SELF DUAL GROUPS AND FINITE SYMMETRIC ALGEBRAS OF LOEWY LENGTH 4

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Abstract. A finite group G is said to be self dual if its group association scheme is self dual. Some examples of self dual groups are known but no general method to construct self dual groups is known. In this paper, we construct self dual groups from symmetric algebras over prime fields whose Loewy lengths are 4.

Let G be a finite group, $\mathrm{Cl}(G) = \{C_1, \cdots, C_k\}$ and $\mathrm{Irr}(G) = \{\chi_1, \cdots, \chi_k\}$ be the complete sets of conjugacy classes and irreducible characters of G, respectively. Let $\{x_i\}$ be a representatives of conjugacy classes. We say G is *self dual* if, after renumbering indices,

$$\frac{|C_j|\chi_i(x_j)}{\chi_i(1)} = \chi_j(1)\chi_j(x_i)$$

for all i, j. This is equivalent to that the group association scheme of G is self dual. Clearly abelian groups are self dual, and some non-abelian self dual groups were constructed in [1], [2], and [5].

For the structure of self dual groups, the next result is known.

Theorem 1 (T.Okuyama [4]). *Self dual groups are nilpotent.*

T.Okuyama suggested that self dual groups are related to some finite algebras. In this point of view, we shall try to construct self dual groups from finite symmetric algebras.

In this paper, we use the next notations. Let F be a prime field $\mathrm{GF}(p)$, and let A be a finite dimensional symmetric algebra over F. $\mathrm{J}(A)$ and $\mathrm{Soc}(A)$ denote the Jacobson radical and the socle of A, respectively. We consider finite groups $G = 1+\mathrm{J}(A)$ and $\overline{G} = G/(1+\mathrm{Soc}(A))$. For $g \in G$, $\overline{g}$ means $g(1+\mathrm{Soc}(A))$, and for $v \in \mathrm{J}(A)$, $\overline{v}$ means $v+\mathrm{Soc}(A)$. Since A is symmetric, there exists a non-degenetate associative symmetric bilinear form $f : A \times A \to F$, and define $\lambda : A \to F$ by $f(a, b) = \lambda(ab)$ then $\ker\lambda$ contains no non-zero right (left) ideal.

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For $u \in \mathrm{J}(A)$ such that $u^p = 0$, we define

$$\ln(1+u) = \sum_{n=1}^{p-1} -\frac{(-u)^n}{n},$$
$$\exp(u) = \sum_{n=0}^{p-1} \frac{u^n}{n!}.$$

These appear in [3]. The next lemma seems to be well known.

Lemma 2. *The followings hold* :

1. $\exp(\ln(1+u)) = 1+u$,
2. $\ln(\exp(u)) = u$,
3. *if* $uv = vu$ *then* $\ln((1+u)(1+v)) = \ln(1+u) + \ln(1+v)$.

Throughout the rest of this paper, we assume that the exponent of $\overline{G}$ is equal to p (then p must be odd if $\overline{G}$ is non-abelian). We fix a primitive p-th root of unity ω in the complex number field. For v, $w \in J(A)$, we define

$$\Phi_{1+u}(1+v) = \frac{1}{|\,\mathrm{C}_{\overline{G}}(\overline{u})|} \sum_{g \in \overline{G}} \omega^{f(\ln(1+u)^g, \ln(1+v))},$$

where $g \in \overline{G}$ can act on G since $1 + \mathrm{Soc}(G)$ is in the center of G. Note that under our assumption u^p may be non-zero, but we can ignore the term of u^p since u^p is in $\mathrm{Soc}(A)$. This is a class function of G, and we can see it is a class function of $\overline{G}$.

The next is our main result.

Theorem 3. *If* $\mathrm{J}^4(A) = 0$ *and the exponent of* $\overline{G}$ *is* p *then* $\overline{G}$ *is self dual.*

I think that the assumption $\mathrm{J}^4(A) = 0$ is not essential, but the assumption about the exponent is essential in a sense.

To show this, we shall show that $\Phi_{1+u} = \chi_{1+u}(1)\chi_{1+u}$ for some $\chi_{1+u} \in \mathrm{Irr}(G)$, and that

$$\frac{\left|\overline{G} : \mathrm{C}_{\overline{G}}(\overline{1+v})\right|}{\Phi_{1+u}(1)} \Phi_{1+u}(1+v) = \Phi_{1+v}(1+u).$$

The second equation holds immediately since

$$f(\ln(1+u)^g, \ln(1+v)) = f(\ln(1+v)^{g^{-1}}, \ln(1+u)).$$

Lemma 4. *If* $\mathrm{J}^4(A) = 0$ *then*

$$\begin{aligned} &f(\ln(1+u), \ln((1+v)(1+w))) \\ =&f(\ln(1+u), \ln(1+v)) + f(\ln(1+u), \ln(1+w)) \\ &\quad + f(\ln(1+u), (vw - wv)/2) \end{aligned}$$

for all $u,\ v,\ w \in \mathrm{J}(A)$.

Proof. By direct calculation, we have

$$\ln((1+v)(1+w)) = \ln(1+v) + \ln(1+w) + \frac{1}{2}(vw - wv)$$
$$+ \frac{1}{6}\{(wv^2 - v^2w) + (wv^2 - vwv) + (w^2v - wvw) + (w^2v - vw^2)\}.$$

Since $\ln(1+u) \in \mathrm{J}(A)$ and $\mathrm{J}^4(A) = 0$, we have the result. □

Lemma 5. *For* $u \in \mathrm{J}(A)$, $\Phi_{1+u} = \chi_{1+u}(1)\chi_{1+u}$ *for some* $\chi_{1+u} \in \mathrm{Irr}(G)$.

Proof. If $u \in \mathrm{J}^2(A)$, the result holds immediately by Lemma 2 (3). Put $\overline{H} = \mathrm{C}_{\overline{G}}(\overline{u})$, and put $\varphi(1+v) = \omega^{f(\ln(1+u),\ln(1+v))}$ for $1+v \in H$. Then, for $1+v,\ 1+w \in H$, $f(\ln(1+u),(vw-wv)/2) = 0$ since f is symmetric and $\ln(1+u)$ and v commute. Thus φ is a linear character of $\overline{H}$ by Lemma 4. We shall show that $\varphi^{\overline{G}} = \Phi_{1+v}$.

Clearly $\Phi_{1+u} = \varphi^{\overline{G}}$ on $\overline{H}$. $\overline{H}$ is normal in $\overline{G}$ since H contains $1 + J^2(A)$, and thus $\varphi^{\overline{G}}(1+v) = 0$ for $1+v \notin H$. We shall show $\Phi_{1+u}(1+v) = 0$ for $1+v \notin H$. Put $a = \ln(1+u)$ and $b = \ln(1+v)$. We have

$$\begin{aligned}\sum_{g\in\overline{G}} \omega^{f(a^g,b)} &= \omega^{f(a,b)} \sum_{g_0\in \mathrm{J}(A)/\operatorname{Soc}(A)} \omega^{f(ag_0 - g_0a,b)} \\ &= \omega^{f(a,b)} \sum_{g_0\in \mathrm{J}(A)/\operatorname{Soc}(A)} \omega^{f(ba-ab,g_0)}.\end{aligned}$$

The map $g_0 \mapsto f(ba-ab,g_0)$ is F-linear and non-zero since $ba-ab \in \operatorname{Soc}(A)$ if and only if $uv - vu \in \operatorname{Soc}(A)$ by Lemma 2. Thus the sum is zero. This means $\Phi_{1+u}(1+v) = 0$ for $1+v \notin H$. Now $\Phi_{1+u} = \varphi^{\overline{G}}$.

Let χ be an irreducible constituent of Φ_{1+u}. For $1+v,\ g = 1+g_0 \in G$,

$$\begin{aligned}\chi((1+v)^g) &= \chi(1+v)\varphi([1+v,g]) \\ &= \chi(1+v)\omega^{f(u,vg_0-g_0v)} \\ &= \chi(1+v)\omega^{f(uv-vu,g_0)},\end{aligned}$$

here we use that $[1+v,g] \in \mathrm{Z}(\chi)$. If $1+v \notin H$ then there exists $g_0 \in \mathrm{J}(A)$ such that $f(uv-vu,g_0) \neq 0$. Thus $\chi(1+v) = 0$. Now φ is fully ramified. The result follows. □

Finally we must show that

Lemma 6. $\Phi_{1+u} = \Phi_{1+v}$ *if and only if* $\overline{1+u}$ *is conjugate to* $\overline{1+v}$ *in* $\overline{G}$.

Proof. If u or v is in $\mathrm{J}^2(A)$ then the result is clear. Assume $u, v \notin \mathrm{J}^2(A)$. Then $\mathrm{C}_{\overline{G}}(\overline{u}) = \mathrm{C}_{\overline{G}}(\overline{v})$ and $f(u-v,x) = 0$ for all $x \in \mathrm{C}_{\overline{G}}(\overline{u})$. We denote $\mathrm{C}_{\overline{G}}(\overline{u})$ by $\overline{H}$. Write $H = 1 + U$. We must show $\overline{1+u} =_{\overline{G}} \overline{1+v}$ and this is equivalent to $\overline{u-v} = \overline{ua-au}$ for some $a \in \mathrm{J}(A)$.

Put $U^{\perp} = \{x \in \mathrm{J}(A) \bigm| f(x,w) = 0 \text{ for all } w \in U\}$. Then $u - v \in U^{\perp}$. Also put $\overline{V} = \{\overline{ub-bu} \bigm| b \in \mathrm{J}(A)\}$. It is enough to show taht $\overline{U^{\perp}} = \overline{V}$, and it is easy to see $\overline{U^{\perp}} \supseteq \overline{V}$. Since f is non-degenerate, $|\overline{U^{\perp}}| = |\overline{\mathrm{J}(A)} : \overline{U}|$. The map $\overline{b} \mapsto \overline{ub-bu}$ is F-linear, its image is $\overline{V}$, and its kernel is U. Thus $|\overline{V}| = |\overline{\mathrm{J}(A)} : \overline{U}|$. The proof is complete. $\square$

Now Theorem 3 holds clearly.

We remark that Theorem 3 holds over arbitrary finite fields. If A is a finite dimensional sysmetric algebra over a finite field, we can regard A as a symmetric algebra over the prime field of the same characteristic and the Jacobson radicals and the socles are coinside, respectively.

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