(x) - \phi_j^n(x)| \leq C \min\left\{1, \frac{j}{n}\right\}, \quad j \geq 1, \quad n \geq 1,$$

into (4.24) and noting that

$$\sum_{j=1}^{\infty} \min\left\{\frac{1}{j^2}, \frac{1}{n^2}\right\} = \sum_{j \leq n} \frac{1}{n^2} + \sum_{j > n} \frac{1}{j^2} \leq \frac{C}{n},$$

we obtain (4.20).

Noting that for any $x \in [x_k, x_{k+1}]$,

$$\bar{G}^n(t, x, y) = (1 - \theta)G(t, x_k, y) + \theta G(t, x_{k+1}, y), \quad \theta := nx - k \in [0, 1],$$

we obtain (4.21) and (4.22) easily by using the well-known properties of $G(t, x, y)$, see (2.8)-(2.10) [9]. $\square$

Remark 4.7. (1) Let $\Delta_n^*(t, x, y) := G^n(t, x, y) - \bar{G}^n(t, x, y)$. Then the estimates (4.20), (4.21) and (4.22) still hold replaced $\Delta_n(t, x, y)$ by $\Delta_n^*(t, x, y)$.

(2) The estimates (4.21) and (4.22) can be improved to the following:

(i) For every $\gamma \in (0, 1)$ and every $\varepsilon \in (0, \frac{1-\gamma}{2})$, there exists $C > 0$ such that

$$\sup_{x \in [0,1]} \int_s^t \int_0^1 |\Delta_n(t-r, x, y)|^2 dy dr \leq C \min\left\{n^{-1}, n^{-\gamma} |t-s|^{\frac{1}{2}-\frac{\gamma}{2}-\varepsilon}\right\}$$

for all $0 \leq s \leq t \leq T$ and $n \geq 1$.

(ii) For every $\gamma \in [0, 1]$, there exists $C > 0$ such that

$$\int_0^s \int_0^1 |\Delta_n(t-r, x, z) - \Delta_n(s-r, y, z)|^2 dz dr \leq C n^{-\gamma} (|t-s|^{1/2} + |x-y|)^{1-\gamma}$$

for all $0 \leq s \leq t \leq T$, $x, y \in [0, 1]$, $n \geq 1$.

But these estimates cannot contribute to the convergence rate of $\mathbb{E}[\|\bar{u}^n - u\|_{T,\infty}^p]$ because of the reflections. So we only use the simple estimates in Lemma 4.6.

Lemma 4.8. Let $\Sigma(t, x), t \geq 0, x \in [0, 1]$ be an $\mathcal{F}_t$ -adapted random field satisfying that for every $p \geq 1$

$$\sup_{0 \leq s \leq T, 0 \leq y \leq 1} \mathbb{E} [|\Sigma(s, y)|^p] \leq C_p. \quad (4.25)$$

Then the random fields

$$\begin{aligned} A_1^n(t, x) &:= \int_0^t \int_0^1 (G(t-s, x, y) - \bar{G}^n(t-s, x, y)) \Sigma(s, y) dy ds, \\ A_2^n(t, x) &:= \int_0^t \int_0^1 (G(t-s, x, y) - \bar{G}^n(t-s, x, y)) \Sigma(s, y) W(ds, dy), \quad (t, x) \in \mathcal{Q}_T \end{aligned}$$

are well defined. Moreover, for every $0 < \theta < \frac{1}{2}$ and every $p > \frac{6}{1-2\theta}$, there exists a constant $C > 0$ such that

$$\mathbb{E} [\|A_1^n\|_{T,\infty}^p] + \mathbb{E} [\|A_2^n\|_{T,\infty}^p] \leq C n^{-p\theta}.$$

Proof. By Proposition 4.4, it suffices to verify that $A_1^n(t, x)$ and $A_2^n(t, x)$ satisfy the assumptions (4.15) and (4.16).

Let $\Delta_n(t, x, y) = G(t, x, y) - \bar{G}^n(t, x, y)$. For any $(t, x) \in [0, T] \times [0, 1]$, using the weighted Hölder inequality and taking expectation, we have

$$\mathbb{E}[|A_1^n(t, x)|^p] \leq \left(\int_0^t \int_0^1 |\Delta_n(t-s, x, y)| dy ds \right)^{p-1} \int_0^t \int_0^1 |\Delta_n(t-s, x, y)| \mathbb{E}[|\Sigma(s, y)|^p] dy ds.$$

Taking supremum over (t, x) and using (4.25), we have

$$\sup_{(t, x) \in \mathcal{Q}_T} \mathbb{E}[|A_1^n(t, x)|^p] \leq C \left(\sup_{(t, x) \in \mathcal{Q}_T} \int_0^t \int_0^1 |\Delta_n(t-s, x, y)| dy ds \right)^p. \quad (4.26)$$

On the other hand, the Burkholder-Davis-Gundy inequality gives that

$$\mathbb{E}[|A_2^n(t, x)|^p] \leq C \mathbb{E} \left[\left(\int_0^t \int_0^1 |\Delta_n(t-s, x, y)|^2 |\Sigma(s, y)|^2 dy ds \right)^{p/2} \right].$$

Then, by Minkowski's integral inequality again, we have

$$\left\| \int_0^t \int_0^1 |\Delta_n(t-s, x, y)|^2 |\Sigma(s, y)|^2 dy ds \right\|_{L^{p/2}(\Omega)} \leq \int_0^t \int_0^1 |\Delta_n(t-s, x, y)|^2 \|\Sigma(s, y)\|_{L^p(\Omega)}^2 dy ds.$$

Hence, this together with the assumption (4.25) gives that

$$\|A_2^n(t, x)\|_{L^p(\Omega)}^2 \leq C \int_0^t \int_0^1 |\Delta_n(t-s, x, y)|^2 dy ds. \quad (4.27)$$

Therefore, using (4.26), (4.27) and (4.20) in Lemma 4.6, we have there is a constant $C > 0$ such that

$$\sup_{(t, x) \in \mathcal{Q}_T} \mathbb{E}[|A_1^n(t, x)|^p] + \sup_{(t, x) \in \mathcal{Q}_T} \mathbb{E}[|A_2^n(t, x)|^p] \leq C n^{-p/2},$$

which testifies the condition (4.15).

Let us continue to testify (4.16), i.e., there exists a constant $C > 0$ such that for $(s, y), (t, x) \in \mathcal{Q}_T$

$$\sup_{n \in \mathbb{N}} \mathbb{E}[|A_i^n(t, x) - A_i^n(s, y)|^p] \leq C(|t-s|^{1/2} + |x-y|)^{p/2}, \quad i = 1, 2. \quad (4.28)$$

Since the proof of $A_1^n(t, x)$ is easier, we just give the proof for $A_2^n(t, x)$ as the example.

$$\begin{aligned} A_2^n(t, x) - A_2^n(s, y) &= \int_0^s \int_0^1 [\Delta_n(t-r, x, z) - \Delta_n(s-r, y, z)] \Sigma(r, z) W(dr, dz) \\ &\quad + \int_s^t \int_0^1 \Delta_n(t-r, x, z) \Sigma(r, z) W(dr, dz) \\ &=: A_{2,1}^n(s, t, x, y) + A_{2,2}^n(s, t, x). \end{aligned}$$

Similar arguments to (4.27) gives

$$\begin{aligned}\|A_{2,1}^n(s, t, x, y)\|_{L^p(\Omega)}^2 &\leq C \int_0^s \int_0^1 |\Delta_n(t-r, x, z) - \Delta_n(s-r, y, z)|^2 dz dr, \\ \|A_{2,2}^n(s, t, x)\|_{L^p(\Omega)}^2 &\leq C \int_s^t \int_0^1 |\Delta_n(t-r, x, z)|^2 dz dr.\end{aligned}$$

Hence, by (4.21) and (4.22) in Lemma 4.6, (4.28) holds for $i = 2$. $\square$

Using the above results, we obtain the convergence rate of $\|\bar{v}^n - \bar{v}\|_{T,\infty}$.

Proposition 4.9. *Under the assumptions of Theorem 2.2, for every $0 < \theta < \frac{1}{2}$ and every $p \geq 1$, there exists $C > 0$ such that*

$$\mathbb{E}\left[\|\bar{v}^n - \bar{v}\|_{T,\infty}^p\right] \leq C(n^{-\alpha p} + n^{-p\theta}); \quad (4.29)$$

recall that α is the index used in the assumption **(A.0)**.

Proof. By Jensen's inequality, it is sufficient to show (4.29) holds for $p > \frac{6}{1-2\theta}$. In the sequel, we assume this condition on p . Subtracting (4.7) from (4.1), we have

$$\bar{v}(t, x) - \bar{v}^n(t, x) = I_0^n(t, x) + I_1^n(t, x) + I_2^n(t, x),$$

where

$$\begin{aligned}I_0^n(t, x) &:= \int_0^1 \Delta_n(t, x, y) u(0, y) dy, \\ I_1^n(t, x) &:= \int_0^t \int_0^1 \Delta_n(t-s, x, y) f(s, y, u(s, y)) dy ds, \\ I_2^n(t, x) &:= \int_0^t \int_0^1 \Delta_n(t-s, x, y) \sigma(s, y, u(s, y)) W(ds, dy)\end{aligned}$$

with $\Delta_n(t, x, y) := G(t, x, y) - \bar{G}^n(t, x, y)$ as before. Then, by Theorem 2.4 and the linear growth condition **(A.2)**, we know that

$$\mathbb{E}[|f(s, y, u(s, y))|^p] + \mathbb{E}[|\sigma(s, y, u(s, y))|^p] \leq C(1 + \mathbb{E}[\|u\|_{T,\infty}^p]) < \infty.$$

Hence, Lemma 4.8 yields that for every $\theta \in (0, \frac{1}{2})$ and $p > \frac{6}{1-2\theta}$,

$$\mathbb{E}\left[\|I_i^n\|_{T,\infty}^p\right] \leq Cn^{-p\theta}, i = 1, 2.$$

So in order to complete the proof of this theorem, it suffices to show

$$\mathbb{E}\left[\|I_0^n\|_{T,\infty}^p\right] \leq Cn^{-\alpha p}. \quad (4.30)$$

By the definitions of $\mathcal{I}_n$ and $\bar{G}_n$, we see

$$I_0^n(t, x) = \mathcal{I}_n(S_t u_0)(x) - S_t u_0(x);$$

recall that $S_t u_0(x)$ is the semigroup defined by (2.15). As we pointed out in the proof of Theorem 2.5, we have

$$\|S_t u_0\|_{C^\alpha} \leq C \|u_0\|_{C^\alpha}, \quad 0 \leq t \leq T.$$

Applying Lemma 4.10 below to $S_t u_0$, we obtain

$$\sup_{0 \leq x \leq 1} |I_0^n(t, x)| \leq n^{-\alpha} \|S_t u_0\|_{C^\alpha([0,1])}.$$

Therefore

$$\sup_{0 \leq t \leq T, 0 \leq x \leq 1} |I_0^n(t, x)| \leq C n^{-\alpha} \|u_0\|_{C^\alpha}.$$

Hence, (4.30) is proved by the assumption **(A.0)** on the initial value u_0 and taking expectations. $\square$

Finally, let us state the lemma used in the above proof. For a function g defined on $[0, 1]$, let $\mathcal{I}_n g$ denote the piecewise linear interpolation on the mesh x_k . Let us first show a sharp estimate on the interpolation error of $\mathcal{I}_n g$.

Lemma 4.10. *For any $0 < \alpha \leq 1$ and $g \in C^\alpha([0, 1])$, we have*

$$\|\mathcal{I}_n g - g\|_{L^\infty(0,1)} \leq n^{-\alpha} \|g\|_{C^\alpha([0,1])}.$$

Proof. For each $x \in [x_k, x_{k+1}]$, let $\theta = nx - k$. Then $\theta \in [0, 1]$. By the definition of $\mathcal{I}_n$, one easily has that

$$\begin{aligned} |\mathcal{I}_n g(x) - g(x)| &\leq [g]_{C^\alpha} ((1-\theta)|x - x_k|^\alpha + \theta|x_{k+1} - x|^\alpha) \\ &\leq n^{-\alpha} \|g\|_{C^\alpha([0,1])} ((1-\theta)\theta^\alpha + \theta(1-\theta)^\alpha), \end{aligned}$$

which completes the proof. $\square$

4.2 Proof of the main result

In this section, we give the proof of Theorem 2.2. Recall that $\bar{v}^n(t, x)$ and $\bar{z}^n(t, x)$ are continuous random fields defined by (4.6) and (4.5) respectively, and

$$\bar{u}^n(t, x) := \bar{z}^n(t, x) + \bar{v}^n(t, x), \quad t \in [0, T], x \in [0, 1].$$

We first give the uniform moment bound of $\bar{u}^n$.

Lemma 4.11. *For any $p \geq 1$,*

$$\sup_{n \in \mathbb{N}} \mathbb{E}[\|\bar{u}^n\|_{T,\infty}^p] < \infty. \quad (4.31)$$

Proof. Recall that $\bar{z}^n(t) \in \mathbb{R}^{n-1}$ is defined via the random Skorohod-type system (4.4), and $\bar{v}^n(t, x)$ is the linear interpolation of the values $\bar{v}(t, x_k)$.

Since $\bar{u}^n(t, x)$ is piecewise linear in x , its supremum over $x \in [0, 1]$ is attained at the grid points. Hence, we have

$$\|\bar{u}^n\|_{T,\infty} = \sup_{0 \leq t \leq T, 0 \leq x \leq 1} |\bar{u}^n(t, x)| = \sup_{0 \leq t \leq T, 1 \leq k \leq n-1} |\bar{z}_k^n(t) + \bar{v}(t, x_k)|.$$

Applying Lemma 3.3 to the Skorohod-type system (4.4), we obtain

$$\sup_{0 \leq t \leq T, 1 \leq k \leq n-1} |\bar{z}_k^n(t) + \bar{v}(t, x_k)| \leq 2 \sup_{0 \leq t \leq T, 1 \leq k \leq n-1} |\bar{v}(t, x_k)|.$$

Therefore, we have

$$\|\bar{u}^n\|_{T,\infty} \leq 2 \sup_{0 \leq t \leq T, 1 \leq k \leq n-1} |\bar{v}(t, x_k)| = 2 \sup_{0 \leq t \leq T, 0 \leq x \leq 1} |\bar{v}^n(t, x)| \leq 2\|\bar{v}\|_{T,\infty}. \quad (4.32)$$

By Theorem 2.4 and the relation between $u(t, x)$ and $\bar{v}(t, x)$, we know that $\mathbb{E}[\|\bar{v}\|_{T,\infty}^p] < \infty$. Consequently, we obtain (4.31) easily by (4.32). $\square$

Let us now give the proof of Theorem 2.2.

Proof of Theorem 2.2. By Jensen's inequality, it is enough to prove (2.7) holds for large enough p . In the sequel, for each $0 < \kappa < \gamma$, we assume $p > \frac{3}{\gamma-\kappa}$. Set

$$\begin{aligned} \hat{v}^n(t, x) &= \int_0^1 G^n(t, x, y) u(0, k_n(y)) dy \\ &\quad + \int_0^t \int_0^1 G^n(t-s, x, y) f(s, k_n(y), \bar{u}^n(s, k_n(y))) dy ds \\ &\quad + \int_0^t \int_0^1 G^n(t-s, x, y) \sigma(s, k_n(y), \bar{u}^n(s, k_n(y))) W(ds, dy). \end{aligned}$$

Then, according to the proof of Theorem 2.1 in [39], especially (5.37) [39], we know

$$\begin{aligned} \mathbb{E}[\|u - u^n\|_{T,\infty}^p] &\leq C(\mathbb{E}[\|u - \bar{u}^n\|_{T,\infty}^p] + \mathbb{E}[\|u^n - \bar{u}^n\|_{T,\infty}^p]) \\ &\leq C(\mathbb{E}[\|u - \bar{u}^n\|_{T,\infty}^p] + \mathbb{E}[\|\hat{v}^n - \bar{v}^n\|_{T,\infty}^p]). \end{aligned} \quad (4.33)$$

We first note that, by Theorem 4.1,

$$\mathbb{E}[\|u - \bar{u}^n\|_{T,\infty}^p] \leq C(\log n)^{-\frac{\kappa p}{2}}. \quad (4.34)$$

So in order to complete the proof of this theorem, the main task to establish the convergence rate of the second term in (4.33).

Recalling that $\bar{v}^n(t, x)$ satisfies (4.7) and setting

$$\Delta_n^*(t, x, y) := G^n(t, x, y) - \bar{G}^n(t, x, y),$$

as in the proof of Theorem 2.1 in [39], we write

$$\hat{v}^n(t, x) - \bar{v}^n(t, x) = B_1^n(t, x) + B_2^n(t, x) + B_3^n(t, x) + B_4^n(t, x), \quad (4.35)$$

where

$$\begin{aligned} B_1^n(t, x) &:= \int_0^t \int_0^1 \Delta_n^*(t-s, x, y) f(s, k_n(y), \bar{u}^n(s, k_n(y))) dy ds, \\ B_2^n(t, x) &:= \int_0^t \int_0^1 \Delta_n^*(t-s, x, y) \sigma(s, k_n(y), \bar{u}^n(s, k_n(y))) W(ds, dy), \\ B_3^n(t, x) &:= \int_0^t \int_0^1 \bar{G}^n(t-s, x, y) \left[f(s, k_n(y), \bar{u}^n(s, k_n(y))) - f(s, y, u(s, y)) \right] dy ds, \\ B_4^n(t, x) &:= \int_0^t \int_0^1 \bar{G}^n(t-s, x, y) \left[\sigma(s, k_n(y), \bar{u}^n(s, k_n(y))) - \sigma(s, y, u(s, y)) \right] W(ds, dy). \end{aligned}$$

From now on, we aim to give the convergence rate of $\mathbb{E}[\|B_i^n\|_{T,\infty}^p], i = 1, 2, 3, 4$. First, we claim that there exists a constant $C > 0$ such that for all $n \in \mathbb{N}$

$$\mathbb{E}[\|B_1^n\|_{T,\infty}^p] \leq C n^{-p/2}. \quad (4.36)$$

Indeed, by the Cauchy-Schwarz inequality, we have that for each $(t, x) \in \mathcal{Q}_T$,

$$\begin{aligned} & |B_1^n(t, x)|^p \\ & \leq \left(\int_0^t \int_0^1 |\Delta_n^*(t-s, x, y)|^2 dy ds \right)^{p/2} \left(\int_0^t \int_0^1 |f(s, k_n(y), \bar{u}^n(s, k_n(y)))|^2 dy ds \right)^{p/2}. \end{aligned} \quad (4.37)$$

Taking the supremum over $(t, x) \in \mathcal{Q}_T$, and then expectation, we obtain

$$\begin{aligned} & \mathbb{E}[\|B_1^n\|_{T,\infty}^p] \\ & \leq \sup_{0 \leq x \leq 1} \left(\int_0^T \int_0^1 |\Delta_n^*(t-s, x, y)|^2 dy ds \right)^{p/2} \mathbb{E} \left[\left(\int_0^T \int_0^1 |f(s, k_n(y), \bar{u}^n(s, k_n(y)))|^2 dy ds \right)^{p/2} \right] \\ & \leq C n^{-p/2} \left(1 + \mathbb{E} \left[\sup_{t \in [0, T], x \in [0, 1]} |\bar{u}^n(t, k_n(x))|^p \right] \right), \end{aligned}$$

where Lemma 4.6, Remark 4.7-(1), and the linear growth condition **(A.2)** on f have been used for the last inequality. Now we conclude the proof of (4.36) by Lemma 4.11.

In view of the linear growth **(A.2)** on σ and Lemma 4.11, we have

$$\sup_{n \in \mathbb{N}} \sup_{(s, y) \in \mathcal{Q}_T} \mathbb{E}[|\sigma(s, k_n(y), \bar{u}^n(s, k_n(y)))|^p] < \infty.$$

On the other hand, by [39, (5.44)], we know that there exists a constant $C > 0$ depending on $\sup_n \mathbb{E}[\|\bar{u}^n\|_{T,\infty}^p] < \infty$ and $\mathbb{E}[\|u\|_{T,\infty}^p]$ such that

$$E[|B_2^n(t, x) - B_2^n(s, y)|^p] \leq C(|t-s|^{\frac{1}{2}} + |x-y|)^{p/2}$$

for all $s, t \in [0, T], x, y \in [0, 1]$. Hence, Lemma 4.8 yields that for any $0 < \theta < \frac{1}{2}$, there exists a constant $C > 0$ such that

$$E[\|B_2^n\|_{T,\infty}^p] \leq C n^{-p\theta}. \quad (4.38)$$

Let us continue to estimate B_3^n and B_4^n . We claim that

$$\mathbb{E}[\|B_3^n\|_{T,\infty}^p] + \mathbb{E}[\|B_4^n\|_{T,\infty}^p] \leq C(\log n)^{-\kappa p/2}. \quad (4.39)$$

Here we only give details for the proof about B_4^n and a similar argument yields the desired estimate on B_3^n . By [39, (5.46)], it follows that there exists a constant $C > 0$ such that for all $(t, x) \in \mathcal{Q}_T$

$$\mathbb{E}[|B_4^n(t, x)|^p] \leq C \left(n^{-p} + \mathbb{E}[\|\bar{u}^n - u\|_{T,\infty}^p] + \mathbb{E} \left[\sup_{0 \leq t \leq T, 0 \leq x \leq 1} |u(t, k_n(x)) - u(t, x)|^p \right] \right).$$

Then, by Theorem 4.1 and (2.10) in Theorem 2.5, we have

$$\begin{aligned} \sup_{t \in [0, T], x \in [0, 1]} \mathbb{E}[|B_4^n(t, x)|^p] & \leq C \left(n^{-p} + (\log n)^{-\frac{\kappa p}{2}} + n^{-p\bar{\alpha}} \right) \\ & \leq C(\log n)^{-\frac{\kappa p}{2}}. \end{aligned} \quad (4.40)$$

Now let

$$\bar{\sigma}_n(s, y) := \sigma(s, k_n(y), \bar{u}^n(s, k_n(y))) - \sigma(s, y, u(s, y)), \quad (s, y) \in \mathcal{Q}_T.$$

By the Lipschitz continuity assumption **(A.1)** on σ , we have

$$\begin{aligned} |\bar{\sigma}_n(s, y)| &\leq C \left(|k_n(y) - y| + |\bar{u}^n(s, k_n(y)) - u(s, y)| \right) \\ &\leq C \left(\frac{1}{n} + |\bar{u}^n(s, k_n(y)) - u(s, k_n(y))| + |u(s, k_n(y)) - u(s, y)| \right). \end{aligned}$$

Then, a similar argument to (4.40) gives that

$$\sup_{s \in [0, T], y \in [0, 1]} \mathbb{E}[|\bar{\sigma}_n(s, y)|^p] \leq C(\log n)^{-\frac{\kappa p}{2}}. \quad (4.41)$$

We next consider the increment estimate. Let $0 \leq s \leq t \leq T$ and $x, y \in [0, 1]$. Write

$$B_4^n(t, x) - B_4^n(s, y) = B_{4,1}^n(s, t, x, y) + B_{4,2}^n(s, t, x),$$

where

$$B_{4,1}^n(s, t, x, y) := \int_0^s \int_0^1 (\bar{G}^n(t-r, x, z) - \bar{G}^n(s-r, y, z)) \bar{\sigma}_n(r, z) W(dr, dz)$$

and

$$B_{4,2}^n(s, t, x) := \int_s^t \int_0^1 \bar{G}^n(t-r, x, z) \bar{\sigma}_n(r, z) W(dr, dz).$$

By the Burkholder-Davis-Gundy inequality and the integrated Minkowski inequality,

$$\mathbb{E}[|B_{4,1}^n|^p] \leq C \left(\int_0^s \int_0^1 |\bar{G}^n(t-r, x, z) - \bar{G}^n(s-r, y, z)|^2 \|\bar{\sigma}_n(r, z)\|_{L^p(\Omega)}^2 dz dr \right)^{p/2}.$$

Using (4.41) and Remark 4.7-(1), we have

$$\begin{aligned} \mathbb{E}[|B_{4,1}^n|^p] &\leq C(\log n)^{-\kappa p/2} \left(\int_0^s \int_0^1 |\bar{G}^n(t-r, x, z) - \bar{G}^n(s-r, y, z)|^2 dz dr \right)^{p/2} \\ &\leq C(\log n)^{-\kappa p/2} (|t-s|^{1/2} + |x-y|)^{p/2}. \end{aligned} \quad (4.42)$$

Similarly, we have

$$\mathbb{E}[|B_{4,2}^n|^p] \leq C(\log n)^{-\kappa p/2} |t-s|^{p/4}. \quad (4.43)$$

Combining (4.42) and (4.43), we obtain

$$\mathbb{E}[|B_4^n(t, x) - B_4^n(s, y)|^p] \leq C(\log n)^{-\kappa p/2} (|t-s|^{1/2} + |x-y|)^{p/2}. \quad (4.44)$$

Using Remark 4.5 together with (4.40) and (4.44), we obtain the desired estimate (4.39) for B_4^n . Consequently, combining (4.36), (4.38) and (4.39), we have

$$\mathbb{E}[\|\hat{v}^n - \bar{v}^n\|_{T, \infty}^p] \leq C(\log n)^{-\frac{\kappa p}{2}}.$$

Substituting this and (4.34) into (4.33), we obtain

$$\mathbb{E}[\|u - u^n\|_{T, \infty}^p] \leq C(\log n)^{-\frac{\kappa p}{2}}.$$

This completes the proof of Theorem 2.2. $\square$

Remark 4.12. (i) In [39], it is proved that

$$\lim_{n \rightarrow \infty} \mathbb{E} \left[\|B_i^n\|_{T,\infty}^{2p} \right] = 0, \quad i = 1, 2, 3, 4.$$

From the proof of Theorem 2.2, we know that the convergence rates of $\mathbb{E}[\|B_i^n\|_{T,\infty}^{2p}]$ are established respectively.

(ii) Let u^λ be the solution of the corresponding penalized SPDE for (1.1), see [10] for example. Then, using Theorem 3.7 together with the standard construction of solutions to the reflected SPDE (1.1), we can obtain that for every $0 < \kappa < \frac{1}{2} \wedge \alpha$ and $p \geq 1$, there exists a constant $C > 0$, independent of λ , such that

$$\mathbb{E} \left[\|u^\lambda - u\|_{T,\infty}^p \right] \leq C \lambda^{-\kappa p/2}, \quad \lambda \geq 1.$$

Since its proof is simpler than the proof of Theorem 2.2, we omit the details. This gives the explicit polynomial convergence rate of the penalized equation to the reflected SPDE. As far as we know, this is a new result.

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