Frobenius adjacency algebras of association schemes

Akihide Hanaki
Faculty of Science, Shinshu University,
Matsumoto, 390-8621, Japan
hanaki@math.shinshu-u.ac.jp

Abstract

The concept of adjacency algebras of association schemes is a generalization of group algebras of finite groups. But adjacency algebras over a positive characteristic field need not be Frobenius algebras. In this article, we characterize association schemes whose adjacency algebras are Frobenius or symmetric algebras. But our characterization is very elementary and it is difficult to use it. So we consider a special case. If an adjacency algebra is a local algebra, then it is a Frobenius algebra if and only if the scheme is p'-valenced. We also give some examples.

Keywords : association scheme, adjacency algebra, Frobenius algebra, symmetric algebra

1 Introduction

In [3, p.303], Bannai and Ito stated the following problem.

Problem. Determine, hopefully by the parameters, association schemes and fields for which the adjacency algebras are semi-simple, symmetric, Frobenius and quasi-Frobenius.

An answer to the problem for the case "semi-simple" was obtained in [7]. In this article, we give an answer to the problem for the cases "symmetric" and "Frobenius". But the answer is easy corollary to well-known facts in the theory of finite dimensional algebras. So we also consider a special case. Suppose that an adjacency algebra over a field of characteristic $p > 0$ is a local algebra, then it is a Frobenius algebra if and only if the scheme is p'-valenced. Also the algebra is a symmetric algebra in this case. Especially, the order of the scheme is a p-power, then the adjacency algebra is local and we can apply this result. Using this criteria, we will give some examples.

2 Preliminaries

Let K be a field, A a finite dimensional K-algebra. For a finite dimensional K-space V, the *dual module* of V is $\widehat{V} = \mathrm{Hom}_K(V, K)$. If V is a right A-module, then $\widehat{V}$ is a left A-module by

$$(a\varphi)(v) = \varphi(va), \quad a \in A,\ v \in V,\ \varphi \in \widehat{V}.$$

Similarly, if V is a left A-module, then $\widehat{V}$ is a right A-module.

Suppose A has a K-antiautomorphism $a \mapsto a^*$. Then the *contragredient module* $\widetilde{V}$ of a right A-module V is a right A-module defined by the following way. Put $\widetilde{V} = \widehat{V}$ as a K-space. Define the right action of A by

$$(\varphi a)(v) = \varphi(va^*), \quad a \in A,\ v \in V,\ \varphi \in \widetilde{V}.$$

Similarly the contragredient module of a left module is a left module.

A K-algebra A is called a *Frobenius algebra* if the dual of the left regular module ${}_AA$ is isomorphic to the right regular module A_A : $\widehat{{}_AA} \cong A_A$. A K-algebra A is called a *symmetric algebra* if the dual of the regular (A, A)-bimodule ${}_AA_A$ is isomorphic to the regular (A, A)-bimodule ${}_AA_A$: $\widehat{{}_AA_A} \cong {}_AA_A$.

A K-form $\varphi : A \to K$ is called *symmetric* if $\varphi(ab) = \varphi(ba)$ for $a, b \in A$, *non-degenerate* if $\varphi(aA) = 0$ implies $a = 0$. It is well known that A is a Frobenius algebra if and only if it has a non-degenerate K-form, and A is a symmetric algebra if and only if it has a non-degenerate symmetric K-form.

A K-bilinear form $\mu : A \times A \to K$ is called *associative* if $\mu(ab, c) = \mu(a, bc)$ for $a, b, c \in A$. For a K-form $\varphi : A \to K$, we can define an associative K-bilinear form μ by $\mu(a, b) = \varphi(ab)$. Conversely an associative K-bilinear form induces a K-form. A K-bilinear form μ is called *symmetric* if $\mu(a, b) = \mu(b, a)$ for $a, b \in A$. A symmetric associative K-bilinear corresponds to a symmetric K-form, and a non-degenerate associative K-bilinear form corresponds to a non-degenerate K-form.

For association schemes, we use Zieschang's notations in [11]. An association scheme will be denoted by (X, S). The order of (X, S) means the cardinality of X. We denote σ_g for the adjacency matrix of $g \in S$. We consider σ_g as a matrix over a suitable coefficient ring. We denote p_{fg}^h for the structure constant (intersection number), namely $\sigma_f\sigma_g = \sum_{h\in S} p_{fg}^h \sigma_h$. The structure constants are considered as rational integers. The valency of $g \in S$ will be denoted by n_g, that is $n_g = p_{gg^*}^1$. The adjacency algebra $\bigoplus_{s\in G} K\sigma_g$ over a field K will be denoted by KS. The map $\sigma_g \mapsto \sigma_{g^*}$ is an antiautomorphism of KS. So we can consider contragredient modules of KS-modules.

3 A condition for an adjacency algebra to be a Frobenius algebra

Let K be a field. The adjacency algebra KS has a natural basis $\{\sigma_g \mid g \in S\}$. Also the dual $\widehat{KS}$ has a basis $\{\tau_g \mid g \in S\}$, where $\tau_g(\sigma_h) = \delta_{gh}$.

Let $\sum_{g\in S} a_g\tau_g$ ($a_g \in K$) be a K-form, and let μ be the corresponding K-bilinear form. We have

$$\mu(\sigma_f, \sigma_h) = \sum_{g\in S} a_g\tau_g\left(\sum_{\ell\in S} p_{fh}^{\ell}\sigma_\ell\right) = \sum_{g\in S} a_g p_{fh}^{g}.$$

This means that the matrix of μ with respect to the basis $\{\sigma_g\}$ is given by $\left(\sum_{g\in S} a_g p_{fh}^g\right)_{fh}$. Now the following holds.

Theorem 3.1. (1) *The adjacency algebra KS is a Frobenius algebra if and only if there exist $a_g \in K$ ($g \in S$) such that the matrix $\left(\sum_{g\in S} a_g p_{fh}^g\right)_{fh}$ is non-singular.*

(2) *The adjacency algebra KS is a symmetric algebra if and only if there exist $a_g \in K$ ($g \in S$) such that the matrix $\left(\sum_{g\in S} a_g p_{fh}^g\right)_{fh}$ is non-singular and symmetric.*

For a prime number p, an association scheme (X, S) is called *p'-valenced* if $p \nmid n_g$ for every $g \in S$. The following result is easy and has already shown in [8].

Corollary 3.2. *If (X, S) is p'-valenced, then KS is a symmetric algebra.*

Proof. Put $a_g = \delta_{1g}$, namely $c = \tau_1$, and apply Theorem 3.1. □

Put $P_g = \left(p_{fh}^g\right)_{fh}$ for $g \in S$. Let X_g ($g \in S$) be indeterminates. We consider the polynomial $F_S(\mathbf{X}) := \det\left(\sum_{g\in S} X_g P_g\right) \in \mathbb{Z}[\mathbf{X}]$. If K is an infinite field, then a non-zero polynomial defines a non-zero function. So we have the following.

Theorem 3.3. *Let $F_S(\mathbf{X})$ be as above, and let K be an infinite field of characteristic p. Then KS is a Frobenius algebra if and only if $F_S(\mathbf{X}) \notin p\,\mathbb{Z}[\mathbf{X}]$.*

Example 3.4. Let (X, S) be a scheme of class 1. Put $S = \{1, g\}$ and let k be the valency of g. Then

$$P_1 = \begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix}, \quad P_g = \begin{pmatrix} 0 & 1 \\ 1 & k-1 \end{pmatrix}.$$

So

$$F_S(\mathbf{X}) = \begin{vmatrix} X_1 & X_g \\ X_g & kX_1 + (k-1)X_g \end{vmatrix} = kX_1^2 + (k-1)X_1X_g - X_g^2.$$

So the algebra KS is a Frobenius (symmetric) algebra for an arbitrary field K.

Example 3.5. Let (X, S) be the group association scheme [2, Example II.2.1 (2)] of the symmetric group of degree 3. Put $S = \{1, f, g\}$. Then

$$P_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}, \quad P_f = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}, \quad P_g = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 2 \\ 1 & 2 & 0 \end{pmatrix}.$$

Now

$$F_S(\mathbf{X}) = 3(2X_1^3 - X_f^3 + 3X_1^2X_f - 2X_1X_g^2 + X_fX_g^2).$$

So KS is not Frobenius if $\mathrm{char}K = 3$. If $\mathrm{char}K \neq 3$, then KS is a Frobenius algebra. Note that this scheme is not $2'$-valenced.

In general, it is difficult to compute $F_S(\mathbf{X})$. But, if a prime number p does not divide $\det(P_g)$ for some $g \in S$, then KS is a Frobenius algebra for a suitable large field K of characteristic p.

Example 3.6. Let (X, S) be the Schurian scheme [2, Example II.2.1 (1)] defined by the alternating group of degree 5 and its Sylow 2-subgroup. Then

$$P_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 \end{pmatrix}, \quad P_f = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 4 & 0 & 0 \end{pmatrix},$$

$$P_g = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 \end{pmatrix}, \quad P_h = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 \end{pmatrix},$$

$$P_i = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix}, \quad P_j = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 1 \end{pmatrix},$$

and $\det(P_h) = 1$. So KS is a Frobenius algebra for an arbitrary field K. Also KS is symmetric if $\mathrm{char}K = p \neq 2$ since S is p'-valenced. But we can check that KS is not a symmetric algebra if $\mathrm{char}K = 2$.

If an adjacency algebra is a Frobenius algebra, then we can obtain orthogonality relations [6] for its characters.

4 A filtration of adjacency algebras

Let (X, S) be an association scheme, p a prime number, and K a field of characteristic $p > 0$. For a non-negative integer ℓ, put

$$I_\ell = \bigoplus_{p^\ell \mid n_g} K\sigma_g.$$

Lemma 4.1. *For a non-negative integer ℓ, I_ℓ is an ideal of KS.*

Proof. We have $n_h p_{fg}^{h^*} = n_f p_{gh}^{f^*}$ [11, Lemma 1.1.3 (iii)]. So if $p^\ell \mid n_f$ and $p^\ell \nmid n_h$, then $p_{fg}^{h^*} \equiv 0 \pmod p$. □

Lemma 4.2. *As (KS, KS)-bimodules, $I_\ell/I_{\ell+1} \cong \widehat{I_\ell/I_{\ell+1}} \cong \widetilde{I_\ell/I_{\ell+1}}$. Namely $I_\ell/I_{\ell+1}$ is a self-dual and self-contragredient (KS, KS)-bimodule.*

Proof. Put $S_\ell = \{g \in S \mid p^\ell \mid n_g, p^{\ell+1} \nmid n_g\}$. For $g \in S_\ell$, we write $\overline{\sigma_g}$ for $\sigma_g + I_\ell \in I_\ell/I_{\ell+1}$. Then $\{\overline{\sigma_g} \mid g \in S_\ell\}$ is a K-basis of $I_\ell/I_{\ell+1}$. Also we define $\overline{\tau_g} \in \widehat{I_\ell/I_{\ell+1}}$ for $g \in S_\ell$ by $\overline{\tau_g}(\overline{\sigma_f}) = \delta_{fg}$. Then $\{\overline{\tau_g} \mid g \in S_\ell\}$ is a K-basis of $\widehat{I_\ell/I_{\ell+1}}$.

For $g \in S_\ell$, we put $n'_g = n_g/p^\ell$. We will show that the map $\overline{\sigma_g} \mapsto n'_g\overline{\tau_{g^*}}$ gives an isomorphism as (KS, KS)-bimodules. It is clear that this is an isomorphism as K-vector spaces, since $n'_g \neq 0$ in K.

For $g \in S_\ell$ and $f \in S$, we have

$$\overline{\sigma_g}\sigma_f = \sum_{h \in S_\ell} p_{gf}^h \overline{\sigma_h}.$$

Also suppose that $(n'_g\overline{\tau_{g^*}})\sigma_f = \sum_{h \in S_\ell} a_h(n'_h\overline{\tau_{h^*}})$, then

$$\begin{aligned} a_h &= \frac{1}{n'_h}((n'_g\overline{\tau_{g^*}})\sigma_f)(\overline{\sigma_{h^*}}) = \frac{n'_g}{n'_h}\overline{\tau_{g^*}}(\sigma_f\overline{\sigma_{h^*}}) \\ &= \sum_{t \in S_\ell} \frac{n'_g}{n'_h} p_{fh^*}^t \overline{\tau_{g^*}}(\overline{\sigma_t}) = \frac{n'_g}{n'_h} p_{fh^*}^{g^*} = p_{gf}^h \end{aligned}$$

Similar equation holds for the left action. So this gives an isomorphism.

Similarly the map $\overline{\sigma_g} \mapsto n'_g\overline{\tau_g}$ gives an isomorphism $I_\ell/I_{\ell+1} \cong \widetilde{I_\ell/I_{\ell+1}}$ □

Now the following theorem is clear.

Theorem 4.3. *Suppose $I_{r+1} = 0$. If $KS \cong I_0/I_1 \oplus \cdots \oplus I_{r-1}/I_r \oplus I_r$ as an (KS, KS)-bimodule, then KS is a symmetric algebra.*

Corollary 4.4. *If there exists a positive integer ℓ such that $p^\ell \mid n_g$ and $p^{\ell+1} \nmid n_g$ for any $1 \neq g \in S$, then KS is a symmetric algebra.*

Proof. Put $J = \sum_{g \in S} \sigma_g$. Then, by our assumption, we have $KS = KJ \oplus I_\ell \cong I_0/I_1 \oplus I_\ell/I_{\ell+1}$. □

In [1], it is shown that, if the condition in Corollary 4.4 holds for $p = 2$, then KS is semisimple.

5 Local Frobenius adjacency algebras

Let (X,S) be an association scheme, p a prime number, and K a field of characteristic $p>0$. Put $J=\sum_{g\in S}\sigma_g\in KS$. Then KJ is a one-dimensional two-sided ideal of KS. Let $B_0(KS)$ be the *principal block* of KS. Namely $B_0(KS)$ is an indecomposable direct summand of KS as a two-sided ideal and $B_0(KS)J\neq 0$.

Suppose the principal block $B_0(KS)$ is a local Frobenius algebra. Then $B_0(KS)$ has the unique maximal submodule, and so its dual has the unique minimal submodule. Since $B_0(KS)$ is a Frobenius algebra, it has the unique minimal submodule and it must be KJ. Put $I_1=\sum_{p|n_g}K\sigma_g$ as in the previous section. Clearly $KJ\cap I_1=0$, so the inclusion $B_0(KS)\to KS$ induces a monomorphism $B_0(KS)\to KS/I_1$. This means the following.

Proposition 5.1. *Suppose the principal block $B_0(KS)$ is a local Frobenius algebra. Then* $\dim_K(B_0(KS))\leq \sharp\{g\in S\mid p\nmid n_g\}$.

Corollary 5.2. *Suppose KS is a local algebra. Then the following statements are equivalent.*

(1) *S is p'-valenced.*

(2) *KS is a Frobenius algebra.*

(3) *KS is a symmetric algebra.*

Proof. By Corollary 3.2, (1) implies (3), and clearly (3) implies (2). Suppose (2). Then we can apply Proposition 5.1 and we have

$$|S|=\dim_K KS=\dim_K B_0(KS)\leq \sharp\{g\in S\mid p\nmid n_g\}.$$

This means that KS is p'-valenced. □

In general, it is not so easy to determine whether KS is local. But if $|X|$ is a p-power, then KS is local [8]. So we can apply Corollary 5.2 for schemes of prime power order.

6 Examples

In this section, let K be a field of characteristic $p>0$.

6.1 Hamming schemes

We consider the Hamming scheme $H(n,q)$. We denote $KH(n,q)$ for the adjacency algebra of $H(n,q)$ over K. If $p\nmid q$, then $KH(n,q)$ is semisimple. If $p\mid q$, then $KH(n,q)\cong KH(n,p)$ and $KH(n,q)$ is a local algebra. For details, see [10].

We determine when $KH(n,q)$ is a symmetric algebra. By Corollary 5.2, it is enough to see the valencies. The i-th valency of $H(n,q)$ is given by $\binom{n}{i}(q-1)^i$. By Lucas' theorem [5, Theorem 3.4.1], the following holds.

Lemma 6.1. *Suppose $a \leq n$ and put*

$$n = \sum_{i=0}^{\ell} n_i p^i, \quad a = \sum_{i=0}^{\ell} a_i p^i, \quad 0 \leq n_i,\ a_i < p.$$

Then $\binom{n}{a}$ is prime to p if and only if $a_i \leq n_i$ for all $0 \leq i \leq \ell$. Especially $\binom{n}{a}$ is prime to p for all $0 \leq a \leq n$ if and only if $n_i = p-1$ all $0 \leq i \leq \ell - 1$, namely $n = kp^{\ell} - 1$ for some $1 \leq k < p$.

Proposition 6.2. *Let K be a field of characteristic $p > 0$. Suppose $p \mid q$. Then the adjacency algebra of the Hamming scheme $H(n, q)$ over K is a symmetric algebra if and only if $n = kp^{\ell} - 1$ for some $1 \leq k < p$ and some non-negative integer ℓ.*

6.2 Some distance-regular graphs on vector spaces

We consider some distance-regular graphs on vector spaces over finite fields : bilinear form graphs [4, p.280], alternating form graphs [4, p.282], Hermitian form graphs [4, p.285], and quadratic form graphs [4, p.290]. Suppose the characteristic of the base field of the graph is p. Then easily we can check that the second valency k_2 of the graph is a multiple of p. So the adjacency algebra of the graph over K never be a Frobenius algebra, except for the case of diameter one.

6.3 Schurian schemes of prime power order

Let G be a finite group, and H a subgroup of G. Then we can define a Schurian scheme (X, S) [2, Example II.2.1 (1)]. Suppose $|X| = |G : H|$ is a p-power. Then the adjacency algebra KS is local. In this case, KS is a symmetric algebra if and only if p does not divide $|H : H \cap H^g|$ for every $g \in G$.

6.4 Group association schemes with local adjacency algebras

Let G be a finite group, and (X, S) its group association scheme [2, Example II.2.1 (2)]. Then the adjacency algebra KS is local if and only if the principal block is the unique block of the group algebra KG. Suppose G has a normal p-subgroup Q such that $C_G(Q)$ is a p-group. Then the adjacency algebra KS over a field K of characteristic p is local [9, Exercise 5.2.10]. In this case, KS is a symmetric algebra if and only if G is an abelian p-group.

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