A. Connes の非可換微分幾何学による素粒子理論の標準模型の再構成

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量子力学において物理量は、可換な実数によってではなく、非可換な作用素（演算子）として記述されるようになった。また、量子力学と特殊相対性理論を統合した場の量子論によって、粒子は、場の固有振動として捉えられるようになる。量子物理学が投じた粒子と波動の双対性は、この場の量子論の考え方で理解される。場の量子論は時空の各点における粒子の生成と消滅を表す作用素が、粒子の個数の固有ベクトルで張られるヒルベルト空間に作用すると考える無限自由度の量子力学である。この見方によれば、数の変化しない粒子が空間の中の点に位置づけられ、時間と共にそれが軌道を描いて運動していくという実在性をア・プリオリに措定するができなくなった。量子力学の重ね合わせの原理から導出される量子もつれという現象、古典的な粒子に与えるような実在性を破綻させるような現象も明確になってきた。このようなことが理解されることで、量子系の実在性と環境世界の実在性、われわれの日常的な直観や概念構成とも深く繋がる古典的世界の実在性の関係が明らかにされつつあるとも言える。

他方、数学史を見ると、20 世紀にはいると、関数解析の分野で、関数の集合にユークリッド空間の構造を入れるヒルベルト空間論が生み出される。さらに、そのヒルベルト空間上の作用素の集合に環という代数構造を入れた作用素環論がフォン・ノイマンによって生み出された。また、幾何学的空間の構造をその上の関数の集合のなす可換な代数構造で捉えようとするリーマン＝ロッホの定理を一般化したアティヤ＝シンガーの指数定理は、幾何学的空間の局所性と大域性の双対性を顕わにする。代数構造によって幾何学的空間を捉えようとする考え方はまさに場の理論の幾何学の考え方と言ってもよい。コンヌは、この代数を非可換化し、アティヤ＝シンガーの指数定理を非可換化ないし量子化し、それを非可換微分幾何学の重要な基礎付けとするのである。ここでは、空間の局所性の意味は新たなコホモロジー、サイクリック・コホモロジーのド・ラム・コホモロジーとのアナロジーを通して拡張、再構成されている。

空間を何か実体そのものとして直接的に把握するのではなく、その上の関数の集合の代数構造を通して把握しようとするのは、現代数学にとって本質的なことである。そこには、空間と代数の間に双対的な関係があるのである。例えば、19 世紀半ば以後の代数幾何学においてはコンパクトな Riemann 面から代数関数体へ、Grothendieck 以後の代数幾何学においては affine schemes から可換環へと、反変関手によって圏同値に移されるのである。作用素環論においては、Gelfand-Naimark の定理によって、ルベーグ可測空間は可換 von Neumann 環へ、局所コンパクトなハウスドルフ位相空間は可換 C*環へと反変関手によって圏同値に移される。この時、包含関係は

局所コンパクトなハウスドルフ位相空間 ⊂ ルベーグ可測空間

可換 C*環（ノルム位相閉） ⊃ 可換 von Neumann 環（強位相閉）

のように、空間と代数とで逆になっている。さらに、解析的正則空間や$C^\infty(X)$のような可換関数環を非可換化したようなものと考えられるが、それはノルム位相で閉じていないものであると考えられる。また、Serre-Swan の定理により、可換 C*環上の有限生成される射影加群は位相空間上のベクトル束に同値であることも分かっているので、一般に非可換 C*環上の有限生成される射影加群を非可換位相空間上のベクトル束と考えられるとする。

　Connes は、以上のような非可換微分幾何学によって素粒子理論の標準模型を再構成しようとするのであるが、その基本方針について述べておく。Riemann が、有名な「幾何学についての基礎」

非可換微分幾何学
　解析　C*環の K 理論：位相幾何学的 K 理論のアナロジーないし拡張
　代数　cyclic cohomology：de Rham cohomology のアナロジーないし拡張

ここまでのことについては、以下の文献を参照
Alain Connes, *Noncommutative Geometry*, Academic Press, 1994, Chap. 5 まで。
原田雅樹[2023], 「Atihah-Singer の指数定理とその非可換化」、『数理物理研究 II　特集 量子』、数理物理出版会、pp. 169-233。

以下、Alain Connes, *Noncommutative Geometry*, Academic Press, 1994, Chap. 6 をコメントする。

4. The quantized calculus (Chapter IV)

The basic new idea of noncommutative differential geometry is a new calculus which replaces the usual differential and integral calculus.

This new calculus can be succinctly described by the following dictionary. We fix a pair $(\mathcal{H}, F)$, where $\mathcal{H}$ is an infinite-dimensional separable Hilbert space and F is a selfadjoint operator of square 1 in $\mathcal{H}$. Giving F is the same as giving the decomposition of $\mathcal{H}$ as the direct sum of the two orthogonal closed subspaces

F: s.a

$F^2 = 1$.

$$\{\xi \in \mathcal{H} \; ; \; F\xi = \pm\xi\}.$$

Assuming, as we shall, that both subspaces are infinite-dimensional, we see that all such pairs $(\mathcal{H}, F)$ are unitarily equivalent. The dictionary is then the following:

CLASSICAL	QUANTUM
Complex variable	Operator in $\mathcal{H}$
Real variable	Selfadjoint operator in $\mathcal{H}$
Infinitesimal	Compact operator in $\mathcal{H}$
Infinitesimal of order α	Compact operator in $\mathcal{H}$ whose characteristic values μ_n satisfy $\mu_n = O(n^{-\alpha}), \quad n \to \infty$
Differential of real or complex variable	$df = [F, f] = Ff - fF$
Integral of infinitesimal of order 1	Dixmier trace $\mathrm{Tr}_\omega(T)$

CHAPTER 6

The metric aspect of noncommutative geometry

The geometric spaces of Gauss and Riemann are defined as manifolds in which the metric is given by the formula

$$\delta(p,q) = \text{ infimum of length of paths } \gamma \text{ from } p \text{ to } q \tag{6.1}$$

where the length of a path γ is computed as the integral of the square root of a quadratic form in the differential of the path

$$\text{Length of } \gamma = \int_p^q (g_{\mu\nu}\, dx^\mu \;\; dx^\nu)^{1/2} \tag{6.2}$$

These geometric spaces form a relevant class of metric spaces, inasmuch as:

α) They are general enough to include numerous examples ranging from non-Euclidean geometries through surfaces embedded in $\mathbb{R}^3$ to space like hypersurfaces in general relativity.

β) They are special enough to deserve the name "geometry", since, being determined by local data, all the tools of differential and integral calculus are available to analyse them.

We have developed in Chapter IV a differential and integral calculus of "infinitesimals", given a Fredholm module $(\mathcal{H}, F)$ over the algebra $\mathcal{A}$ of coordinates on a possibly non-commutative space X. The Fredholm module $(\mathcal{H}, F)$ over $\mathcal{A}$ specifies the calculus on X but not the metric structure. For instance, the construction of $(\mathcal{H}, F)$ in the manifold

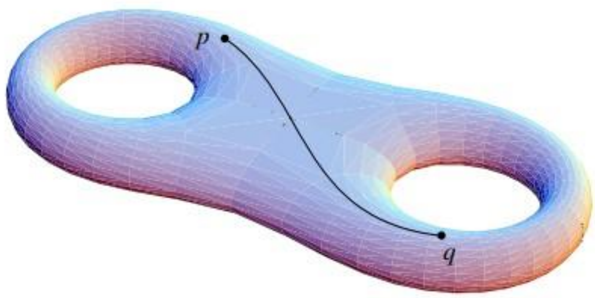

FIGURE 1. Geodesic

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case (Section IV.4) only used the conformal structure. In fact, in the example of Section IV.3, where $X = \mathbb{S}^1$ and $(\mathcal{H}, F)$ is the Hilbert transform, the quantum differential expression

$$\mathrm{d}Z\,\mathrm{d}\overline{Z} = [F, Z][F, \overline{Z}] \quad , \quad Z : \mathbb{S}^1 \rightarrow \mathbb{C} \tag{6.3}$$

where Z is the boundary value of a univalent map, yields an infinitesimal unit of length intimately tied up with the metric on $Z(\mathbb{S}^1)$ induced by the usual Riemannian metric $\mathrm{d}z\mathrm{d}\overline{z}$ of $\mathbb{C}$. If we vary Z, even the *dimension* of $\mathbb{S}^1$ for the "metric" (3) will change (cf. Section IV.3).

Let $\mathcal{A}$ be an involutive algebra and $(\mathcal{H}, F)$ a Fredholm module over $\mathcal{A}$. To define a "unit of length" in the corresponding space X, we shall consider an operator of the form

$$G = \sum_1^q (\mathrm{d}x^\mu)^* \, g_{\mu\nu} (\mathrm{d}x^\nu) \tag{6.4}$$

where $\mathrm{d}x = [F, x]$ for any $x \in \mathcal{A}$, the x^μ are elements of $\mathcal{A}$ and where $g = (g_{\mu\nu})_{\mu,\nu=1,\ldots,q}$ is a positive element of the matrix algebra $M_q(\mathcal{A})$.

We want to think of G as the $\mathrm{d}s^2$ of Riemannian geometry. It is by construction a positive "infinitesimal", i.e. a positive compact operator on $\mathcal{H}$. The unit of length is its positive square root

$$\mathrm{d}s = G^{1/2}. \tag{6.5}$$

To measure distances in the possibly noncommutative space X we first replace the points $p, q \in X$ by the corresponding pure states φ, ψ on the C^*-algebra closure of $\mathcal{A}$

$$\varphi(f) = f(p) \ , \ \psi(f) = f(q) \quad \forall f \in \mathcal{A}. \tag{6.6}$$

We then dualise the basic formula (1) as follows

$$\operatorname{dist}(p, q) = \operatorname{Sup}\{|f(p) - f(q)| \ ; \ f \in \mathcal{A} \ , \ \|\mathrm{d}f/\mathrm{d}s\| \le 1\} \tag{6.7}$$

which only involves p, q through the associated pure states (6). Since we are in the noncommutative set up we need to deal with the ambiguity in the order of the terms in an expression such as $\mathrm{d}f/\mathrm{d}s$ which can be either $\mathrm{d}f(\mathrm{d}s)^{-1}$ or $(\mathrm{d}s)^{-1}\,\mathrm{d}f$ or $(\mathrm{d}s)^{-\alpha}\,\mathrm{d}f(\mathrm{d}s)^{-(1-\alpha)}$ for instance. Instead of handling this problem directly we shall assume that G commutes with F, i.e. that $\mathrm{d}G = 0$, a condition similar to the Kähler condition, and introduce the following selfadjoint operator

$$D = FG^{-1/2} = F\ \mathrm{d}s^{-1}, \tag{6.8}$$

$\mathrm{d}s^{-1} = |D|$

例

$F = \gamma^\mu$

$= \begin{pmatrix} 0 & (1, -i\sigma) \\ (1, i\sigma) & 0 \end{pmatrix}$

whose existence assumes that G is nonsingular, i.e. $\ker G = 0$. We shall then formulate (7) as follows

$$\mathrm{d}(p, q) = \operatorname{Sup}\{|f(p) - f(q)| \ ; \ f \in \mathcal{A} \ , \ \|[D, f]\| \le 1\}\,. \tag{6.9}$$

Now the operator F is by construction the sign of D, while G is obtained from D by the formula

$$G = D^{-2}. \tag{6.10}$$

spectral triple

Thus it is more economical to take as our basic data the triple $(\mathcal{A}, \mathcal{H}, D)$ consisting of a Hilbert space $\mathcal{H}$, an involutive algebra $\mathcal{A}$ of operators on $\mathcal{H}$ and an unbounded selfadjoint operator D on $\mathcal{H}$. The conditions satisfied by such triples are

$$\begin{aligned} &[D, a] \text{is bounded for any} a \in \mathcal{A} \\ &(D - \lambda)^{-1} \text{is compact for any} \lambda \notin \mathbb{R} \end{aligned} \tag{6.11}$$

and were already formalised in Chapter IV Definition 2.11. In the present chapter we shall begin a systematic investigation of those geometric spaces. Besides Riemannian manifolds (see below) and spaces of non integral Hausdorff dimension (Section IV.3) the following are examples of geometric spaces described by our data:

a) Discrete spaces
b) Duals of discrete subgroups of Lie groups
c) Configuration space in supersymmetric quantum field theory.

We shall deal with Example a in Section 3 below. We have described already in great detail the triples $(\mathcal{A}, \mathcal{H}, D)$ corresponding to b) and c) in Section IV.9.

Our first task in this chapter will be to show that the Riemannian spaces are special cases of the above notion of geometric spaces. This will be done using an elliptic differential operator of order one, the Dirac operator (or the signature operator in the non-spin case). We shall first see that formula (9) applied to the triple (algebra of functions, Hilbert space of spinors, Dirac operator) readily gives back the geodesic distance (1) on the Riemannian manifold. Our next task will be to develop the analogue of the Lagrangian formulation of electrodynamics involving matter fields and gauge bosons for our more general geometric spaces. This will be done using the tools of the quantized calculus developed in Chapter IV Section 2. As mentioned above the commutator $[D, f]$, $f \in \mathcal{A}$ will play the role of the differential quotient df/ds. As a central result we shall prove the inequality between the second Chern number of a "vector bundle" and the minimum of the Yang-Mills action on vector potentials. We shall see that our new notion of geometric space treats on an equal footing the continuum and the discrete, while the action for electrodynamics on the simplest mixture of continuum and discrete–the product of 4-dimensional continuum by a discrete 2-point space–gives the Glashow-Weinberg-Salam model for leptons. The notion of manifold in noncommutative geometry will be reached only after an understanding of Poincaré duality, i.e., that the K-homology cycle $(\mathfrak{H}, D)$ yields the fundamental class of the space under consideration. The notion of manifold obtained is directly inspired by the work of D. Sullivan **[543]** who discovered the basic role played by the K-homology fundamental class of a manifold.

The main example of a space to which all these considerations will be applied is Euclidean space-time in physics, i.e., space-time but with imaginary time. What we shall give is a geometric interpretation of the now experimentally confirmed effective low-energy model of particle physics, namely the Glashow–Weinberg–Salam standard model. This model is a gauge theory model with gauge group $U(1) \times SU(2) \times SU(3)$ and a pair of complex Higgs fields providing masses by the symmetry-breaking mechanism. We interpret this model geometrically as a pure gauge theory, i.e. electrodynamics, but on a more elaborate space-time $E' = E \times F$, the product of ordinary Euclidean space-time by a finite space F. The geometry of this finite space is specified by a pair $(\mathfrak{H}, D)$ as above, where $\mathfrak{H}$ is finite-dimensional and the selfadjoint operator D encodes the nine fermion masses and the four Kobayashi–Maskawa mixing parameters of the standard model.

The values of the hypercharges do not have to be fitted artificially to their experimental values but come out right from a simple unimodularity condition on the space E'.

Our analysis is limited to the classical context and does not at the moment address the questions related to renormalization, such as the existence of relations between coupling constants or the naturalness problem. Nevertheless, our more geometric and conceptual interpretation of the standard model gives a clear indication that particle physics is not so much a long list of elementary particles as the unveiling of the fine geometric structure of space-time.

The content of this Chapter Is organized as follows:

1. Riemannian geometry and the Dirac operator.
2. Positivity in Hochschild cohomology and inequalities for the Yang Mills action.
3. Product of continuum by discrete and the symmetry breaking mechanism.
4. The commutant and Poincaré duality.
5. The standard $U(1)\times SU(2)\times SU(3)$ model.

1. Riemannian Manifolds and the Dirac Operator

Let M be a compact Riemannian spin manifold, and let $D = \partial_M$ be the corresponding Dirac operator (cf. [**227**]). Thus, D is an unbounded selfadjoint operator acting in the Hilbert space $\mathfrak{H}$ of L^2-spinors on the manifold M.

We shall give four formulas below that show how to reconstruct the *metric space* (M, d), where d is the geodesic distance, the *volume measure* dv on M, the space of *gauge potentials*, and, finally, the *Yang–Mills* action functional, from the purely operator-theoretic data

$$(\mathcal{A}, \mathfrak{H}, D),$$

where D is the Dirac operator on the Hilbert space $\mathfrak{H}$ and where $\mathcal{A}$ is the abelian von Neumann algebra of multiplication by bounded measurable functions on M.

Thus, $\mathcal{A}$ is an abelian von Neumann algebra on $\mathfrak{H}$, and knowing the pair $(\mathfrak{H}, \mathcal{A})$ yields essentially no information (cf. Chapter V) except for the multiplicity, which is here the constant $2^{[d/2]}$, where $d = \dim M$. Similarly, the mere knowledge of the operator D on $\mathfrak{H}$ is equivalent to giving its list of eigenvalues $(\lambda_n)_{n\in\mathbb{N}}$, $\lambda_n \in \mathbb{R}$, and is an impractical point of departure for reconstructing M. The growth of these eigenvalues, i.e., the behavior of $|\lambda_n|$ as $n \to \infty$, is again governed by the dimension d of M, namely, $|\lambda_n| \sim Cn^{1/d}$ as $n \to \infty$.

The proof is straightforward, but it is relevant to go through it to see what is involved. The operator $[D, a]$, which by Lemma 1 is bounded iff a is Lipschitz, is then given by the Clifford multiplication $i^{-1}\gamma(\mathrm{d}a)$ by the gradient $\mathrm{d}a$ of a. This gradient is ([**202**]) a bounded measurable section of the cotangent bundle T^*M of M, and we have

$$\|[D, a]\| = \operatorname{ess\,sup} \|\mathrm{d}a\| = \text{the Lipschitz norm of } a.$$

It follows at once that the right-hand side of Formula 1 is less than or equal to the geodesic distance $d(p, q)$. However, fixing the point p and considering the function $a(q) = d(q, p)$, one checks that a is Lipschitz with constant 1, so that $\|[D, a]\| \leq 1$, which yields the desired equality. Note that Formula 1 is in essence dual to the original formula

$$(*) \qquad d(p, q) = \text{infimum of the length of paths } \gamma \text{ from } p \text{ to } q,$$

in the sense that, instead of involving arcs, namely copies of $\mathbb{R}$ inside the manifold M, it involves *functions* a, that is, maps from M to $\mathbb{R}$ (or to $\mathbb{C}$).

This is an essential point for us since, in the case of discrete spaces or of noncommutative spaces X, there are no interesting arcs in X but there are plenty of *functions*, namely, the elements $a \in \mathcal{A}$ of the defining algebra. We note at once that the right-hand side of Formula 1 is meaningful in that general context and it defines a metric on the space of *states* of the C^*-algebra A, the norm closure of $\mathbf{A} = \{a \in \mathcal{A};\ [D, a] \text{ is bounded}\}$

$$\boxed{d(\varphi, \psi) = \sup\{|\varphi(a) - \psi(a)|;\ \|[D, a]\| \leq 1\}.}$$

Finally, we also note that, although both Formula 1 and the formula $(*)$ give the same result for Riemannian manifolds, they are of quite different nature if we try to use them in actual measurements of distances. The formula $(*)$ uses the idealized notion of a path, and quantum mechanics teaches us that there is nothing like "the path followed by a particle". Thus, for measurements of very small distances, it is more natural to use wave functions and Formula 1.

We have now recovered from our original data $(\mathcal{A}, \mathfrak{H}, D)$ the metric space (M, d), where d is the geodesic distance. Let us now deal with the tools of differential and integral calculus, the first obvious example being the measure given by the volume form

$$f \mapsto \int_M f \mathrm{d}v,$$

where, in local coordinates $x^\mu, g_{\mu\nu}$, we have

$$\mathrm{d}v = \left(\det(g_{\mu\nu})\right)^{1/2} |\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n|.$$

This takes us to our second formula, which is nothing more than a restatement of H. Weyl's theorem about the asymptotic behavior of elliptic differential operators

([**247**], [**227**]). It does, however, involve a new tool, *the Dixmier trace* Tr_ω (cf. Chapter IV.2), which, unlike asymptotic expansions, makes sense in full generality in our context and is the correct operator-theoretic substitute for integration:

Formula 2. *For every* $f \in A$, *we have* $\int_M f dv = c(d)\ \mathrm{Tr}_\omega(f|D|^{-d})$, *where* $d = \dim M$, $c(d) = 2^{(d-[d/2])}\ \pi^{d/2}\ \Gamma\left(\frac{d}{2}+1\right)$.

By convention we let D^{-1} be equal to 0 on the finite-dimensional subspace $\mathrm{Ker} D$.

Let us refer to Section IV.2 for the detailed definition and properties of the Dixmier trace Tr_ω. We can interpret the right-hand side of the equality as the limit of the sequence

$$\frac{1}{\log N}\sum_{j=0}^{N}\lambda_j,$$

where the λ_j are the eigenvalues of the compact operator $f|D|^{-d}$, or, equivalently, as the residue, at the point $s = 1$, of the function

$$\zeta(s) = \mathrm{Trace}(f|D|^{-ds})\ (\Re s > 1).$$

The crucial fact for us is that the Dixmier trace makes sense independently of the context of pseudodifferential operators and that all properties of the integral $\int_M f dv$, such as positivity, finiteness, covariance, etc., become obvious corollaries of the general properties of the Dixmier trace:

A) *Positivity*: $\mathrm{Tr}_\omega(T) \geq 0$ if T is a positive operator.

B) *Finiteness*: $\mathrm{Tr}_\omega(T) < \infty$ if the eigenvalues of $|T|$ satisfy $\sum_0^N \mu_n(T) = O(\log N)$. Dixmier trace.

C) *Covariance*: $\mathrm{Tr}_\omega(UTU^*) = \mathrm{Tr}_\omega(T)$ for every unitary U.

D) *Vanishing*: $\mathrm{Tr}_\omega(T) = 0$ if T is of trace class.

Property D is the counterpart of locality in our framework; it shows that the Dixmier trace of an operator is unaffected by a finite-rank perturbation, and allows many identities to hold, as we have seen in Chapter IV.

Now, setting up the integral of functions, i.e., the Riemannian volume form, is a good indication but quite far from the full story. In particular, many distinct Riemannian metrics yield the same volume form. Since our aim is to investigate physical space-time at the scale of elementary particle physics, we shall now make a deliberate choice: instead of focusing on the intrinsic Riemannian curvature, which would drive us towards general relativity, we shall concentrate on the measurement (using $(\mathfrak{H}, D)$) of the curvature of connections on vector bundles, and on the Yang–Mills functional, which takes us to the theory of matter fields. This line is of course easier since it does not involve derivatives of the $g_{\mu\nu}$.

Let us state our aim clearly: to recover the Yang–Mills functional on connections on vector bundles, making use of only the following data (IV.2.11):

後にConnesはDixmier traceを$\int$と書く。

Definition 2. *A* K-cycle $(\mathfrak{H}, D)$, *over an algebra* $\mathcal{A}$ *with involution* $*$, *consists of a* $*$*-representation of* $\mathcal{A}$ *on a Hilbert space* $\mathfrak{H}$ *together with an unbounded selfadjoint operator* D *with compact resolvent, such that* $[D, a]$ *is bounded for every* $a \in \mathcal{A}$.

We shall assume that $\mathcal{A}$ is unital and that the unit $1 \in \mathcal{A}$ acts as the identity on $\mathfrak{H}$ (cf. Remark 12 for the nonunital case).

If the eigenvalues λ_n of $|D|$ are of the order of $n^{1/d}$ as $n \to \infty$, we say that the K-cycle is (d, ∞)-summable (cf. Section 2 of Chapter IV). On the algebra of functions on a compact Riemannian spin manifold, the Dirac operator determines a K-cycle that is (d, ∞)-summable, where $d = \dim M$. Finer regularity of functions, such as infinite differentiability, is easily expressed using the domains of powers of the derivation δ, $\delta(a) = [|D|, a]$.

We shall not be too specific about the choice of regularity; our discussion applies to any degree of regularity higher than Lipschitz.

The value of the following construction is that it will also apply when the $*$-algebra $\mathcal{A}$ is noncommutative, or when D is no longer the Dirac operator (cf. Section 3). The reader can have in mind both the Riemannian case and the slightly more involved case where the algebra $\mathcal{A}$ is the $*$-algebra of matrices of functions on a Riemannian manifold, just to bear in mind that the notion of exterior product no longer makes sense over such an algebra.

We shall begin with the notion of connection on the trivial bundle, i.e., the case of "electrodynamics", and define vector potentials and the Yang–Mills action in that case. We shall then treat the general case of arbitrary Hermitian bundles, i.e., in algebraic terms, of arbitrary Hermitian, finitely generated projective modules over $\mathcal{A}$.

We wish to define k-forms over $\mathcal{A}$ as operators on $\mathfrak{H}$ of the form

$$\omega = \sum a_0^j [D, a_1^j] \cdots [D, a_k^j],$$

where the a_i^j are elements of $\mathcal{A}$ represented as operators on $\mathfrak{H}$. This idea arises because, although the operator D fails to be invariant under the representation on $\mathfrak{H}$ of the unitary group $\mathcal{U}$ of $\mathcal{A}$,

$$\mathcal{U} = \{u \in \mathcal{A};\ u^*u = uu^* = 1\},$$

the following equality shows that the failure of invariance is governed by a 1-form in the above sense: by $\omega_u = u[D, u^*]$, that is,

$$uDu^* = D + \omega_u.$$

Note that ω_u is selfadjoint as an operator on $\mathfrak{H}$. Thus, it is natural to adopt the following definition:

Definition 3. *A* vector potential V *is a selfadjoint element of the space of* 1*-forms* $\sum a_0^j [D, a_1^j]$, *where* $a_k^j \in \mathcal{A}$.

$\omega_u = uDu^* - D$

$- u[D,u^*]u = uD - Du$

We start with the left-hand side; it is equal to

$\cdot uDVu^* - DuVu^*$

$$
\begin{aligned}
& \mathrm{d}\gamma_u(V) + (u[D,u^*] + uVu^*)^2 \\
= \; & \mathrm{d}\gamma_u(V) + u[D,u^*]u[D,u^*] + u[D,u^*]uVu^* \\
& + uVu^*u[D,u^*] + uV^2u^* \\
= \; & \mathrm{d}\gamma_u(V) - [D,u][D,u^*] - [D,u]Vu^* + uV[D,u^*] + uV^2u^* \\
= \; & \sum [D, ua_0^j][D, a_1^j u^*] - \sum [D, ua_0^j a_1^j][D,u^*] \\
& - [D,u]Vu^* + uV[D,u^*] + uV^2u^* \\
= \; & u\mathrm{d}Vu^* + uV^2u^*,
\end{aligned}
$$

where the last equality follows from

$$
\begin{aligned}
& \sum [D,u]a_0^j[D,a_1^j u^*] - \sum [D,u]a_0^j a_1^j [D,u^*] = [D,u]Vu^*, \\
& \sum u[D,a_0^j][D,a_1^j u^*] - \sum u[D,a_0^j]a_1^j[D,u^*] = u\mathrm{d}Vu^*, \\
& \sum ua_0^j[D,a_1^j][D,u^*] = uV[D,u^*].
\end{aligned}
$$

The difficulty that we overlooked is the following: the same vector potential V might be written in several ways as $V = \sum a_0^j[D, a_1^j]$, so that the definition of $\mathrm{d}V$ as

$$\mathrm{d}V = \sum [D, a_0^j][D, a_1^j]$$

is ambiguous.

To understand the nature of the problem, let us introduce some algebraic notation. We let $\Omega^*\mathcal{A}$ be the reduced universal differential graded algebra over $\mathcal{A}$ (Chapter III.1). It is by definition equal to $\mathcal{A}$ in degree 0 and is generated by symbols $\mathrm{d}a$ $(a \in \mathcal{A})$ of degree 1 with the following presentation:

α) $\mathrm{d}(ab) = (\mathrm{d}a)b + a\mathrm{d}b$ $(\forall a, b \in \mathcal{A})$,
β) $\mathrm{d}1 = 0$.

One can check that $\Omega^1\mathcal{A}$ is isomorphic as an $\mathcal{A}$-bimodule to the kernel $\ker(m)$ of the multiplication mapping $m : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$, the isomorphism being given by the mapping

$$\sum a_i \otimes b_i \in \ker(m) \mapsto \sum a_i \mathrm{d}b_i \in \Omega^1\mathcal{A}.$$

The involution $*$ of $\mathcal{A}$ extends uniquely to an involution on Ω^* with the rule

$$(\mathrm{d}a)^* = -\mathrm{d}a^*.$$

The differential d on $\Omega^*\mathcal{A}$ is defined *unambiguously* by

$$\mathrm{d}(a^0\mathrm{d}a^1 \cdots \mathrm{d}a^n) = \mathrm{d}a^0\mathrm{d}a^1 \cdots \mathrm{d}a^n \;\; \forall a^j \in \mathcal{A},$$

and it satisfies the relations

One can immediately check that in the basic example of the Dirac operator on a spin Riemannian manifold, a vector potential in the above sense is exactly a 1-form ω on the manifold M and that this form is imaginary, the corresponding operator in the space of spinors being given by the Clifford multiplication:

$$V = i^{-1}\gamma(\omega)\ (i = \sqrt{-1}).$$

The action of the unitary group $\mathcal{U}$ on vector potentials is such that it replaces the operator $D + V$ by $u(D+V)u^*$; thus it is given by the algebraic formula

$$\gamma_u(V) = u[D, u^*] + uVu^*\ (u \in \mathcal{U}).$$

We now need only define the curvature or field strength θ for a vector potential, and use the analogue of the above Formula 2 to integrate the square of θ: the formula

$$\mathrm{YM}\,(V) = \mathrm{Tr}_\omega(\theta^2|D|^{-d})$$

should give us the Yang–Mills action.

The formula for θ should be of the form $\theta = \mathrm{d}V + V^2$; the only difficulty is in defining properly the "differential" $\mathrm{d}V$ of a vector potential, as an operator on $\mathfrak{H}$.

Let us examine what happens; the naive formulation is

$$\text{If } V = \sum a_0^j[D, a_1^j] \text{ then } \mathrm{d}V = \sum [D, a_0^j][D, a_1^j].$$

Before we point out what the difficulty is, let us check that if we replace V by $\gamma_u(V)$, where

$$\gamma_u(V) = u[D, u^*] + \sum u a_0^j[D, a_1^j]u^*,$$

then the curvature is transformed covariantly:

$$\mathrm{d}(\gamma_u(V)) + \gamma_u(V)^2 = u(\mathrm{d}V + V^2)u^*.$$

As this computation is instructive, we shall carry it out in detail. First, in order to write $\gamma_u(V)$ in the same form as V, we use the equality

$$[D, a_1^j]u^* = [D, a_1^j u^*] - a_1^j[D, u^*].$$

Thus, $\gamma_u(V) = u[D, u^*] + \sum u a_0^j[D, a_1^j u^*] - \sum u a_0^j a_1^j[D, u^*]$, and we have

$$\mathrm{d}\gamma_u(V) = [D, u][D, u^*] + \sum [D, u a_0^j][D, a_1^j u^*] - \sum [D, u a_0^j a_1^j][D, u^*].$$

We now claim that the following operators on $\mathfrak{H}$ are indeed equal:

$$\mathrm{d}\gamma_u(V) + \gamma_u(V)^2 = u(\mathrm{d}V + V^2)u^*.$$

$\mathrm{d}^2\omega = 0 \ \forall \omega \in \Omega^*\mathcal{A}$,
$\mathrm{d}(\omega_1\omega_2) = (\mathrm{d}\omega_1)\omega_2 + (-1)^{\partial\omega_1}\omega_1\mathrm{d}\omega_2 \ \forall \omega_j \in \Omega^*\mathcal{A}$.

Proposition 4.

1) *The following equality defines a $*$-representation π of the reduced universal algebra $\Omega^*(\mathcal{A})$ on $\mathfrak{H}$:*
$$\pi(a^0\mathrm{d}a^1\cdots\mathrm{d}a^n) = a^0[D,a^1]\cdots[D,a^n] \ \forall a^j \in \mathcal{A}.$$

2) *Let $J_0 = \ker\pi \subset \Omega^*$ be the graded two-sided ideal of Ω^* given by $J_0^{(k)} = \{\omega \in \Omega^k;\ \pi(\omega) = 0\}$; then $J = J_0 + \mathrm{d}J_0$ is a graded differential two-sided ideal of $\Omega^*(\mathcal{A})$.*

The first statement is obvious; let us discuss the second. By construction, J_0 is a two-sided ideal but it is not, in general, a *differential* ideal, i.e., if $\omega \in \Omega^k(\mathcal{A})$ and $\pi(\omega) = 0$, one does not in general have $\pi(\mathrm{d}\omega) = 0$. This is exactly the reason why the above definition of $\sum[D,a_0^j][D,a_1^j]$ as the differential of $\sum a_0^j[D,a_1^j]$ was ambiguous.

Let us show, however, that $J = J_0 + \mathrm{d}J_0$ is still a two-sided ideal. Since $\mathrm{d}^2 = 0$ it is obvious that J is then a differential ideal. Let $\omega \in J^{(k)}$ be a homogeneous element of J; then ω is of the form $\omega = \omega_1 + \mathrm{d}\omega_2$, where $\omega_1 \in J_0 \cap \Omega^k$, $\omega_2 \in J_0 \cap \Omega^{k-1}$. Let $\omega' \in \Omega^{k'}$, and let us show that $\omega\omega' \in J^{(k+k')}$. We have
$$\begin{aligned}\omega\omega' &= \omega_1\omega' + (\mathrm{d}\omega_2)\omega' = \omega_1\omega' + \mathrm{d}(\omega_2\omega') - (-1)^{k-1}\omega_2\mathrm{d}\omega' \\ &= \big(\omega_1\omega' + (-1)^k\omega_2\mathrm{d}\omega'\big) + \mathrm{d}(\omega_2\omega').\end{aligned}$$
But, the first term belongs to $J_0 \cap \Omega^{k+k'}$ and $\omega_2\omega' \in J_0 \cap \Omega^{k+k'-1}$.

Using 2) of Proposition 4, we can now introduce the graded differential algebra
$$\Omega_D^* = \Omega^*(\mathcal{A})/J.$$

Let us first investigate Ω_D^0, Ω_D^1 and Ω_D^2.

We have $J \cap \Omega^0 = J_0 \cap \Omega^0 = \{0\}$ provided that we assume, as we shall, that $\mathcal{A}$ is a *subalgebra* of $\mathcal{L}(\mathfrak{H})$. Thus, $\Omega_D^0 = \mathcal{A}$.

Next, $J \cap \Omega^1 = J_0 \cap \Omega^1 + \mathrm{d}(J_0 \cap \Omega^0) = J_0 \cap \Omega^1$; thus Ω_D^1 is the quotient of Ω^1 by the kernel of π, and it is thus exactly the $\mathcal{A}$-bimodule $\pi(\Omega^1)$ of operators ω of the form
$$\omega = \sum a_j^0[D,a_j^1] \ (a_j^k \in \mathcal{A}).$$

Finally, $J \cap \Omega^2 = J_0 \cap \Omega^2 + \mathrm{d}(J_0 \cap \Omega^1)$ and the representation π gives an isomorphism

$(*)$
$$\boxed{\Omega_D^2 \cong \pi(\Omega^2)/\pi\big(\mathrm{d}(J_0 \cap \Omega^1)\big).}$$

More precisely, this means that we can view an element ω of Ω_D^2 as a class of elements ρ of the form
$$\rho = \sum a_j^0[D,a_j^1][D,a_j^2] \ (a_j^k \in \mathcal{A})$$

modulo the sub-bimodule of elements of the form

$$\rho_0 = \sum [D, b_j^0][D, b_j^1] \quad ; \quad b_j^k \in \mathcal{A}, \ \sum b_j^0 [D, b_j^1] = 0.$$

It is now clear that since we work modulo this subspace $\pi\big(\mathrm{d}(J_0 \cap \Omega^1)\big)$, the question of ambiguity in the definition of $\mathrm{d}\omega$ for $\omega \in \pi(\Omega^1)$ no longer arises.

The equality $(*)$ makes sense for all k,

$$(*) \qquad \Omega_D^k \cong \pi(\Omega^k)/\pi\big(\mathrm{d}(J_0 \cap \Omega^{k-1})\big),$$

and allows us to define the following inner product on Ω_D^k: for each k let $\mathfrak{H}_k$ be the Hilbert space completion of $\pi(\Omega^k)$ with the inner product

$$\langle T_1, T_2 \rangle_k = \mathrm{Tr}_\omega (T_2^* T_1 |D|^{-d}) \ \forall T_j \in \pi(\Omega^k).$$

Let P be the orthogonal projection of $\mathfrak{H}_k$ onto the orthogonal complement of the subspace $\pi\big(\mathrm{d}(J_0 \cap \Omega^{k-1})\big)$. By construction, the inner product $\langle P\omega_1, \omega_2 \rangle = \langle P\omega_1, P\omega_2 \rangle$ for $\omega_j \in \pi(\Omega^k)$ depends only on their classes in Ω_D^k. We denote by Λ^k the Hilbert space completion of Ω_D^k for this inner product; it is, of course, equal to $P\mathfrak{H}_k$.

Proposition 5.

1) *The actions of $\mathcal{A}$ on Λ^k by left and right multiplication define commuting unitary representations of $\mathcal{A}$ on Λ^k.*
2) *The functional* $\mathrm{YM}(V) = \langle \mathrm{d}V + V^2, \mathrm{d}V + V^2 \rangle$ *is positive, quartic and invariant under gauge transformations,*

$$\gamma_u(V) = u\mathrm{d}u^* + uVu^* \ \forall u \in \mathcal{U}(\mathcal{A}).$$

3) *The functional* $I(\alpha) = \mathrm{Tr}_\omega(\theta^2 |D|^{-d})$, with $\theta = \pi(\mathrm{d}\alpha + \alpha^2)$, *is positive, quartic and gauge invariant on* $\{\alpha \in \Omega^1(\mathcal{A});\ \alpha = \alpha^*\}$.
4) *One has* $\mathrm{YM}(V) = \inf\{I(\alpha);\ \pi(\alpha) = V\}$.

Let us say a few words about the easy proof. First, the left and right actions of $\mathcal{A}$ on $\mathfrak{H}_k$ are unitary. The unitarity of the right action of $\mathcal{A}$ follows from the equality $\mathrm{Tr}_\omega(Ta|D|^{-d}) = \mathrm{Tr}_\omega(aT|D|^{-d}) \quad \forall T \in \mathcal{L}(\mathcal{H}),\ a \in \mathcal{A}$. Since $\pi\big(\mathrm{d}(J_0 \cap \Omega^{k-1})\big) \subset \pi(\Omega^k)$ is a sub-bimodule of $\pi(\Omega^k)$ it follows that P is a bimodule morphism:

$$P(a\xi b) = aP(\xi)b \ \forall a, b \in \mathcal{A},\ \xi \in \mathfrak{H}_k.$$

Thus 1) follows. As for 2), one merely notes that by the above calculation, with $\mathrm{d}V$ now unambiguous, $\theta = \mathrm{d}V + V^2$ is covariant under gauge transformations, whence the result. For 3), one again uses the above calculation to show that $\mathrm{d}\alpha + \alpha^2$ transforms covariantly under gauge transformations.

Finally, 4) follows from the property of the orthogonal projection P: as an element of Λ^2, $\mathrm{d}V + V^2$ is equal to $P\big(\pi(\mathrm{d}\alpha + \alpha^2)\big)$ for any α with $\pi(\alpha) = V$, and since the ambiguity in $\pi(\mathrm{d}\alpha)$ is exactly $\pi\big(\mathrm{d}(J_0 \cap \Omega^1)\big)$ one gets 4).

Stated in simpler terms, the meaning of Proposition 5 is that the ambiguity that we met above in the definition of the operator curvature $\theta = \mathrm{d}V + V^2$ can be ignored by taking the infimum

$$\mathrm{YM}(V) = \inf \mathrm{Tr}_\omega(\theta^2|D|^{-d})$$

over all possibilities for $\theta = \mathrm{d}V + V^2$, $\mathrm{d}V = \sum[D, a_j^0][D, a_j^1]$ being ambiguous. The action obtained is nevertheless quartic by 2).

We shall now check that in the case of Riemannian manifolds with the Dirac K-cycle, the graded differential algebra Ω_D^* is canonically isomorphic to the *de Rham algebra of ordinary forms on M with their canonical pre-Hilbert space structure*. The whole point is that Propositions 4 and 5 give us these concepts in far greater generality, and the formula in 4) will allow extending to this generality, in the case $d = 4$, the inequality between the topological action and the Yang–Mills action YM (cf. Section 2). We refer the skeptical reader to the examples of Section 3.

3. Product of the Continuum by the Discrete and the Symmetry Breaking Mechanism

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We have shown how to extend, to our context of finitely summable K-cycles $(\mathfrak{H}, D)$ over an algebra $\mathcal{A}$, the concepts of gauge potentials and Yang–Mills action, as well as the way in which this action is related to a topological action in the case of dimension 4. In this section we shall give several examples of computations of this action. We first briefly recall its definition and use the opportunity to add to it a fermionic part.

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We are given a $*$-algebra $\mathcal{A}$ and a (d, ∞)-summable K-cycle $(\mathfrak{H}, D)$ over $\mathcal{A}$. This gives us a representation on $\mathfrak{H}$ of the reduced universal differential algebra $\Omega^*\mathcal{A}$:

$$\boxed{\pi(a^0 \mathrm{d}a^1 \cdots \mathrm{d}a^k) = a^0[D, a^1] \cdots [D, a^k] \ \forall a^j \in \mathcal{A},}$$

which defines a quotient differential graded algebra

$$\Omega_D^*(\mathcal{A}) = \Omega^*(\mathcal{A})/J, \ J = J_0 + \mathrm{d}J_0, \ J_0^{(k)} = \Omega^k \cap \mathrm{Ker}\, \pi.$$

A compatible connection ∇ on a Hermitian, finitely generated projective module $\mathcal{E}$ over $\mathcal{A}$ is given by a linear mapping

$$\nabla : \mathcal{E} \to \mathcal{E} \otimes_\mathcal{A} \Omega_D^1$$

which satisfies the Leibniz rule and is compatible with the inner product. The affine space $C(\mathcal{E})$ of such connections is acted on by the unitary group $\mathcal{U}(\mathcal{E})$ of the $*$-algebra of endomorphisms $\mathrm{End}_\mathcal{A}(\mathcal{E})$. This action transforms the curvature $\theta = \nabla^2$ of such connections covariantly, and

$$\boxed{\mathrm{YM}(\nabla) = \mathrm{Tr}_\omega\big(\pi(\theta)^2|D|^{-d}\big)}$$

is a gauge invariant quartic positive action on $C(\mathcal{E})$ (cf. Section 1).

2. Example a). The space we are dealing with has *two points* a and b. Thus, the algebra $\mathcal{A}$ is just the direct sum $\mathbb{C} \oplus \mathbb{C}$ of two copies of $\mathbb{C}$. An element $f \in \mathcal{A}$ is given by two complex numbers $f(a), f(b) \in \mathbb{C}$. Let $(\mathfrak{H}, D, \gamma)$ be a 0-dimensional K-cycle

over $\mathcal{A}$; then $\mathfrak{H}$ is *finite-dimensional* and the representation of $\mathcal{A}$ in $\mathfrak{H}$ corresponds to a decomposition of $\mathfrak{H}$ as a direct sum $\mathfrak{H} = \mathfrak{H}_a + \mathfrak{H}_b$, with the action of $\mathcal{A}$ given by

$$f \in \mathcal{A} \mapsto \begin{bmatrix} f(a) & 0 \\ 0 & f(b) \end{bmatrix}.$$

If we write D as a 2×2 matrix in this decomposition,

$$D = \begin{bmatrix} D_{aa} & D_{ab} \\ D_{ba} & D_{bb} \end{bmatrix},$$

we can ignore the diagonal elements since they commute exactly with the action of $\mathcal{A}$. We shall thus take D to be of the form

$$D = \begin{bmatrix} 0 & D_{ab} \\ D_{ba} & 0 \end{bmatrix},$$

where $D_{ba} = D_{ab}^*$ and D_{ba} is a linear mapping from $\mathfrak{H}_a$ to $\mathfrak{H}_b$. We shall denote this linear mapping by M and take for γ the $\mathbb{Z}/2$-grading given by the matrix

$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \gamma$. We thus have

$$\mathcal{A} = \mathbb{C} \oplus \mathbb{C}, \ \mathfrak{H} = \mathfrak{H}_a \oplus \mathfrak{H}_b, \ D = \begin{bmatrix} 0 & M^* \\ M & 0 \end{bmatrix}, \ \gamma = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Let us first compute the metric on the space $X = \{a, b\}$, given by Formula 1 of Section 1. Given $f \in \mathcal{A}$, we have

$$\begin{aligned} [D, f] &= \left[\begin{bmatrix} 0 & M^* \\ M & 0 \end{bmatrix}, \begin{bmatrix} f(a) & 0 \\ 0 & f(b) \end{bmatrix} \right] \\ &= \begin{bmatrix} 0 & M^*\big(f(b) - f(a)\big) \\ -M\big(f(b) - f(a)\big) & 0 \end{bmatrix} = \big(f(b) - f(a)\big) \begin{bmatrix} 0 & M^* \\ -M & 0 \end{bmatrix}. \end{aligned}$$

Thus, the norm of this commutator is $|f(b) - f(a)|\lambda$, where λ is the largest eigenvalue $\|M\|$ of $|M|$. Therefore

$$d(a, b) = \sup\{|f(a) - f(b)|; \ \|[D, f]\| \leq 1\} = 1/\lambda.$$

Let us now determine the space of gauge potentials, the curvature and the action in two cases.

$\boxed{\textit{Case } \alpha)}$ $\mathcal{E} = \mathcal{A}$ (i.e., the trivial bundle over X)

The space $\Omega^1(\mathcal{A})$ of universal 1-forms over $\mathcal{A}$ is given by the kernel of the multiplication $m : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$, $m(f \otimes g) = fg$. These are functions on $X \times X$ that vanish on the diagonal. Thus, $\Omega^1(\mathcal{A})$ is a 2-dimensional space; if $e \in \mathcal{A}$ is the idempotent $e(a) = 1$, $e(b) = 0$, this space has as basis

$$e de, \ (1 - e) de,$$

so that every element of $\Omega^1(\mathcal{A})$ is of the form $\lambda e\mathrm{d}e + \mu(1-e)\mathrm{d}(1-e)$. The differential $\mathrm{d} : \mathcal{A} \to \Omega^1(\mathcal{A})$ is the finite difference

$$\mathrm{d}f = (\Delta f)e\mathrm{d}e - (\Delta f)(1-e)\mathrm{d}(1-e), \ \ \Delta f = f(a) - f(b);$$

it is a derivation with values in the bimodule $\Omega^1(\mathcal{A})$, which *fails to be commutative since* $f\omega \neq \omega f$ for $\omega \in \Omega^1$, $f \in \mathcal{A}$.

Also, if $M \neq 0$ then the representation $\pi : \Omega^*(\mathcal{A}) \to \mathcal{L}(\mathfrak{H})$ is injective on $\Omega^1(\mathcal{A})$, so that $\Omega^1(\mathcal{A}) = \Omega^1_D(\mathcal{A})$. We have

$$\pi\big(\lambda e\mathrm{d}e + \mu(1-e)\mathrm{d}e\big) = \begin{bmatrix} 0 & -\lambda M^* \\ \mu M & 0 \end{bmatrix} \in \mathcal{L}(\mathfrak{H}).$$

A vector potential is given by a selfadjoint element of Ω^1_D, i.e., by a single complex number Φ, with

$$\pi(V) = \begin{bmatrix} 0 & \overline{\Phi} M^* \\ \Phi M & 0 \end{bmatrix}.$$

$e = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $de = \begin{pmatrix} 0 & -M^* \\ M & 0 \end{pmatrix}$

Since $V = -\overline{\Phi}e\mathrm{d}e + \Phi(1-e)\mathrm{d}e$, its curvature is

$$\theta = \mathrm{d}V + V^2 = -\overline{\Phi}\mathrm{d}e\mathrm{d}e - \Phi\mathrm{d}e\mathrm{d}e + \big(\overline{\Phi}e\mathrm{d}e - \Phi(1-e)\mathrm{d}e\big)^2,$$

and, using the equalities $e\mathrm{d}e(1-e) = e\mathrm{d}e$, $e(\mathrm{d}e)e = 0$, $(1-e)\mathrm{d}e(1-e) = 0$, we have

$$\theta = -(\Phi + \overline{\Phi})\mathrm{d}e\mathrm{d}e - (\Phi\overline{\Phi})\mathrm{d}e\mathrm{d}e.$$

Under the representation π, we have $\boxed{\pi(\mathrm{d}e) = \begin{bmatrix} 0 & -M^* \\ M & 0 \end{bmatrix}}$ and $\boxed{\pi(\mathrm{d}e\mathrm{d}e) = \begin{bmatrix} -M^*M & 0 \\ 0 & -MM^* \end{bmatrix}}$.

This yields the formula for the Yang-Mills action

$$\mathrm{trace}(\theta^2) = \mathrm{YM}(V) = 2(|\Phi + 1|^2 - 1)^2 \mathrm{Trace}\big((M^*M)^2\big),$$

where Φ is an arbitrary complex number. The action of the gauge group $\mathcal{U} = U(1) \times U(1)$ on the space of vector potentials, i.e., on Φ, is given by

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$$\gamma_u(V) = u\mathrm{d}u^* + uVu^*;$$

$\pi(ede) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & -M^* \\ M & 0 \end{pmatrix} = \begin{pmatrix} 0 & -M^* \\ 0 & 0 \end{pmatrix}$

for $u = u_a e + u_b(1-e)$, this gives

$$\begin{aligned}\gamma_u(V) &= \big(u_a e + u_b(1-e)\big)(\overline{u}_a\mathrm{d}e - \overline{u}_b\mathrm{d}e) \\ &\quad + \big(u_a e + u_b(1-e)\big)\big(-\overline{\Phi}e\mathrm{d}e + \Phi(1-e)\mathrm{d}e\big)\big(\overline{u}_a e + \overline{u}_b(1-e)\big) \\ &= e\mathrm{d}e + u_b\overline{u}_a(1-e)\mathrm{d}e - u_a\overline{u}_b e\mathrm{d}e - (1-e)\mathrm{d}e \\ &\quad - u_a\overline{u}_b\overline{\Phi}e\mathrm{d}e + u_b\overline{u}_a\Phi(1-e)\mathrm{d}e,\end{aligned}$$

which, on the variable $1 + \Phi$, just means multiplication by $u_b\overline{u}_a$.

In this very simple case, our action $\mathrm{YM}(V)$ reproduces the usual situation of broken symmetries (Figure 3); it has a non-unique minimum, $|\Phi + 1| = 1$, which is acted upon nontrivially by the gauge group.

$\Phi = e^{i\theta} - 1$ で上のYM(V)が minimal になる.

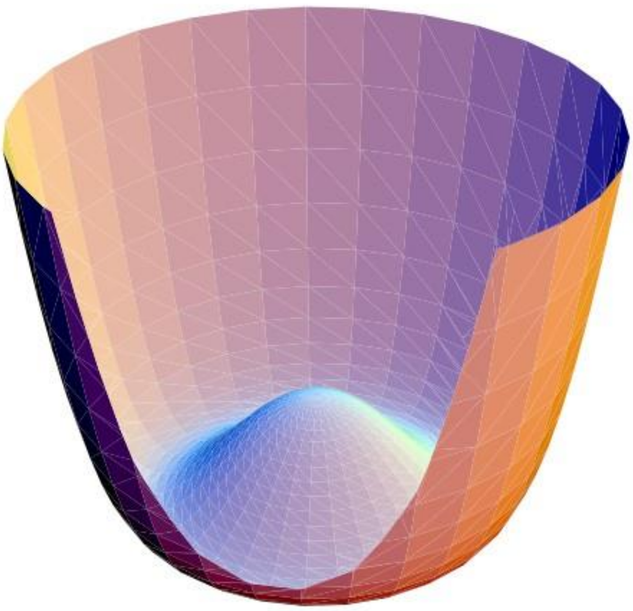


FIGURE 3. The potential $YM(V)$

The fermionic action is in this case given by

$$\langle \psi, \big(D + \pi(V)\big)\psi \rangle,$$

where the operator $D + \pi(V)$ is equal to

$$\begin{bmatrix} 0 & M^* \\ M & 0 \end{bmatrix} + \begin{bmatrix} 0 & \overline{\Phi} M^* \\ \Phi M & 0 \end{bmatrix} = \begin{bmatrix} 0 & (1+\overline{\Phi})M^* \\ (1+\Phi)M & 0 \end{bmatrix},$$

which is a term of Yukawa type coupling the fields $(1 + \Phi)$ and ψ.

$\boxed{\textit{Case } \beta):}$ Let us take for $\mathcal{E}$ the nontrivial bundle over $X = \{a, b\}$ with fibers of dimensions n_a and n_b, respectively, over a and b. This bundle is nontrivial if and only if $n_a \neq n_b$; we shall consider the simplest case $n_a = 2$, $n_b = 1$. The finitely generated projective module $\mathcal{E}$ of sections is of the form

$$\mathcal{E} = f\mathcal{A}^2,$$

where the idempotent $f \in M_2(\mathcal{A})$ is given by the formula

$$f = \begin{bmatrix} (1,1) & 0 \\ 0 & (1,0) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & e \end{bmatrix}$$

in terms of the notation of α).

$$= \begin{bmatrix} \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} & 0 \\ 0 & \begin{matrix} 1 & 0 \\ 0 & 0 \end{matrix} \end{bmatrix}$$

To the idempotent f there corresponds a particular compatible connection on $\mathcal{E}$, given by $\nabla_0\xi = f\mathrm{d}\xi$ with the obvious notation. An arbitrary compatible connection on $\mathcal{E}$ has the form

$$\nabla\xi = \nabla_0\xi + \rho\xi,$$

$\nabla = fd + \rho$

ρ: 1-form

where $\rho = \rho^*$ is a selfadjoint element of $M_2(\Omega^1_D(\mathcal{A}))$ such that $f\rho = \rho f = \rho$. If we write ρ as a matrix,

$$\rho = \begin{bmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{bmatrix},$$

$dee = \begin{pmatrix} 0 & 0 \\ M & 0 \end{pmatrix}$

$f = \begin{pmatrix} I & 0 \\ 0 & e \end{pmatrix}$

these conditions read:

$$e\rho_{21} = \rho_{21},\ e\rho_{22} = \rho_{22} = \rho_{22}e,\ \rho_{12}e = \rho_{12};$$

$de = \begin{pmatrix} 0 & -M^* \\ M & 0 \end{pmatrix}$

thus we get

$$\rho_{11} = -\overline{\Phi}_1 e\mathrm{d}e + \Phi_1(1-e)\mathrm{d}e,\ \rho_{21} = \overline{\Phi}_2 e\mathrm{d}e,\ \rho_{12} = \rho_{21}^*,\ \rho_{22} = 0,$$

where Φ_1 and Φ_2 are arbitrary complex numbers.

$ede = \begin{pmatrix} 0 & -M^* \\ 0 & 0 \end{pmatrix}$

The curvature θ is given by

$(fd+\rho)^2 f$

$= fd\rho f + fdfdf + \rho^2 + \rho f df$

$f(d\rho)f$

$$\begin{aligned} \theta &= f\mathrm{d}f\mathrm{d}f + f\mathrm{d}\rho f + \boxed{\rho^2} \\ &= \begin{bmatrix} 0 & 0 \\ 0 & e\mathrm{d}e\mathrm{d}e \end{bmatrix} + \begin{bmatrix} \mathrm{d}\rho_{11} & (\mathrm{d}\rho_{12})e \\ e\mathrm{d}\rho_{21} & 0 \end{bmatrix} + \begin{bmatrix} \rho_{11}\rho_{11} + \rho_{12}\rho_{21} & \rho_{11}\rho_{12} \\ \rho_{21}\rho_{11} & \rho_{21}\rho_{12} \end{bmatrix}. \end{aligned}$$

$(ede)^* = \begin{pmatrix} 0 & 0 \\ -M & 0 \end{pmatrix}$

An easy calculation gives the action $\mathrm{YM}(\nabla)$ in terms of the variables Φ_1, Φ_2:

$$\mathrm{YM}(\nabla) = \Big(1 + 2\big(1 - (|\Phi_1 + 1|^2 + |\Phi_2|^2)\big)^2\Big)\mathrm{Tr}\big((M^*M)^2\big).$$

$(ede)^* = -(dee)$

It is, by construction, invariant under the gauge group $U(1) \times U(2)$. What we learn in Example β) rather than in α) is that the choice of vacuum corresponds to a choice of connection minimizing the action, and in case β) there is really no preferred choice of ∇_0, the point 0 of the space of vector potentials (case α)) having no intrinsic meaning. In fact, the space of connections realizing the minimum of the action YM is a 3-sphere – N-G bosons

$$\{(\Phi_1, \Phi_2) \in \mathbb{C}^2;\ |\Phi_1 + 1|^2 + |\Phi_2|^2 = 1\}$$

whose elements have the following meaning. Let E_a (resp. E_b) be the fiber of our Hermitian bundle over the point a (resp. b) of X; then $\dim E_a = 2$, $\dim E_b = 1$. As we saw above, the differential $\mathrm{d} : \mathcal{A} \to \Omega^1(\mathcal{A})$ is the finite difference. One way to extend it to the bundle E is to use an isometry $u : E_b \to E_a$ and the formula

$$(\Delta\xi)_a = \xi_a - u\xi_b,\ (\Delta\xi)_b = \xi_b - u^*\xi_a.$$

All minimal connections ∇ are of the form

$$\nabla\xi = (\Delta\xi)_a \otimes e\mathrm{d}e + (\Delta\xi)_b \otimes (1-e)\mathrm{d}(1-e).$$

shall now return to the 4-dimensional case and work out the case of the product space in detail.

a: left handed
b: right handed

3. Example b). (4-dimensional Riemannian manifold V) $\times$ (2-point space X).

Let us fix the notation: V is a compact Riemannian spin 4-manifold, $\mathcal{A}_1$ the algebra of functions on V and $(\mathfrak{H}_1, D_1, \gamma_5)$ the Dirac K-cycle on $\mathcal{A}_1$, with its canonical $\mathbb{Z}/2$-grading γ_5 given by the orientation, let $\mathcal{A}_2$, $\mathfrak{H}_2$, D_2 be as in Example a) above, that is, $\mathcal{A}_2 = \mathbb{C} \oplus \mathbb{C}$, $\mathfrak{H}_2$ is the direct sum $\mathfrak{H}_{2,a} \oplus \mathfrak{H}_{2,b}$, and D_2 is given by the matrix

$$D_2 = \begin{bmatrix} 0 & M^* \\ M & 0 \end{bmatrix}.$$

Let $\mathcal{A} = \mathcal{A}_1 \otimes \mathcal{A}_2$, $\mathfrak{H} = \mathfrak{H}_1 \otimes \mathfrak{H}_2$ and $D = D_1 \otimes 1 + \gamma_5 \otimes D_2$.

The algebra $\mathcal{A}$ is commutative; it is the algebra of complex-valued functions on the space $Y = V \times X$, which is the union of two copies of the manifold V: $Y = V_a \cup V_b$.

Let us first compute the metric on Y associated with the K-cycle $(\mathfrak{H}, D)$:

$$d(p,q) = \sup_{f \in \mathcal{A}} \{ |f(p) - f(q)|;\ \|[D,f]\| \leq 1 \}.$$

To the decomposition $Y = V_a \cup V_b$ there corresponds a decomposition of $\mathcal{A}$ as $\mathcal{A}_a \oplus \mathcal{A}_b$, so that every $f \in \mathcal{A}$ is a pair (f_a, f_b) of functions on V. Also, to the decomposition

$$\mathfrak{H}_2 = \mathfrak{H}_{2,a} \oplus \mathfrak{H}_{2,b},$$

there corresponds a decomposition $\mathfrak{H} = \mathfrak{H}_a \oplus \mathfrak{H}_b$, in which the action of $f = (f_a, f_b) \in \mathcal{A}$ is diagonal:

a: left handed
b: right handed.

$$f \mapsto \begin{bmatrix} f_a & 0 \\ 0 & f_b \end{bmatrix} \in \mathcal{L}(\mathfrak{H}).$$

In this decomposition the operator D becomes

$(V)\otimes(X)$

$$D = \begin{bmatrix} \partial_V \otimes 1 & \gamma_5 \otimes M^* \\ \gamma_5 \otimes M & \partial_V \otimes 1 \end{bmatrix},$$

where ∂_V is the Dirac operator on V and γ_5 is the $\mathbb{Z}/2$-grading of its spinor bundle.

This gives us the formula for the "differential" of a function $f \in \mathcal{A}$:

$$[D,f] = \begin{bmatrix} i^{-1}\gamma(\mathrm{d}f_a) \otimes 1 & (f_b - f_a)\gamma_5 \otimes M^* \\ (f_a - f_b)\gamma_5 \otimes M & i^{-1}\gamma(\mathrm{d}f_b) \otimes 1 \end{bmatrix}.$$

$\gamma_5 = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{bmatrix}$

(*)

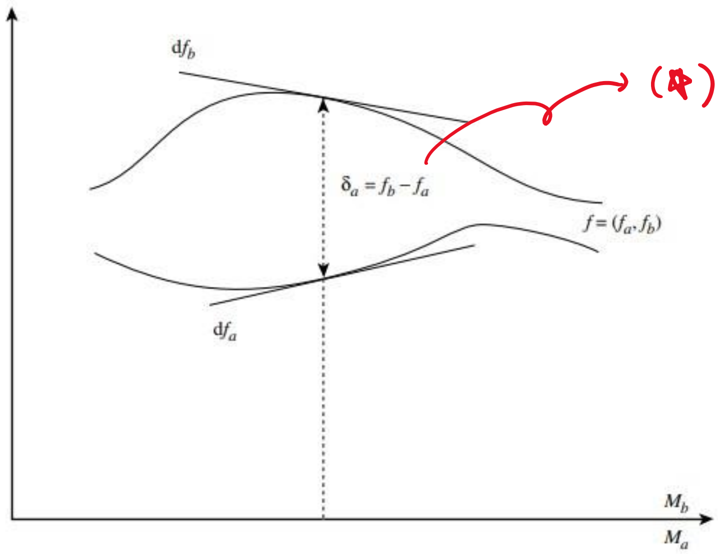
The differential $[D,f]$ thus contains three parts:

α) the usual differential $\mathrm{d}f_a$ of the restriction of f to the copy V_a of V;
β) the usual differential $\mathrm{d}f_b$ of the restriction of f to the copy V_b of V;
γ) the finite difference $\Delta f = f(p_a) - f(p_b)$, where p_a and p_b are the points of V_a and V_b above a given point p of V.

$\gamma^\mu = \begin{pmatrix} 0 & (1, -i\sigma) \\ (1, i\sigma) & 0 \end{pmatrix}$

$\gamma(\mathrm{d}f_a) = \gamma^\mu \partial_\mu f_a,$

Euclid.



(*)

FIGURE 4. Differential of a function on a double-space

The corresponding operator in $\mathfrak{H}$ is given by (**)

$$\begin{bmatrix} i^{-1}\gamma(\omega_a)\otimes 1 & \delta_a\gamma_5\otimes M^* \\ \delta_b\gamma_5\otimes M & i^{-1}\gamma(\omega_b)\otimes 1 \end{bmatrix} = \alpha;$$

$\begin{pmatrix} f_a & 0 \\ 0 & f_b \end{pmatrix} := (f_a, f_b)$

the bimodule structure over $\mathcal{A}$ is given, with obvious notation, by

$$\begin{aligned} (f_a, f_b)(\omega_a, \omega_b, \delta_a, \delta_b) &= (f_a\omega_a, f_b\omega_b, f_a\delta_a, f_b\delta_b), \\ (\omega_a, \omega_b, \delta_a, \delta_b)(f_a, f_b) &= (f_a\omega_a, f_b\omega_b, f_b\delta_a, f_a\delta_b). \end{aligned}$$

$\begin{pmatrix} \omega_a & \delta_a \\ \delta_b & \omega_b \end{pmatrix} := (\omega_a, \omega_b, \delta_a, \delta_b)$

The involution $*$ is given by $(\omega_a, \omega_b, \delta_a, \delta_b)^* = (-\overline{\omega}_a, -\overline{\omega}_b, \overline{\delta}_b, \overline{\delta}_a)$.

The terms δ_a and δ_b correspond to the bimodule of finite differences on passing from one copy V_a to the other copy V_b of V. Note that even though $\mathcal{A}$ is commutative, this bimodule is *not commutative*; for, if it were commutative then the finite difference would fail to be a derivation. With the above notation, the differential $f \in \mathcal{A} \mapsto \pi(\mathrm{d}f)$ reads as follows:

$$f = (f_a, f_b) \mapsto (\mathrm{d}f_a, \mathrm{d}f_b, f_b - f_a, f_a - f_b) \in \Omega^1_D.$$

When we project on V, the bimodule Ω^1_D can be viewed as a 10-dimensional bundle over V, given by two copies of the complexified cotangent bundle, and a trivial 2-dimensional bundle

$$T^*_p(V)_{\mathbb{C}} \oplus T^*_p(V)_{\mathbb{C}} \oplus \mathbb{C} \oplus \mathbb{C};$$

however, one must keep in mind the nontrivial bimodule structure in the last two terms. Figure IV.4 illustrates the situation.

As in the case of the Dirac operator on Riemannian manifolds (Section 1, Lemma 6), let us compute the pairs of operators of the form $\pi(\rho) = T_1$, $\pi(\mathrm{d}\rho) = T_2$ for $\rho \in \Omega^1(\mathcal{A})$. Given $\rho = \sum f_j \mathrm{d}g_j \in \Omega^1(\mathcal{A})$, with $f_j, g_j \in \mathcal{A}$, we have

← (※※) より

$$\pi(\rho) = \begin{bmatrix} i^{-1}\gamma(\omega_a) \otimes 1 & \delta_a \gamma_5 \otimes M^* \\ \delta_b \gamma_5 \otimes M & i^{-1}\gamma(\omega_b) \otimes 1 \end{bmatrix},$$

where $\omega_a = \sum f_{ja} \mathrm{d}g_{ja}$, $\omega_b = \sum f_{jb} \mathrm{d}g_{jb}$ and

$$\delta_a = \sum f_{ja}(g_{jb} - g_{ja}), \ \delta_b = \sum f_{jb}(g_{ja} - g_{jb}).$$

We have $\pi(\mathrm{d}\rho) = \sum \pi(\mathrm{d}f_j)\pi(\mathrm{d}g_j)$, which gives the 2×2 matrix

$$\pi(\mathrm{d}\rho) = \begin{bmatrix} -\gamma(\xi_a) \otimes 1 + (\delta_a + \delta_b) \otimes M^*M & \gamma_5 i^{-1}\gamma(\eta_a) \otimes M^* \\ \gamma_5 i^{-1}\gamma(\eta_b) \otimes M & -\gamma(\xi_b) \otimes 1 + (\delta_a + \delta_b) \otimes MM^* \end{bmatrix},$$

where $\xi_a = \sum \mathrm{d}f_{ja}\mathrm{d}g_{ja}$ and $\xi_b = \sum \mathrm{d}f_{jb}\mathrm{d}g_{jb}$ are sections of the Clifford algebra bundle C^2 over V, whereas

$$\begin{aligned} \eta_b &= \sum \big((f_{ja} - f_{jb})\mathrm{d}g_{ja} - (g_{ja} - g_{jb})\mathrm{d}f_{jb}\big), \\ \eta_a &= \sum \big((f_{jb} - f_{ja})\mathrm{d}g_{jb} - (g_{jb} - g_{ja})\mathrm{d}f_{ja}\big). \end{aligned}$$

Using the equalities

$$\begin{aligned} \mathrm{d}\delta_a &= \sum \big(f_{ja}(\mathrm{d}g_{jb} - \mathrm{d}g_{ja}) + (g_{jb} - g_{ja})\mathrm{d}f_{ja}\big), \\ \mathrm{d}\delta_b &= \sum \big(f_{jb}(\mathrm{d}g_{ja} - \mathrm{d}g_{jb}) + (g_{ja} - g_{jb})\mathrm{d}f_{jb}\big), \\ \omega_a &= \sum f_{ja}\mathrm{d}g_{ja}, \ \omega_b = \sum f_{jb}\mathrm{d}g_{jb}, \end{aligned}$$

we can rewrite η_a and η_b as

$$\eta_a = \omega_b - \mathrm{d}\delta_a - \omega_a, \ \eta_b = \omega_a - \mathrm{d}\delta_b - \omega_b.$$

As in the Riemannian case (Lemma 6 of Section 1), the sections ξ_a and ξ_b of C^2 are arbitrary except for $\sigma_2(\xi_a) = \mathrm{d}\omega_a$ and $\sigma_2(\xi_b) = \mathrm{d}\omega_b$. This shows that the subspace $\pi\big(\mathrm{d}(J_0 \cap \Omega^1)\big)$ of $\pi(\Omega^2)$ is the space of 2×2 matrices of operators of the form

$$T = \begin{bmatrix} \gamma(\xi_a) \otimes 1 & 0 \\ 0 & \gamma(\xi_b) \otimes 1 \end{bmatrix},$$

where ξ_a and ξ_b are sections of C^0, i.e., are just arbitrary scalar-valued functions on V, so that $\gamma(\xi_a) = \xi_a$, $\gamma(\xi_b) = \xi_b$.

A general element of $\pi(\Omega^2)$ is a 2×2 matrix of operators of the form

$$T = \begin{bmatrix} -\gamma(\alpha_a) \otimes 1 + h_a \otimes M^*M & \gamma_5 i^{-1}\gamma(\beta_a) \otimes M^* \\ \gamma_5 i^{-1}\gamma(\beta_b) \otimes M & -\gamma(\alpha_b) \otimes 1 + h_b \otimes MM^* \end{bmatrix},$$

where α_a, α_b are arbitrary sections of C^2, h_a, h_b are arbitrary functions on V and β_a, β_b are arbitrary sections of C^1 (i.e., 1-forms).

Lemma 6. *Assume that M^*M is not a scalar multiple of the identity matrix. Then, an element of Ω_D^2 is given by:*

1) *a pair of ordinary 2-forms α_a, α_b on V;*
2) *a pair of ordinary 1-forms β_a, β_b on V;*
3) *a pair of scalar functions h_a, h_b on V.*

The hypothesis $M^*M \neq \lambda$ is important since otherwise the functions h_a, h_b are eliminated by $\pi\big(\mathrm{d}(J_0 \cap \Omega^1)\big)$.

Using the above computation of $\pi(\mathrm{d}\rho)$ we can, moreover, compute the sextuple $(\alpha_a, \alpha_b, \beta_a, \beta_b, h_a, h_b)$; then for the differential $\mathrm{d}\omega$ of an element $\omega = (\omega_a, \omega_b, \delta_a, \delta_b)$ of Ω_D^1, we get:

微分
2-form

1) $\alpha_a = \mathrm{d}\omega_a$, $\alpha_b = \mathrm{d}\omega_b$;
2) $\beta_a = \omega_b - \omega_a - \mathrm{d}\delta_a$, $\beta_b = \omega_a - \omega_b - \mathrm{d}\delta_b$;
3) $h_a = \delta_a + \delta_b$, $h_b = \delta_a + \delta_b$.

Thus, we see that the differential $\mathrm{d}\omega \in \Omega_D^2$ involves the differential terms $\mathrm{d}\omega_a$, $\mathrm{d}\omega_b$, $\mathrm{d}\delta_a$ and $\mathrm{d}\delta_b$ as well as the finite-difference terms $\omega_a - \omega_b$ and $\delta_a + \delta_b$, but in combinations such as $\omega_b - \omega_a - \mathrm{d}\delta_a$ imposed by $\mathrm{d}(\mathrm{d}f) = 0$. Let us compute the product $\omega\omega' \in \Omega_D^2$ of two elements $\omega = (\omega_a, \omega_b, \delta_a, \delta_b)$, $\omega' = (\omega'_a, \omega'_b, \delta'_a, \delta'_b)$ of Ω_D^1; we get:

積
2-form

1) $\alpha_a = \omega_a \wedge \omega'_a$, $\alpha_b = \omega_b \wedge \omega'_b$;
2) $\beta_a = \delta_a\omega'_b - \delta'_a\omega_a$, $\beta_b = \delta_b\omega'_a - \delta'_b\omega_b$;
3) $h_a = \delta_a\delta'_b$, $h_b = \delta_b\delta'_a$.

The next step is to determine the inner product on the space Ω_D^2 of 2-forms given in Section 1. By definition, we take the orthogonal complement of $\pi\big(\mathrm{d}(J_0 \cap \Omega^1)\big)$ in $\pi(\Omega^2)$ with the inner product $\langle T_1, T_2\rangle = \mathrm{Tr}_\omega(T_1^* T_2 |D|^{-4})$. An easy calculation then gives:

Lemma 7. *Let $\lambda(M^*M)$ be the orthogonal projection (for the Hilbert-Schmidt scalar product) of the matrix M^*M onto the scalar matrices λid. Then, the squared norm of an element $(\alpha_a, \alpha_b, \beta_a, \beta_b, h_a, h_b)$ of Ω_D^2 is given by $(8\pi^2)^{-1}$ times*

$$\int_V (N_a\|\alpha_a\|^2 + N_b\|\alpha_b\|^2)\mathrm{d}v + \mathrm{tr}(M^*M)\int_V (\|\beta_a\|^2 + \|\beta_b\|^2)\mathrm{d}v$$
$$+ \;\; \mathrm{tr}\Big(\big(M^*M - \lambda(M^*M)\big)^2\Big) \times \int_V (\|h_a\|^2 + \|h_b\|^2)\mathrm{d}v,$$

where $N_a = \dim \mathfrak{H}_{2,a}$ and $N_b = \dim \mathfrak{H}_{2,b}$.

Since the restriction of E to V_a is trivial with fiber $\mathbb{C}^2$, we may as well describe ∇_a by a 2×2 matrix $\begin{bmatrix} \omega_{11}^a & \omega_{12}^a \\ \omega_{21}^a & \omega_{22}^a \end{bmatrix}$ of 1-forms on V that is skew-adjoint. Similarly, ∇_b is given by a single skew-adjoint 1-form $[\omega_{11}^b]$, and u by a pair of complex fields $(1+\varphi_1, \varphi_2) = u(1)$. With these notations, the connection ∇ is given by the equality

$$\nabla \xi = f d\xi + \rho \xi \in \mathcal{E} \otimes_{\mathcal{A}} \Omega_D^1 \ \forall \xi \in \mathcal{E},$$

where $\mathcal{E} = f\mathcal{A}^2$, $f \in M_2(\mathcal{A})$ being the idempotent $f = \begin{bmatrix} 1 & 0 \\ 0 & e \end{bmatrix}$, $e = \overset{(1,0)?}{\cancel{(0,1)}} \in \mathcal{A}$, and where $\rho \in M_2(\Omega_D^1)$ is the 2×2 matrix whose entries are the following elements of Ω_D^1:

$$\begin{aligned} \rho_{11} &= (\omega_{11}^a, \omega_{11}^b, \varphi_1, \overline{\varphi}_1), \\ \rho_{12} &= (\omega_{12}^a, 0, 0, \overline{\varphi}_2), \\ \rho_{21} &= (\omega_{21}^a, 0, \varphi_2, 0), \\ \rho_{22} &= (\omega_{22}^a, 0, 0, 0), \end{aligned}$$

$$\rho = \begin{pmatrix} \omega_{11}^a & \omega_{12}^a & \varphi_1 & 0 \\ \omega_{21}^a & \omega_{22}^a & \varphi_2 & 0 \\ \overline{\varphi}_1 & \overline{\varphi}_2 & \omega_{11}^b & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} \omega^a & \delta^a \\ \delta^b & \omega^b \end{pmatrix}$$

or, equivalently,

$$\rho = \left[\begin{bmatrix} \omega_{11}^a & \omega_{12}^a \\ \omega_{21}^a & \omega_{22}^a \end{bmatrix}, \begin{bmatrix} \omega_{11}^b & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} \varphi_1 & 0 \\ \varphi_2 & 0 \end{bmatrix}, \begin{bmatrix} \overline{\varphi}_1 & \overline{\varphi}_2 \\ 0 & 0 \end{bmatrix} \right].$$

$$\delta_a = \begin{pmatrix} \varphi_1 & 0 \\ \varphi_2 & 0 \end{pmatrix}$$

The curvature θ is then the following element of $fM_2(\Omega_D^2)f$:

aとbの違いから出る.

$$\theta = f d f d f + f d\rho f + \rho^2,$$

$$\delta_b = \begin{pmatrix} \overline{\varphi}_1 & \overline{\varphi}_2 \\ 0 & 0 \end{pmatrix}$$

which is easily determined using the above computation of

$$d : \Omega_D^1 \to \Omega_D^2, \ \wedge : \Omega_D^1 \times \Omega_D^1 \to \Omega_D^2.$$

As we saw in Lemma 6, elements of Ω_D^2 have a differential degree and a finite-difference degree (α, β) adding up to 2. Let us thus begin with terms in θ of bi-degree $(2,0)$. To compute them we just use the formulas 1) following Lemma 6:

$$\boxed{\alpha_a = d\omega_a, \ \alpha_b = d\omega_b} \ \boxed{\alpha_a = \omega_a \wedge \omega_a', \ \alpha_b = \omega_b \wedge \omega_b'.}$$

We thus see that the component $\theta^{(2,0)}$ of bi-degree $(2,0)$ is the following 2×2 matrix of 2-forms on $V_a \cup V_b$:

$$\theta_a^{(2,0)} = d\omega^a + \omega^a \wedge \omega^a, \ \theta_b^{(2,0)} = d\omega^b + \omega^b \wedge \omega^b = \begin{bmatrix} d\omega_{11}^b & 0 \\ 0 & 0 \end{bmatrix}.$$

Next, we look at the component $\theta^{(1,1)}$ of bi-degree $(1,1)$ and use the formulas 2):

$$\boxed{\begin{aligned} \beta_a &= \omega_b - \omega_a - d\delta_a, \\ \beta_b &= \omega_a - \omega_b - d\delta_b, \end{aligned}} \quad \boxed{\begin{aligned} \beta_a &= \delta_a \omega_b' - \omega_a \delta_a', \\ \beta_b &= \delta_b \omega_a' - \omega_b \delta_b'. \end{aligned}}$$

$\rho = \sum f_j d g_j \in \Omega^1(\mathcal{A})$ with $f_j, g_j \in \mathcal{A}$

$$\pi(\rho) = \begin{bmatrix} i^{-1}\gamma(\omega_a)\otimes 1, & \delta_a \gamma_5 \otimes M^* \\ \delta_b \gamma_5 \otimes M, & i^{-1}\gamma(\omega_b)\otimes 1 \end{bmatrix}.$$

where $\underset{SO(2)\text{ field}}{\omega_a = \sum_j f_{ja}\, d g_{ja}}$, $\underset{U(1)\text{ field}}{\omega_b = \sum_j f_{jb}\, d g_{jb}}$

$$\underbrace{\delta_a = \sum f_{ja}(g_{jb} - g_{ja}),\ \delta_b = \sum f_{jb}(g_{ja} - g_{jb})}_{\text{Higgs}}$$

$\pi(d\rho) =$

$[D, \pi(\rho)] = D\pi(\rho) + \pi(\rho) D$

$$= \begin{bmatrix} -\gamma(d\omega_a)\otimes 1 + (\delta_a+\delta_b)\otimes M^*M, & \gamma_5 i^{-1}\gamma(\omega_b - d\delta_a - \omega_a)\otimes M^* \\ \gamma_5 i^{-1}\gamma(\omega_a - d\delta_b - \omega_b)\otimes M, & -\gamma(d\omega_b)\otimes 1 + (\delta_a+\delta_b)\otimes M M^* \end{bmatrix}$$

$\pi(\rho)\wedge\pi(\rho)$

$$= \begin{bmatrix} -\gamma(\omega_a\wedge\omega_a)\otimes 1 + \delta_a\delta_b\otimes M^*M, & (-i^{-1}\gamma_5\gamma(\omega_a)\delta_a + i^{-1}\gamma_5\delta_a\gamma(\omega_b))\otimes M^* \\ (i^{-1}\gamma_5\delta_b\gamma(\omega_a) - i^{-1}\gamma_5\gamma(\omega_b)\delta_b)\otimes M, & \delta_b\delta_a\otimes MM^* - \gamma(\omega_b\wedge\omega_b)\otimes 1 \end{bmatrix}$$

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Let $\omega_a = \begin{pmatrix} \omega_a^{11} & \omega_a^{12} \\ \omega_a^{21} & \omega_a^{22} \end{pmatrix}$, $\omega_b = \begin{pmatrix} \omega_b^{11} & 0 \\ 0 & 0 \end{pmatrix}$

$\delta_a = \begin{pmatrix} \varphi_1 & 0 \\ \varphi_2 & 0 \end{pmatrix}$, $\delta_b = \begin{pmatrix} \bar{\varphi}_1 & \bar{\varphi}_2 \\ 0 & 0 \end{pmatrix}$

for $\pi(\rho)$.

For such ρ, $f\rho = \rho f = \rho$

$\Rightarrow f = \begin{pmatrix} I & 0 \\ 0 & e \end{pmatrix}$, where $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

- connection
  $fd + \rho$

- curvature
  $\underline{\theta = (fd+\rho)(fd+\rho)f = fdfdf + f(d\rho)f + \rho^2}$

$D_M = \begin{pmatrix} 0 & \tilde{M}^* \\ \tilde{M} & 0 \end{pmatrix}$, where $\tilde{M} = \begin{pmatrix} M & 0 \\ 0 & M \end{pmatrix}$, $\tilde{M}^* = \begin{pmatrix} M^* & 0 \\ 0 & M^* \end{pmatrix}$.

$df = [D_M, f] = \begin{pmatrix} 0 & -\tilde{M}^*(1-e) \\ (1-e)\tilde{M}, & 0 \end{pmatrix}$

$\therefore fdfdf = \begin{pmatrix} 0 & & \\ & -1 & \\ & & 0 \\ & & & 0 \end{pmatrix} M^*M$

calculation of $\Theta = \begin{pmatrix} \theta^{(1,1)} & \theta^{(1,2)} \\ \theta^{(2,1)} & \theta^{(2,2)} \end{pmatrix}$

n.b. $\left(\begin{array}{l} \omega: \text{skew adjoint} \quad \text{ie } \omega_a^* = -\omega_a \\ \gamma^\mu \gamma_5 = -\gamma_5 \gamma^\mu \end{array} \right.$

$$\theta^{(1,1)} = -\gamma(\alpha_a)\otimes 1 + h_a \otimes M^*M \longrightarrow \theta_a^{(11)}$$

$$\theta^{(1,2)} = \gamma_5 i^{-1}\gamma(\beta_a)\otimes M^* \longrightarrow \theta_a^{(02)}$$

$$\theta^{(2,1)} = \gamma_5 i^{-1}\gamma(\beta_b)\otimes M \longrightarrow \theta_b^{(02)}$$

$$\theta^{(2,2)} = -\gamma(\alpha_b)\otimes 1 + h_b \otimes MM^* \longrightarrow \theta_b^{(11)}$$

where

$$\alpha_a = d\omega_a + \omega_a \wedge \omega_a, \qquad \alpha_b = d\omega_b$$

$$\beta_a = \begin{pmatrix} -d\varphi_1 - (\omega_a^{11} - \omega_b^{11})(\varphi_1 + 1) - \omega_a^{12}\varphi_2 & , 0 \\ -d\varphi_2 - \omega_a^{21}(\varphi_1 + 1) - (\omega_a^{22} - \omega_b^{11})\varphi_2 & , 0 \end{pmatrix}$$

$$\beta_b = \begin{pmatrix} -d\bar\varphi_1 + (\omega_a^{11} - \omega_b^{11})(\bar\varphi_1 + 1) + \omega_a^{21}\bar\varphi_2 & , -d\bar\varphi_2 + \omega_a^{12}(\bar\varphi_1 + 1) + (\omega_a^{22} - \omega_b^{11})\bar\varphi_2 \\ 0 & , 0 \end{pmatrix}$$

$$h_a = \begin{pmatrix} \varphi_1 + \bar\varphi_1 + \varphi_1\bar\varphi_1 & , \bar\varphi_2 + \varphi_1\bar\varphi_2 \\ \varphi_2 + \varphi_2\bar\varphi_1 & , \varphi_2\bar\varphi_2 - 1 \end{pmatrix}.$$

$$h_b = \begin{pmatrix} \varphi_1 + \bar\varphi_1 + \bar\varphi_1\varphi_1 + \bar\varphi_2\varphi_2 & , 0 \\ 0 & , 0 \end{pmatrix}$$

We can get I_0, I_1, I_2.

Lemma*

$\lambda(M^*M)$: orthogonal projection (for the Hilbert-Schmidt scalar product) of the matrix M^*M onto the scalar matrices $\lambda\,\mathrm{id}$.
Then, the squared norm of an element $(\alpha_a, \alpha_b, \beta_a, \beta_b, h_a, h_b)$ of Ω_D^2 is given by $(8\pi^2)^{-1}$ times

$$\int_V (N_a\|\alpha_a\|^2 + N_b\|\alpha_b\|^2)\,dv + \mathrm{tr}(M^*M)\int_V (\|\beta_a\|^2 + \|\beta_b\|^2)\,dv$$
$$+ \mathrm{tr}\left((M^*M - \lambda(M^*M))^2\right)\int_V (\|h_a\|^2 + \|h_b\|^2)\,dv$$

where $N_a = \dim \mathfrak{H}_{2,a}$ and $N_b = \dim \mathfrak{H}_{2,b}$

$$YM(\rho) = \int_V \mathrm{trace}(\theta^2)\,dv$$

Thus, $\theta^{(1,1)}$ is the following 2×2 matrix of 1-forms on $V_a \cup V_b$:

$$\begin{aligned}
\theta_a^{(1,1)} &= \left(\begin{bmatrix}\omega^b_{11} & 0\\ 0 & 0\end{bmatrix} - \begin{bmatrix}\omega^a_{11} & \omega^a_{12}\\ \omega^a_{21} & \omega^a_{22}\end{bmatrix} - \begin{bmatrix}\mathrm{d}\varphi_1 & 0\\ \mathrm{d}\varphi_2 & 0\end{bmatrix}\right)\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\\
&\quad + \begin{bmatrix}\varphi_1 & 0\\ \varphi_2 & 0\end{bmatrix}\begin{bmatrix}\omega^b_{11} & 0\\ 0 & 0\end{bmatrix} - \begin{bmatrix}\omega^a_{11} & \omega^a_{12}\\ \omega^a_{21} & \omega^a_{22}\end{bmatrix}\begin{bmatrix}\varphi_1 & 0\\ \varphi_2 & 0\end{bmatrix}\\
&= \begin{bmatrix}-\mathrm{d}\varphi_1 - (\omega^a_{11}-\omega^b_{11})(\varphi_1+1) - \omega^a_{12}\varphi_2 & 0\\ -\mathrm{d}\varphi_2 - \omega^a_{21}(\varphi_1+1) - (\omega^a_{22}-\omega^b_{11})\varphi_2 & 0\end{bmatrix}.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
\theta_b^{(1,1)} &= \begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\left(\begin{bmatrix}\omega^a_{11} & \omega^a_{12}\\ \omega^a_{21} & \omega^a_{22}\end{bmatrix} - \begin{bmatrix}\omega^b_{11} & 0\\ 0 & 0\end{bmatrix} - \begin{bmatrix}\mathrm{d}\overline{\varphi}_1 & \mathrm{d}\overline{\varphi}_2\\ 0 & 0\end{bmatrix}\right)\\
&\quad + \begin{bmatrix}\overline{\varphi}_1 & \overline{\varphi}_2\\ 0 & 0\end{bmatrix}\begin{bmatrix}\omega^a_{11} & \omega^a_{12}\\ \omega^a_{21} & \omega^a_{22}\end{bmatrix} - \begin{bmatrix}\omega^b_{11} & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}\overline{\varphi}_1 & \overline{\varphi}_2\\ 0 & 0\end{bmatrix}\\
&= \begin{bmatrix}\mathrm{d}\overline{\varphi}_1+(\omega^a_{11}-\omega^b_{11})(\overline{\varphi}_1+1)+\omega^a_{21}\overline{\varphi}_2 & -\mathrm{d}\overline{\varphi}_2+\omega^a_{12}(\overline{\varphi}_1+1)+(\omega^a_{22}-\omega^b_{11})\varphi_2\\ 0 & 0\end{bmatrix}.
\end{aligned}$$

Finally, we have to compute the component $\theta^{(0,2)}$; we use the formulas 3):

$$h_a = \delta_a + \delta_b, \qquad h_a = \delta_a\delta'_b,$$

?

$$h_b = \delta_a + \delta_b, \qquad h_b = \delta_b\delta'_a.$$

We then have

$$\begin{aligned}
\theta_a^{(0,2)} &= \begin{bmatrix}\varphi_1 & 0\\ \varphi_2 & 0\end{bmatrix} + \begin{bmatrix}\overline{\varphi}_1 & \overline{\varphi}_2\\ 0 & 0\end{bmatrix} + \begin{bmatrix}\varphi_1 & 0\\ \varphi_2 & 0\end{bmatrix}\begin{bmatrix}\overline{\varphi}_1 & \overline{\varphi}_2\\ 0 & 0\end{bmatrix} + \begin{bmatrix}0 & 0\\ 0 & -1\end{bmatrix}\\
&= \begin{bmatrix}\varphi_1+\overline{\varphi}_1+\varphi_1\overline{\varphi}_1 & \overline{\varphi}_2(1+\varphi_1)\\ \varphi_2(1+\overline{\varphi}_1) & \cancel{0}\ \varphi_2\overline{\varphi}_2 - 1?\end{bmatrix},
\end{aligned}$$

~ fdfdf

fdpf.

$\delta_a + \delta_b$

$\delta_b\delta_a$

?

$$\begin{aligned}
\theta_b^{(0,2)} &= \cancel{\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}} + \begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}\varphi_1+\overline{\varphi}_1 & \overline{\varphi}_2\\ \varphi_2 & 0\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\\
&\quad + \begin{bmatrix}\overline{\varphi}_1 & \overline{\varphi}_2\\ 0 & 0\end{bmatrix}\begin{bmatrix}\varphi_1 & 0\\ \varphi_2 & 0\end{bmatrix}\\
&= \begin{bmatrix}\cancel{1}+\varphi_1+\overline{\varphi}_1+\varphi_1\overline{\varphi}_1+\varphi_2\overline{\varphi}_2 & 0\\ 0 & 0\end{bmatrix}.
\end{aligned}$$

これで
1+ω倍する
となって
つじつま
があう.

Thus,

$$\mathrm{YM}(\nabla) = I_2 + I_1 + I_0,$$

where each I_j is the integral over M of a Lagrangian density given by the following formulas.

First, for I_2:

$$|\mathrm{d}\omega^a + \omega^a\wedge\omega^a|^2 N_a + |\mathrm{d}\omega^b|^2 N_b,$$

pure gauge boson

?

$$\theta_2^{(0,2)} = \begin{bmatrix}\varphi_1+\overline{\varphi}_1+\varphi_1\overline{\varphi}_1, & \overline{\varphi}_2(1+\varphi_1)\\ \varphi_2(1+\overline{\varphi}_1), & \varphi_2\overline{\varphi}_2 - 1\end{bmatrix}$$

$$\theta_b^{(0,2)} = \begin{bmatrix}\varphi_1+\overline{\varphi}_1+\varphi_1\overline{\varphi}_1+\varphi_2\overline{\varphi}_2, & 0\\ 0 & 0\end{bmatrix}$$

where $N_a = \dim \mathfrak{H}_{2,a}$, $N_b = \dim \mathfrak{H}_{2,b}$ and the norms are the squared norms for the curvatures of the connections ∇^a and ∇^b, respectively.

Next, for I_1:

kinetic term for Higgs field → weak boson : mass を与える.

$$2\left|\nabla\begin{pmatrix}1+\varphi_1\\ \varphi_2\end{pmatrix}\right|^2 \operatorname{tr}(M^*M),$$

where ∇ is the covariant differentiation of a pair of scalar fields, given by

$$\mathrm{d} + \begin{bmatrix}\omega^a_{11} - \omega^b_{11} & \omega^a_{12}\\ \omega^a_{21} & \omega^a_{22} - \omega^b_{11}\end{bmatrix}.$$

Higgs self-interaction

Finally, for I_0:

$$\Big(1 + 2\big(1 - (|1+\varphi_1|^2 + |\varphi_2|^2)\big)^2\Big)\operatorname{Tr}\Big(\big(\lambda^\perp(M^*M)\big)^2\Big),$$

where $\lambda^\perp$ is the orthogonal projection in the Hilbert–Schmidt space of matrices onto the orthogonal complement of the scalar multiples of the identity. These terms are obtained, with the right coefficients, from the computation of the Hilbert space norm on Ω^2_D in Lemma 7.

The fermionic action is even easier to compute. We have

$$\langle\psi, D_\nabla\psi\rangle = J_0 + J_1,$$

where $\psi \in \mathcal{E}\otimes_{\mathcal{A}}\mathfrak{H}$, $\gamma\psi = \psi$, is given by a pair of left-handed sections of $S\otimes\mathfrak{H}_{2,a}$ denoted by $\begin{bmatrix}\psi^a_1\\ \psi^a_2\end{bmatrix}$, and a right-handed section of $S\otimes\mathfrak{H}_{2,b}$ denoted by ψ^b. Both J_0 and J_1 are given by Lagrangian densities:

1 ↑
2 ↓

fermion kinetic term

$$J_0 \;:\; \overline{\psi}^a(\partial + i^{-1}\gamma(\omega^a))\psi^a + \overline{\psi}^b(\partial + i^{-1}\gamma(\omega^b))\psi^b,$$

Yukawa. a? b? mass を与える係数

$$J_1 \;:\; \overline{\psi}_b M[(1+\varphi_1), \varphi_2]\psi_a + \text{h.c.}$$

We can now make the point concerning this example b): modulo a few nuances that we shall deal with, the five terms of our action

$$I_0 + I_1 + I_2 + J_0 + J_1$$

are the five terms of the Glashow–Weinberg–Salam unification of electromagnetic and weak forces for N generations of leptons (where $N = N_a = N_b$ is the dimension of $\mathfrak{H}_{2,a}$ and $\mathfrak{H}_{2,b}$).

Let us describe for instance, from [**197**], and using the conventional notation of physics, the five constituents of the G.W.S. Lagrangian, which we write directly in the Euclidean (i.e., imaginary time) framework. For each constituent, we give the corresponding fields and Lagrangian:

$$\mathcal{L} = \mathcal{L}_G + \mathcal{L}_f + \mathcal{L}_\Phi + \mathcal{L}_Y + \mathcal{L}_V,$$

where

1) $\mathcal{L}_G$: The *pure gauge boson* part is just

$$\mathcal{L}_G = \frac{1}{4}(G_{\mu\nu a}G_a^{\mu\nu}) + \frac{1}{4}(F_{\mu\nu}F^{\mu\nu}),$$

where $G_{\mu\nu a} = \partial_\mu W_{\nu a} - \partial_\nu W_{\mu a} + g\varepsilon_{abc}W_{\mu b}W_{\nu c}$ and $F_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$ are the field strength tensors of an $SU(2)$ gauge field $W_{\mu a}$ and a $U(1)$ gauge field B_μ. (Einstein summation over repeated indices is used here.)

2) $\mathcal{L}_f$: The *fermion kinetic term* has the form

$$\mathcal{L}_f = -\sum\left[\overline{f}_L\gamma^\mu\left(\partial_\mu + ig\frac{\tau_a}{2}W_{\mu a} + ig'\frac{Y_L}{2}B_\mu\right)f_L + \overline{f}_R\gamma^\mu\left(\partial_\mu + ig'\frac{Y_R}{2}B_\mu\right)f_R\right],$$

where the f_L (resp. f_R) are the left-handed (resp. right-handed) fermion fields, which for leptons and for each generation are given by a pair, i.e., an isodoublet, of left-handed spinors (such as $\begin{bmatrix}\nu_L\\ e_L\end{bmatrix}$), and a singlet ($e_R$), i.e., a right-handed spinor.

We shall return later to the hypercharges Y_L and Y_R, which for leptons are given by $Y_L = -1$, $Y_R = -2$.

3) $\mathcal{L}_\Phi$: The *kinetic terms for the Higgs fields* are

$$\mathcal{L}_\Phi = -\left|\left(\partial_\mu + ig\frac{\tau_a}{2}W_{\mu a} + i\frac{g'}{2}B_\mu\right)\Phi\right|^2,$$

where $\Phi = \begin{bmatrix}\Phi_1\\ \Phi_2\end{bmatrix}$ is an $SU(2)$ doublet of complex scalar fields Φ_1 and Φ_2 with hypercharge $Y_\Phi = 1$.

4) $\mathcal{L}_Y$: The *Yukawa coupling* of Higgs fields with fermions is

$$\mathcal{L}_Y = -\sum[H_{ff'}(\overline{f}_L\cdot\Phi)f'_R + H^*_{ff'}\overline{f'}_R(\Phi^+\cdot f_L)],$$

where $H_{ff'}$ is a general coupling matrix in the space of different families.

5) $\mathcal{L}_V$: The *Higgs self-interaction* is the potential

$$\mathcal{L}_V = \mu^2(\Phi^+\Phi) - \frac{1}{2}\lambda(\Phi^+\Phi)^2,$$

where $\lambda > 0$ and $\mu^2 > 0$ are scalars.

All of these terms are deeply rooted in both experimental and theoretical physics, but we postpone an elaboration of this point to the complete model invoking quarks and strong interactions as well. For the moment we shall establish a dictionary, or change-of-variables, between our action and the Glashow–Weinberg–Salam action.

The first obvious nuance between the two actions is that our action involves a $U(1)$ and a $U(2)$ gauge field while the G.W.S. action involves a $U(1)$ and an $SU(2)$ gauge field (however, cf. [**579**] for an interesting perspective on this point). We shall thus, *in an artificial manner*, reduce our theory to $U(1) \times SU(2)$ by imposing on the connections $\nabla^a = \mathrm{d} + \omega^a$ and $\nabla^b = \mathrm{d} + \omega^b$ the following condition:

$$\text{Ad hoc condition: } \operatorname{tr}(\omega^a) = \omega^b.$$

Let us now spell out the dictionary.

Noncommutative geometry	**Classical field theory**
vector $\psi \in \mathcal{E} \otimes_{\mathcal{A}} \mathfrak{H}, \gamma\psi = \psi$	chiral fermion f
differential components of connection ω^a, ω^b	pure gauge bosons W, B
finite-difference component of connection $(1 + \delta^a), \delta^b$	Higgs field Φ
I_2	$\mathcal{L}_G$
I_1	$\mathcal{L}_\Phi$
I_0	$\mathcal{L}_V$
J_0	$\mathcal{L}_f$
J_1	$\mathcal{L}_Y$

It is moreover straightforward, using the above "ad hoc condition", to work out the change of variables from our fields ψ, ω, δ to the fields f, W, B, Φ which gives the equality

$$g^{-2}\mathrm{YM}(\nabla) + \langle \psi, D_\nabla \psi \rangle = \mathcal{L}(f, W, B, \Phi),$$

where the right-hand side is a special case of the G.W.S. Lagrangian, with a few constraints. These relations are of limited use for three reasons. The first is that the model incorporates neither the quarks nor the strong interaction, the second is the artificial nature of the "ad hoc condition", and the third and most important is that, due to renormalization, the coupling constants such as g, g', μ, λ, $H_{ff'}$ that appear in the G.W.S. model are all functions of an effective energy Λ to which, for the moment, we can give no preferred value.

In Section 5 we shall remove the first two defects by explaining how to incorporate quarks and strong interactions, as well as how the hypercharges of physics occur, from a conceptual point of view.

Lagrangian of boson fields

$\mathcal{L}_G$: pure gauge boson

$$\mathcal{L}_G = \frac{1}{4}(G_{\mu\nu a} G_a^{\mu\nu}) + \frac{1}{4}(F_{\mu\nu} F^{\mu\nu})$$

$$G_{\mu\nu a} = \partial_\mu W_{\nu a} - \partial_\nu W_{\mu a} + g\,\varepsilon_{abc} W_{\mu b} W_{\nu c}$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

$W_{\mu a}$: $SU(2)$ gauge field.

A_μ : $U(1)$ gauge field.

$\mathcal{L}_\Phi$: kinetic terms for the Higgs fields

$$\mathcal{L}_\Phi = -\left|\left(\partial_\mu + i g \frac{\tau_a}{2} W_{\mu a} + i \frac{g'}{2} A_\mu\right)\Phi\right|^2$$

$\Phi = \begin{bmatrix} \Phi_1 \\ \Phi_2 \end{bmatrix}$: $SU(2)$ doublet of complex scalar fields Φ_1, Φ_2.

$\mathcal{L}_V$: Higgs self interaction

$$\mathcal{L}_V = \mu^2(\Phi^\dagger\Phi) - \frac{1}{2}\lambda(\Phi^\dagger\Phi)^2$$

$\lambda > 0$, $\mu^2 > 0$

I_2 : $\mathcal{L}_G$

$$I_2 = |d\omega_a + \omega_a \wedge \omega_a|^2 N_a + |d\omega_b|^2 N_b,$$

where ω_a : $SU(2)$ gauge field
ω_b : $U(1)$ gauge field.) 1-form
$N_a = \dim \mathfrak{H}_{2,a}$
$N_b = \dim \mathfrak{H}_{2,b}$.

I_1 : $\mathcal{L}_\Phi$

$$I_1 = 2\left|\nabla\begin{pmatrix}1+\varphi_1\\ \varphi_2\end{pmatrix}\right|^2 \mathrm{tr}(M^*M),$$

M: mass term of Higgs fields related to λ, μ. → Weak boson に mass を与える.

where $\nabla = d + \begin{bmatrix}\omega_a^{11} - \omega_b^{11} & \omega_a^{12} \\ \omega_a^{21} & \omega_a^{22} - \omega_b^{11}\end{bmatrix}$

$\begin{pmatrix}1+\varphi_1\\ \varphi_2\end{pmatrix}$: doublet of complex scalar fields Φ_1, Φ_2, whose symmetry is spontaneously broken.

I_0 : $\mathcal{L}_V$

$$I_0 = (1 + 2(1 - (|1+\varphi_1|^2 + |\varphi_2|^2))^2\ \mathrm{tr}((\lambda^\perp(M^*M))^2)$$

$\lambda^\perp$: orthogonal projection in the Hilbert-Schmit space of matrices onto the orthogonal complement of the scalar multiples of the identity.

Lagrangean of fermion fields

$\mathcal{L}_f$: fermion kinetic term

$$\mathcal{L}_f = -\sum\Big[\bar{f}_L \gamma^\mu\Big(\partial_\mu + ig\frac{\tau_a}{2}W_{\mu a} + ig'\frac{Y_L}{2}B_L\Big)f_L$$

$$+ \bar{f}_R\gamma^\mu\Big(\partial_\mu + ig'\frac{Y_R}{2}B_\mu\Big)f_R$$

f_L, f_R : left-handed and right-handed fermion fields

$Y_L = -1$, $Y_R = -2$: hypercharges for leptons

$\mathcal{L}_Y$: Yukawa coupling of Higgs fields with fermions.

$$\mathcal{L}_Y = -\sum\big[H_{ff'}(\bar{f}_L \cdot \Phi)f'_R + H^*_{ff'}\bar{f}'_R(\Phi^+ \cdot f_L)\big]$$

where $H_{ff'}$: general coupling matrix

$J_0 \sim \mathcal{L}_f$. $\quad \not{\partial} = \gamma^\mu\partial_\mu$

$$J_0 = \bar{\psi}_a(i^{-1}\not{\partial} + i^{-1}\gamma(\omega_a))\psi_a + \bar{\psi}_b(i^{-1}\not{\partial} + i^{-1}\gamma(\omega_b))\psi_b$$

$J_1 \sim \mathcal{L}_Y$

$$J_1 = \bar{\psi}_a M[(1+\varphi_1), \varphi_2]\psi_b + h.c.$$

N generations of leptons.

where $N = N_a = N_b$ is the dimension of $\mathfrak{H}_{2,a}$ and $\mathfrak{H}_{2,b}$

point a : left-handed fermion fields

point b : right-handed fermion fields

5. The Standard $U(1) \times SU(2) \times SU(3)$ Model

In this section, we shall start from the main point of the computation of our Yang–Mills functional in Example b) of Section 3 (referred to briefly as Example 3b)), i.e., in the case of the product of a continuum by a discrete 2-point space. The point is that we recovered the Glashow–Weinberg–Salam model for leptons, with the five different pieces of its Lagrangian, from this simple modification of the (4-dimensional) continuum. The question we shall answer in the present section is the following: Can one, by a similar procedure, incorporate the quarks as well as strong interactions?

Before embarking on this problem, some preparation is required to explain better what our aim is. First, there is at present (1993) no question that the standard model of electro-weak and strong interactions is a remarkably successful phenomenological model of particle physics. Since I did not take any part in its elaboration, I shall refrain from a survey of the experimental roots of this model or of the long history of its elaboration. I refer the reader to the beautiful book of A. Pais [**429**] or to the more technical papers [**197**]. This seems an important prerequisite for a mathematician reader of the present section, who might otherwise underestimate the depth of the physical roots of the model.

Next, by the work of 't Hooft [**284**] this model is renormalizable ([**57**], [**213**]), a necessary requirement for applying the only known perturbative recipe for quantizing the theory. It nevertheless has problems, such as the naturalness problem [**197**], which make specialists doubt that it is really of fundamental significance, thus leading them to look for alternative routes, grand unification, technicolor… These alternate routes all share a common feature: they deny any fundamental significance to the Higgs boson.

Our contribution does not throw any new light on the above theoretical problems of the standard model, since it is limited to the *classical level*. However, it specifies very precisely which modification of the continuum, in fact its replacement by a product with a *finite space*, entails that the Lagrangian of quantum electrodynamics becomes the Lagrangian of the standard model. As we shall see, the geometry of the finite set will be, as advocated above, specified by its *Dirac operator*, and this will be an operator in a finite-dimensional Hilbert space encoding both the masses of the elementary particles and the Kobayashi–Maskawa mixing parameters.

Once the structure of this finite space F is given, we merely apply our general action to the space (continuum) $\times F$ to get the standard model action. In many ways, our contribution should be regarded as an *interpretation*, of a geometric nature, of all the intricacies of the most accurate phenomenological model of high-energy physics; if it makes the model more intelligible to a mathematical audience, then our purpose will in some small measure be achieved. It does undoubtedly confirm that high-energy (i.e., small-distance) physics is in fact unveiling the fine structure of space-time. Finally, it

gives a status to the Higgs boson as just another gauge field, but corresponding to a finite difference rather than a differential.

5.β The standard model. Just as for the Glashow–Weinberg–Salam model for leptons, the Lagrangian of the standard model contains five different terms,

$$\mathcal{L} = \mathcal{L}_G + \mathcal{L}_f + \mathcal{L}_\varphi + \mathcal{L}_Y + \mathcal{L}_V,$$

which we now describe together with the field content of the theory. (As before, we shall use the Einstein convention of summation over repeated indices.)

1) The pure gauge boson part $\mathcal{L}_G$

U(1) SU(3) 611

SU(2)

$$\mathcal{L}_G = \frac{1}{4}(G_{\mu\nu a}G_a^{\mu\nu}) + \frac{1}{4}(F_{\mu\nu}F^{\mu\nu}) + \frac{1}{4}(H_{\mu\nu b}H_b^{\mu\nu}),$$

where $G_{\mu\nu a}$ is the field strength tensor of an $SU(2)$ gauge field $W_{\mu a}$, F_μ is the field strength tensor of a $U(1)$ gauge field B_μ, and $H_{\mu\nu b}$ is the field strength tensor of an $SU(3)$ gauge field $V_{\mu b}$. This last gauge field, the *gluon field*, is the carrier of the strong force; the gauge group $SU(3)$ is the *color group*, and is thus the essential new ingredient. The respective coupling constants for the fields W, B, and V will be denoted g, g', and g'', consistent with the previous notation.

2) The fermion kinetic term $\mathcal{L}_f$

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To the leptonic terms

$$-\sum_f [\overline{f}_L \gamma^\mu (\partial_\mu + ig\frac{\tau_a}{2}W_{\mu a} + ig'\frac{Y_L}{2}B_\mu) f_L + \overline{f}_R \gamma^\mu (\partial_\mu + ig'\frac{Y_R}{2}B_\mu) f_R],$$

one adds the following similar terms involving the quarks:

$$\begin{aligned} -\sum_f [\overline{f}_L \gamma^\mu (\partial_\mu &+ ig\frac{\tau_a}{2}W_{\mu a} + ig'\frac{Y_L}{2}B_\mu + ig''\lambda_b V_{\mu b}) f_L \\ &+ \overline{f}_R \gamma^\mu (\partial_\mu + ig'\frac{Y_R}{2}B_\mu + ig''\lambda_b V_{\mu b}) f_R]. \end{aligned}$$

For each of the three generations of quarks $\begin{bmatrix} u \\ d \end{bmatrix}$, $\begin{bmatrix} c \\ s \end{bmatrix}$, and $\begin{bmatrix} t \\ b \end{bmatrix}$ one has a left-handed isodoublet (such as $\begin{bmatrix} u_L \\ d_L \end{bmatrix}$), two right-handed $SU(2)$ singlets (such as $\begin{bmatrix} u_R \\ d_R \end{bmatrix}$), and each quark field appears in 3 colors so that, for instance, there are three u_R fields: u_R^r, u_R^y, u_R^b. All of these quark fields are thus in the fundamental representation $\mathbf{3}$ of $SU(3)$.

The hypercharges Y_L and Y_R are identical for different generations and are given by the following table:

	e, μ, τ	ν_e, ν_μ, ν_τ	u, c, t	d, s, b
Y_L	-1	-1	$\frac{1}{3}$	$\frac{1}{3}$
Y_R	-2		$\frac{4}{3}$	$-\frac{2}{3}$

These numbers are not explained by theory but are set by hand so as to get the correct electromagnetic charges Q_{em} from the formulas

$$2Q_{\text{em}} = Y_L + 2I_3, \;\; 2Q_{\text{em}} = Y_R,$$

where I_3 is the third generator of the weak isospin group $SU(2)$.

3) The kinetic terms for the Higgs fields

$$\mathcal{L}_\varphi = -\left| \left(\partial_\mu + ig\frac{\tau_a}{2}W_{\mu a} + i\frac{g'}{2}B_\mu \right) \varphi \right|^2,$$

where $\varphi = \begin{bmatrix} \varphi_1 \\ \varphi_2 \end{bmatrix}$ is an $SU(2)$ doublet of complex scalar fields with hypercharge $Y_\varphi = 1$. This term is *exactly the same* as in the G.W.S. model for leptons.

4) The Yukawa coupling of Higgs fields with fermions

$$\mathcal{L}_Y = -\sum_{f,f'} [H_{ff'}\, \overline{f}_L \cdot \varphi\, f'_R + H^*_{ff'} \overline{f'_R} (\varphi^* \cdot f_L)],$$

where $H_{ff'}$ is a general coupling matrix in the space of different fermions, about which we must now be more explicit. First, there is no $H_{ff'} \neq 0$ between leptons and quarks, so that $\mathcal{L}_Y$ is a sum of a leptonic and a quark part. Since there is no right-handed neutrino in this model, the leptonic part can always be put into the form

$$\mathcal{L}_{Y,\text{lepton}} = -G_e(\overline{L}_e \cdot \varphi) e_R - G_\mu(\overline{L}_\mu \cdot \varphi)\mu_R - G_\tau(\overline{L}_\tau \cdot \varphi)\tau_R + \text{h.c.},$$

where L_e is the isodoublet $\begin{bmatrix} \nu_{e,L} \\ e_L \end{bmatrix}$, and similarly for the other generations. The coupling constants G_e, G_μ, and G_τ provide the lepton masses through the Higgs vacuum contribution.

The quark Yukawa coupling is more complicated owing to new terms which provide the masses of the up particles, and to the mixing angles. We have three new terms. The first is of the form

$$(*) \qquad G\overline{L} u_R \tilde{\varphi},$$

where the isodoublet $L = \begin{bmatrix} u_L \\ q_L \end{bmatrix}$ is obtained from a left-handed up quark and a mixing q_L of left-handed down quarks (taken from the three families); the two others have a similar structure but with the up quark replaced by the charm and top quarks respectively. Also, $\tilde{\varphi}$ needs to have the same isospin but opposite hypercharge to the Higgs doublet φ and is given by

$$(**) \qquad \tilde{\varphi} = J\varphi^*, \ J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

We refer to [**197**] for more details on this point, to which we shall return later.

5) The Higgs self-interaction

$$\mathcal{L}_V = \mu^2 \varphi^+ \varphi - \frac{1}{2}\lambda(\varphi^+\varphi)^2$$

has exactly the same form as in the previous case.

Thus, we see that there are, essentially, three novel features of the complete standard model with respect to the leptonic case:

A) The new gauge symmetry: *color*, with gluons responsible for the strong interaction.

B) The new values $\frac{1}{3}$, $\frac{4}{3}$, $-\frac{2}{3}$ of the hypercharge for quarks.

C) The new Yukawa coupling terms $(*)$.

We shall now briefly explain how these new features motivate a corresponding modification of Example 3b), which led us above to the G.W.S. model for leptons. First, our model will still be a *product* of an ordinary Euclidean continuum by a finite space.

In Example 3b), for the algebra $\mathcal{A}$ of functions on the finite space, we took the algebra $\mathbb{C}_a \oplus \mathbb{C}_b$. But since we then considered a bundle on $\{a, b\}$ with fiber $\mathbb{C}^2$ over a and $\mathbb{C}$ over b, we could have in an equivalent fashion taken $\mathcal{A} = M_2(\mathbb{C}) \oplus \mathbb{C}$ and then dealt with vector potentials, instead of connections on vector bundles. Let us see how C) leads to replacing $\mathcal{A} = M_2(\mathbb{C}) \oplus \mathbb{C}$ by $\mathcal{A} = \mathbb{H} \oplus \mathbb{C}$, where $\mathbb{H}$ is the Hamiltonian algebra of quaternions. The point is simply that the equation $(**)$ which relates φ and $\tilde{\varphi}$ is the same as the unitary equivalence $\mathbf{2} \sim \overline{\mathbf{2}}$ of the fundamental representation $\mathbf{2}$ of $SU(2)$ with the complex-conjugate or contragradient representation $\overline{\mathbf{2}}$, i.e., we have

$$g \in U(2),\ JgJ^{-1} = \overline{g} \ \Leftrightarrow \ g \in SU(2).$$

Let us simply remark that $x \in M_2(\mathbb{C})$, $JxJ^{-1} = \overline{x}$ defines an algebra, the quaternion algebra $\mathbb{H}$.

Next, let us see how A) leads us to the formalism of bimodules and Poincaré duality of Section 4. Indeed, let us look at any isodoublet of the form $\begin{bmatrix} u_L \\ d_L \end{bmatrix}$ of left-handed quarks. It appears in 3 colors,

$$\begin{matrix} u_L^r & u_L^y & u_L^b \\ \\ d_L^r & d_L^y & d_L^b \end{matrix},$$

which makes it clear that the corresponding representation of $SU(2) \times SU(3)$ is the external tensor product $\mathbf{2}_{SU(2)} \otimes \mathbf{3}_{SU(3)}$ of their fundamental representations. It is easy to convince oneself that even if one neglects the nuance between $U(n)$ and $SU(n)$ in general, there is no way to obtain such groups and representations from a single algebra and its unitary group. The solution that we found, namely to take $(\mathcal{A}, \mathcal{B})$-bimodules, with $\mathcal{B} = \mathbb{C} \oplus M_3(\mathbb{C})$ (and $\mathcal{A} = \mathbb{C} \oplus \mathbb{H}$ as above) is in fact already suggested by the following picture in a paper of J. Ellis [**197**], very close to that of the diagonal $\Delta \subset X \times X$ in Poincaré duality:
The apparent "generation" or "family" structure of fundamental fermions.

We just refine it by taking algebras—$\mathbb{C} \oplus \mathbb{H}$ for the y-axis, $\mathbb{C} \oplus M_3(\mathbb{C})$ for the x-axis—instead of groups, which allows us to account better for the leptons (by using the $\mathbb{C}$ of $\mathbb{C} \oplus M_3(\mathbb{C})$).

Finally, we shall get a conceptual understanding of the numbers B) from a general unimodularity condition that makes sense in noncommutative geometry, but we need not anticipate that point.

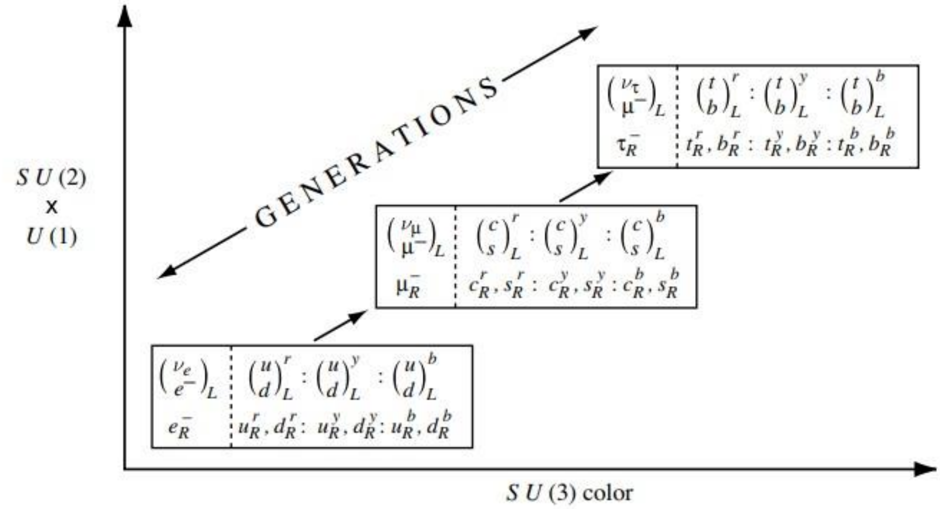

FIGURE 5. The apparent "generation" or "family" structure of fundamental fermions. The horizontal axis corresponds to $SU(3)$ color properties, the vertical axis to $SU(2)\times U(1)$ representation contents

We are now ready to describe in detail the geometric structure of the finite space F which, once crossed by $\mathbb{R}^4$, gives the standard model.

5.γ Geometric structure of the finite space F. This structure is given by an $(\mathcal{A}, \mathcal{B})$-module $(\mathfrak{H}, D, \gamma)$, where $\mathcal{A}$ is the $*$-algebra $\mathbb{C} \oplus \mathbb{H}$ while $\mathcal{B}$ is the $*$-algebra $\mathbb{C} \oplus M_3(\mathbb{C})$. Unlike $\mathcal{B}$, the algebra $\mathcal{A}$ is only an algebra over $\mathbb{R}$. The $*$-representations π of $\mathcal{A}$ on a finite-dimensional Hilbert space are characterized (up to unitary equivalence) by three multiplicities: n_+, n_-, and m, where $\mathfrak{H}_\pi = \mathbb{C}^{n_+} \oplus \mathbb{C}^{n_-} \oplus \mathbb{C}^{2m}$; if $a = (\lambda, q) \in \mathcal{A} = \mathbb{C} \oplus \mathbb{H}$, then $\pi(a)$ is the block diagonal matrix

$$\pi(a) = (\lambda \otimes \mathrm{id}_{n^+}) \oplus (\overline{\lambda} \otimes \mathrm{id}_{n^-}) \oplus \left(\begin{bmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{bmatrix} \otimes \mathrm{id}_m \right),$$

where the quaternion q is $q = \alpha + \beta \mathrm{j}$ with $\alpha, \beta \in \mathbb{C} \subset \mathbb{H}$. The representation of the complex $*$-algebra $\mathcal{B}$ on $\mathfrak{H}$ gives a decomposition

$$\mathfrak{H} = \mathfrak{H}_0 \oplus (\mathfrak{H}_1 \otimes \mathbb{C}^3)$$

in which $\mathcal{B}$ acts by $\pi(b) = b_0 \oplus (1 \otimes b_1)$ for $b = (b_0, b_1) \in \mathbb{C} \oplus M_3(\mathbb{C})$; thus the commuting representation of $\mathcal{A}$ is given by *a pair* π_0, π_1 of representations on $\mathfrak{H}_0$ and $\mathfrak{H}_1$. The $(\mathcal{A}, \mathcal{B})$-bimodule $\mathfrak{H}$ is thus completely described by six multiplicities: (n^0_+, n^0_-, m^0) for π_0 and (n^1_+, n^1_-, m^1) for π_1. We shall take these to be of the form

$$(n^0_+, n^0_-, m^0) = N(1, 0, 1), \; (n^1_+, n^1_-, m^1) = N(1, 1, 1)$$

(where N will eventually be the number of generations $N = 3$). We shall take the $\mathbb{Z}/2$-grading γ in $\mathfrak{H}$ to be given, as in Example 3b), by the element $\gamma = (1, -1)$ of the center of $\mathcal{A}$. Finally, we shall take for D the most general selfadjoint operator in $\mathfrak{H}$ that anticommutes with γ ($D\gamma = -\gamma D$) and commutes with $\mathbb{C} \otimes \mathcal{B}$, where $\mathbb{C} \subset \mathcal{A}$ is the diagonal subalgebra $\{(\lambda, \lambda);\ \lambda \in \mathbb{C}\}$. (As we shall see, D encodes both the masses of the fermions and the Kobayashi–Maskawa mixing parameters.) It follows that the action of $\mathcal{A}$ and the operator D in $\mathfrak{H}_0$ (resp. $\mathfrak{H}_1$) have the following general form (with $q = \alpha + \beta \mathrm{j} \in \mathbb{H}$):

$$\pi_0(f,q) = \begin{bmatrix} f & 0 & 0 \\ 0 & \alpha & \beta \\ 0 & -\overline{\beta} & \overline{\alpha} \end{bmatrix}, \quad D_0 = \begin{bmatrix} 0 & M_e^* & 0 \\ M_e & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$Md \quad 3\times 3$

$Mu \quad 3 \times 3$

$$\pi_1(f,q) = \begin{bmatrix} f & 0 & 0 & 0 \\ 0 & \overline{f} & 0 & 0 \\ 0 & 0 & \alpha & \beta \\ 0 & 0 & -\overline{\beta} & \overline{\alpha} \end{bmatrix}, \quad D_1 = \begin{bmatrix} 0 & 0 & M_d^* & 0 \\ 0 & 0 & 0 & M_u^* \\ M_d & 0 & 0 & 0 \\ 0 & M_u & 0 & 0 \end{bmatrix},$$

where M_e, M_u, M_d are arbitrary $N \times N$ complex matrices.

Since π_0 is a degenerate case ($M_u = 0$) of π_1, we just restrict to π_1 in order to determine $\Omega_D^1(\mathcal{A})$.

A straightforward computation gives $\pi_1(\sum a_j \mathrm{d}a_j')$ with $a_j, a_j' \in \mathcal{A}$, $a_j = (\lambda_j, q_j)$, $q_j = \alpha_j + \beta_j \mathrm{j}$, $q_j' = \alpha_j' + \beta_j' \mathrm{j}$; we have

$$\pi_1(\sum a_j \mathrm{d}a_j') = \begin{bmatrix} 0 & X \\ Y & 0 \end{bmatrix},$$

where X and Y are the matrices

$$X = \begin{bmatrix} M_d^* \varphi_1 & M_d^* \varphi_2 \\ -M_u^* \overline{\varphi}_2 & M_u^* \overline{\varphi}_1 \end{bmatrix}, \quad Y = \begin{bmatrix} M_d \varphi_1' & M_u \varphi_2' \\ -M_d \overline{\varphi_2'} & M_u \overline{\varphi_1'} \end{bmatrix},$$

with

$$\begin{aligned} \varphi_1 &= \sum \lambda_i (\alpha_i' - \lambda_i'), \ \varphi_2 = \sum \lambda_i \beta_i', \\ \varphi_1' &= \sum \alpha_i (\lambda_i' - \alpha_i') + \beta_i \overline{\beta_i'}, \ \varphi_2' = \sum -\alpha_i \beta_i' + \beta_i (\overline{\lambda_i'} - \overline{\alpha_i'}). \end{aligned}$$

It follows that $\Omega_D^1(\mathcal{A}) = \mathbb{H} \oplus \mathbb{H}$ with the $\mathcal{A}$-bimodule structure given by

$$\begin{aligned} (\lambda, q)(q_1, q_2) &= (\lambda q_1, q q_2) \ \forall q_1, q_2 \in \mathbb{H}, \\ (q_1, q_2)(\lambda, q) &= (q_1 q, q_2 \lambda) \ \forall \lambda \in \mathbb{C}, \ q \in \mathbb{H}, \end{aligned}$$

and the differential d again being the *finite difference*:

$$\mathrm{d}(\lambda, q) = (q - \lambda, \lambda - q) \in \mathbb{H} \oplus \mathbb{H}$$

(just set $q_1 = \varphi_1 + \varphi_2 \mathrm{j}$, $q_2 = \varphi_1' + \varphi_2' \mathrm{j}$ with the above φ's).

Finally, the involution $*$ on $\Omega_D^1(\mathcal{A})$ is given by

$$(q_1, q_2)^* = (\overline{q}_2, \overline{q}_1) \ \forall q_j \in \mathbb{H}.$$

The space $\mathcal{U}$ of vector potentials is thus naturally isomorphic to $\mathbb{H}$, and a similar computation shows that $\Omega_D^2(\mathcal{A}) = \mathbb{H} \oplus \mathbb{H}$ with the $\mathcal{A}$-bimodule structure

$$(\lambda, q)(q_1, q_2)(\lambda', q') = (\lambda q_1 \lambda', q q_2 q') \ \forall \lambda, \lambda' \in \mathbb{C}, \ q, q' \in \mathbb{H};$$

the product $\Omega_D^1 \times \Omega_D^1 \to \Omega_D^2$ is given by

$$(q_1, q_2) \wedge (q_1', q_2') = (q_1 q_2', q_2 q_1'),$$

and the differential $\mathrm{d} : \Omega_D^1 \to \Omega_D^2$ by

$$\mathrm{d}(q_1, q_2) = (q_1 + q_2, q_1 + q_2).$$

Thus, the curvature θ of a vector potential $V = (q, \overline{q})$ is

$$\theta = \mathrm{d}V + V^2 = (q + q^* + qq^*, q + q^* + q^* q) = (|1 + q|^2 - 1)(1, 1),$$

where $q \mapsto |q|$ denotes the norm of quaternions.

We thus see that the action $\mathrm{YM}(V) = \mathrm{Trace}\big(\pi(\theta)^2\big)$ (we are in the 0-dimensional case) is the same symmetry-breaking quartic potential for a pair of complex numbers as in Example 3b).

The detailed expression for the Hilbert space norm on $\Omega_D^2(\mathcal{A}) = \mathbb{H} \oplus \mathbb{H}$ is given, for $\omega = (q_1, q_2)$, $q_j = \alpha_j + \beta_j \mathrm{j}$, by

$$\|\omega\|^2 = \lambda_1 |\alpha_1|^2 + \mu_1 |\beta_1|^2 + \lambda_2 (|q_2|^2),$$

where

$$\begin{aligned}
\lambda_1 &= \mathrm{Trace}(|M_e|^4) + 3\mathrm{Trace}(|M_d|^4 + |M_u|^4), \\
\mu_1 &= 6\,\mathrm{Trace}(|M_d|^2 |M_u|^2), \\
\lambda_2 &= \frac{1}{2}\mathrm{Trace}\big(|M_e|^4 + 3(|M_d|^4 + |M_u|^4 + 2|M_d|^2 |M_u|^2)\big).
\end{aligned}$$

Finally, we shall investigate what freedom we have in the choice of the selfadjoint operators D_0, D_1 on $\mathfrak{H}_0$, $\mathfrak{H}_1$ in the above example. Two pairs $(\mathfrak{H}_j, D_j)$ and $(\mathfrak{H}_j', D_j')$ give identical results if there exist unitaries $U_j : \mathfrak{H}_j \to \mathfrak{H}_j'$ such that:

α) $U_j D_j U_j^* = D_j$ $(j = 1, 2)$,

β) $U_j \pi_j(a) U_j^* = \pi_j'(a)$ $\forall a \in \mathcal{A}$, $j = 1, 2$.

Making use of this freedom, we can assume that D_0 is diagonal in $\mathfrak{H}_0$ and has positive eigenvalues e_1, e_2, e_3. Thus, the situation for D_0 is described by these 3 positive numbers.

For $\mathfrak{H}_1$, a general element of the commutant of $\pi_1(\mathcal{A})$ is of the form

$$U_1 = \begin{bmatrix} V_1 & 0 & 0 & 0 \\ 0 & V_2 & 0 & 0 \\ 0 & 0 & V_3 & 0 \\ 0 & 0 & 0 & V_3 \end{bmatrix},$$

where the V_j are unitary operators when U_1 is unitary.

Conjugating D_1 by U_1 replaces M_d and M_u, respectively, by

$$M_d' = V_1 M_d V_3^*, \;\; M_u' = V_2 M_u V_3^*.$$

We thus see that we can assume that both M_u and M_d are positive matrices and that one of them, say M_u, is diagonal.

The invariants are thus the eigenvalues of M_u and M_d, i.e., a total of 6 positive numbers, and the pair of maximal abelian subalgebras generated by M_u and M_d. Since any pair $\mathcal{A}_1$, $\mathcal{A}_2$ of maximal abelian subalgebras of $M_3(\mathbb{C})$ are conjugate by a unitary W, $W\mathcal{A}_1 W^* = \mathcal{A}_2$, which is given modulo the unitary groups $\mathcal{U}(\mathcal{A}_j)$, there remain 4 parameters with which to specify W so that $W M_d W^*$ is also diagonal. Such a W corresponds to the Kobayashi–Maskawa mixing matrix in the standard model.

5.δ Geometric structure of the standard model. We shall show in this section how the standard model is obtained from the product geometry of the usual 4-dimensional continuum by the above finite geometry F. Thus, we let V be a 4-dimensional spin manifold and $(L^2, \partial_V, \gamma_5)$ its Dirac K-cycle. The product geometry is, according to the general rule for forming products, described by the algebras

$$\mathcal{A} = C^\infty(V) \otimes (\mathbb{C} \oplus \mathbb{H}), \;\; \mathcal{B} = C^\infty(V) \otimes \big(\mathbb{C} \oplus M_3(\mathbb{C})\big).$$

The Hilbert space $H = L^2(V, S) \otimes \mathfrak{H}_F$, where $\mathfrak{H}_F$ is described in γ) above, i.e., $\mathfrak{H}_F = \mathfrak{H}_0 \oplus (\mathfrak{H}_1 \otimes \mathbb{C}^3)$. There is a corresponding decomposition $H = H_0 \oplus (H_1 \otimes \mathbb{C}^3)$, with corresponding representations π_j of $\mathcal{A}$ on H_j.

Then $D = \partial_V \otimes 1 + \gamma_5 \otimes D_F$, where D_F is as above. This gives a decomposition $D = D_0 \oplus (D_1 \otimes 1)$, where, according to γ), we take M_e, M_u and M_d to be *positive*

F : finite.

matrices:

mass of ν_e is zero

$$D_0 = \begin{bmatrix} \partial_V \otimes 1 & \gamma_5 \otimes M_e & 0 \\ \gamma_5 \otimes M_e & \partial_V \otimes 1 & 0 \\ 0 & 0 & \partial_V \otimes 1 \end{bmatrix},$$

lepton

$$D_1 = \begin{bmatrix} \partial_V \otimes 1 & 0 & \gamma_5 \otimes M_d & 0 \\ 0 & \partial_V \otimes 1 & 0 & \gamma_5 \otimes M_u \\ \gamma_5 \otimes M_d & 0 & \partial_V \otimes 1 & 0 \\ 0 & \gamma_5 \otimes M_u & 0 & \partial_V \otimes 1 \end{bmatrix}$$

quark

We shall first restrict attention to the algebra $\mathcal{A}$, the case of $\mathcal{B}$ being easier. Note that $\mathcal{A} = C^\infty(V, \mathbb{C}) \oplus C^\infty(V, \mathbb{H})$, so that every $a \in \mathcal{A}$ is given by a pair (f, q) consisting of a $\mathbb{C}$-valued function f on V and an $\mathbb{H}$-valued function q on V.

Let us first compute $\Omega^1_D(\mathcal{A})$. Given $\rho = \sum a_s da'_s \in \Omega^1(\mathcal{A})$, with $a_s, a'_s \in \mathcal{A}$, we have $a_s = (f_s, q_s)$, $a'_s = (f'_s, q'_s)$, where f_s, f'_s are complex-valued functions on V and q_s, q'_s are $\mathbb{H}$-valued functions on V, of the form

$$q_s = \alpha_s + \beta_s \mathrm{j}, \; q'_s = \alpha'_s + \beta'_s \mathrm{j}.$$

su(2)

Then

SU(1)

$$\pi_1(\rho) = \begin{bmatrix} i^{-1}\gamma(A) \otimes 1 & 0 & \varphi_1 \gamma_5 \otimes M_d & \varphi_2 \gamma_5 \otimes M_d \\ 0 & i^{-1}\gamma(\overline{A}) \otimes 1 & -\overline{\varphi}_2 \gamma_5 \otimes M_u & \overline{\varphi}_1 \gamma_5 \otimes M_u \\ \varphi'_1 \gamma_5 \otimes M_d & \varphi'_2 \gamma_5 \otimes M_u & i^{-1}\gamma(W_1) \otimes 1 & i^{-1}\gamma(W_2) \otimes 1 \\ -\overline{\varphi'_2} \gamma_5 \otimes M_d & \overline{\varphi'_1} \gamma_5 \otimes M_u & i\gamma(\overline{W}_2) \otimes 1 & i^{-1}\gamma(\overline{W}_1) \otimes 1 \end{bmatrix},$$

quark.

SU(2)

where $A = \sum f_s df'_s$ is a $\mathbb{C}$-valued 1-form on V, and $W_1 + W_2 \mathrm{j} = W = \sum q_s dq'_s$ is an $\mathbb{H}$-valued 1-form on V (cf. [**20**]).

Also, φ_j and φ'_j are complex-valued functions on V given by the same formulas as above for the finite geometry, namely,

$$\varphi_1 = \sum f_s(\alpha'_s - f'_s), \; \varphi_2 = \sum f_s \beta'_s,$$

$$\varphi'_1 = \sum \left(\alpha_s(f'_s - \alpha'_s) + \beta_s \overline{\beta'_s}\right), \; \varphi'_2 = \sum \left(\beta_s(\overline{f'_s} - \overline{\alpha'_s}) - \alpha_s \beta'_s\right).$$