COHEN-MACAULAY MODULES AND HOLONOMIC MODULES OVER FILTERED RINGS

HIROKI MIYAHARA(∗) AND KENJI NISHIDA(**)

Abstract. We study Gorenstein dimension and grade of a module $M$ over a filtered ring whose associated graded ring is a commutative Noetherian ring. An equality or an inequality between these invariants of a filtered module and its associated graded module is the most valuable property for an investigation of filtered rings. We prove an inequality $G$-dim$M \leq G$-dim$\text{gr}M$ and an equality $\text{grade}M = \text{grade}\text{gr}M$, whenever Gorenstein dimension of $\text{gr}M$ is finite (Theorems 2.3 and 2.8). We would say that the use of $G$-dimension adds a new viewpoint for studying filtered rings and modules. We apply these results to a filtered ring with a Cohen-Macaulay or Gorenstein associated graded ring and study a Cohen-Macaulay, perfect or holonomic module.

1. Introduction

Homological theory of filtered (non-commutative) rings grew in studying, among others, $D$-modules, i.e., rings of differential operators (cf. [4], [17] etc.). The use of an invariant ‘grade’ is a core of the theory for Auslander regular or Gorenstein filtered rings ([4], [5], [6], [7], [14]). In particular, its invariance under forming associated graded modules is essential. Using Gorenstein dimension ([1], [9]), we extend the class of rings for which the invariance holds.

Let $\Lambda$ be a left and right Noetherian ring. Let $\text{mod} \Lambda$ (respectively, $\text{mod} \Lambda^{op}$) be the category of all finitely generated left (respectively, right) $\Lambda$-modules. We denote the stable category by $\text{mod}_f \Lambda$, the syzygy functor by $\Omega: \text{mod}_f \Lambda \to \text{mod}_f \Lambda$, and the transpose functor by $\text{Tr}: \text{mod}_f \Lambda \to \text{mod}_f \Lambda^{op}$ (see [2], Chapter 4, §1 or [1], Chapter 2, §1). For $M \in \text{mod}_f \Lambda$, we put $M^* := \text{Hom}_\Lambda(M, \Lambda) \in \text{mod}_f \Lambda^{op}$.

Gorenstein dimension, one of the most valuable invariants of the homological study of rings and modules, is introduced in [1]. A $\Lambda$-module $M$ is said to have Gorenstein dimension zero, denoted by $G$-dim$\Lambda M = 0$, if $M^{**} \cong M$ and $\text{Ext}_\Lambda^k(M, \Lambda) = \text{Ext}^{k\Lambda^{op}}_{\Lambda^{op}}(M^*, \Lambda) = 0$ for $k > 0$. It follows from [1], Proposition 3.8 that $G$-dim$M = 0$ if and only if $\text{Ext}_\Lambda^k(M, \Lambda) = \text{Ext}_\Lambda^{k\Lambda^{op}}(\text{Tr}M, \Lambda) = 0$ for $k > 0$. For a positive integer $k$, $M$ is said to have Gorenstein dimension less than or equal to $k$, denoted by $G$-dim$\Lambda M \leq k$, if there exists an exact sequence $0 \to G_k \to \cdots \to G_0 \to M \to 0$ with $G$-dim$G_i = 0$ for $0 \leq i \leq k$. We have that G-dim $M \leq k$ if and only if $G$-dim $\Omega^kM = 0$ by [1], Theorem 3.13. It is also proved in [1] that if $G$-dim $M < \infty$ then $G$-dim $M = \sup\{k: \text{Ext}_\Lambda^k(M, \Lambda) \neq 0\}$. In the following, we abbreviate ‘Gorenstein dimension’ to $G$-dimension.

We define another important invariant ‘grade’. Let $M \in \text{mod}_f \Lambda$. We put $\text{grade}_\Lambda M := \inf\{k: \text{Ext}_\Lambda^k(M, \Lambda) \neq 0\}$.

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In this paper we study G-dimension and grade of a filtered module over a filtered ring whose associated graded ring is commutative and Noetherian and apply the results to a filtered ring with a Gorenstein or Cohen-Macaulay associated graded ring.

In section two, we study G-dimension and grade of modules over a filtered ring. As usual, we analyze them by using the properties of associated graded modules. We start from studying G-dimension. When an associated graded ring \( \text{gr}A \) of a filtered ring \( A \) is commutative and Noetherian, a filtered \( A \)-module \( M \) whose associated graded module \( \text{gr}M \) has finite G-dimension has also finite G-dimension and an inequality \( \text{G-dim}M \leq \text{G-dim} \text{gr}M \) holds true (Theorem 2.3). We see that if an associated graded ring is regular then an equality holds for every \( M \). However, it is open whether an equality holds or not in general. As for G-dimension zero, we show that if \( \text{G-dim} \text{gr}M = 0 \), then \( \text{G-dim}M = 0 \) and the converse holds whenever some additional conditions for \( M \) are assumed (Theorem 2.5). Assume further that \( \text{gr}A \) is a \(^\ast\)local ring with the condition (P) (see Appendix), then ‘Auslander-Bridger formula’ holds for a filtered module \( M \) such that \( \text{gr}M \) has finite G-dimension and \( \text{G-dim}M = \text{G-dim} \text{gr}M \). \( \text{G-dim}M + \text{depth} \text{gr}M = \text{depth} \text{gr}A \) (Proposition 2.6).

To handle grade in the literatures, a kind of ‘finitary’ condition over a ring such as ‘regularity’ or ‘Gorensteiness’ is setted ([7], §5 and [14], Chapter III, §2, 2.5). We find out that only the finiteness of G-dimension of \( \text{gr}M \) implies \( \text{grade}M = \text{grade} \text{gr}M \) for a filtered module with a good filtration (Theorem 2.8). Suppose that \( \text{gr}A \) is Gorenstein. Then all finite \( \text{gr}A \)-modules have finite G-dimension. Thus all filtered modules with a good filtration satisfy the equality. Since regularity implies Gorensteiness, our results also cover regular filtered rings.

In section three, we apply the results obtained in the previous section to Cohen-Macaulay modules over filtered rings with a Cohen-Macaulay associated graded ring and holonomic modules over Gorenstein filtered rings. When \( \text{gr}A \) is a Cohen-Macaulay \(^\ast\)local ring with the condition (P), we define Cohen-Macaulay filtered modules and see that they are perfect. Then they satisfy a duality (Theorem 3.2). Moreover, assume that \( A \) is Gorenstein. Then injective dimension of \( A \) is finite, say \( d \), so that we can define a holonomic module. A filtered module \( M \) with a good filtration is holonomic, if \( \text{grade}M = d \). We generalize some results in [14], Chapter III, §4 and give a characterization of a holonomic module \( M \) by a property of \( \text{Min}(\text{gr}M) \). An example of a filtered (non-regular) Gorenstein ring is given in 3.8.

The summary of commutative graded Noetherian rings, especially, \(^\ast\)local rings are stated in Appendix.

### 2. Gorenstein dimension and grade for modules over filtered Noetherian rings

Let \( A \) be a ring. A family \( \mathcal{F} = \{ \mathcal{F}_pA : p \in \mathbb{N} \} \) of additive subgroups of \( A \) is called a filtration of \( A \), if

(i) \( 1 \in \mathcal{F}_0A \),

(ii) \( \mathcal{F}_pA \subset \mathcal{F}_{p+1}A \),

(iii) \( (\mathcal{F}_pA)(\mathcal{F}_qA) \subset \mathcal{F}_{p+q}A \).
(iv) $\Lambda = \bigcup_{p \in \mathbb{N}} \mathcal{F}_p \Lambda$.

A pair $(\Lambda, \mathcal{F})$ is called a filtered ring. In the following, a ring $\Lambda$ is always a filtered ring for some filtration $\mathcal{F}$, so that we only say that $\Lambda$ is a filtered ring.

Let $\sigma_p : \mathcal{F}_p \Lambda \to \mathcal{F}_p \Lambda / \mathcal{F}_{p-1} \Lambda$ be a natural homomorphism. Put

$$\text{gr} \Lambda = \text{gr}_\mathcal{F} \Lambda := \bigoplus_{p=0}^{\infty} \mathcal{F}_p \Lambda / \mathcal{F}_{p-1} \Lambda \quad (\mathcal{F}_{-1} \Lambda = 0).$$

Then $\text{gr} \Lambda$ is a graded ring with multiplication

$$\sigma_p(a) \sigma_q(b) = \sigma_{p+q}(ab), \quad a \in \mathcal{F}_p \Lambda, \; b \in \mathcal{F}_q \Lambda.$$

We always assume that $\text{gr} \Lambda$ is a commutative Noetherian ring. Therefore, $\Lambda$ is a right and left Noetherian ring. Our main objective is to study $\Lambda$ by relating $G$-dimension and grade of $\text{mod} \Lambda$ and those of $\text{mod} (\text{gr} \Lambda)$. Sometimes we assume further that $\text{gr} \Lambda$ is a *local* ring with the condition (P) (see Appendix).

Let $\mathcal{F}$ be a (left) $\Lambda$-module. A family $\mathcal{F} = \{ \mathcal{F}_p M : p \in \mathbb{Z} \}$ of additive subgroups of $M$ is called a filtration of $M$, if

(i) $\mathcal{F}_p M \subset \mathcal{F}_{p+1} M$,
(ii) $\mathcal{F}_{-p} M = 0$ for $p >> 0$,
(iii) $(\mathcal{F}_p \Lambda)(\mathcal{F}_q M) \subset \mathcal{F}_{p+q} M$,
(iv) $M = \bigcup_{p \in \mathbb{Z}} \mathcal{F}_p M$.

A pair $(\mathcal{F}, \Lambda)$ is called a filtered $\Lambda$-module. Similar to $\Lambda$, we sometimes abbreviate and say that $M$ is a filtered module. Let $\tau_p : \mathcal{F}_p M \to \mathcal{F}_p M / \mathcal{F}_{p-1} M$ be a natural homomorphism. Put

$$\text{gr} M = \text{gr}_\mathcal{F} M := \bigoplus_{p \in \mathbb{Z}} \mathcal{F}_p M / \mathcal{F}_{p-1} M.$$ 

Then $\text{gr} M$ is a graded $\text{gr} \Lambda$-module by

$$\sigma_p(a) \tau_q(x) = \tau_{p+q}(ax), \quad a \in \mathcal{F}_p \Lambda, \; x \in \mathcal{F}_q M.$$

As for filtered rings and module, the reader is referred to [14] or [20]. We only state here some definitions and facts. For a filtered module $(M, \mathcal{F})$, we call $\mathcal{F}$ to be a good filtration, if there exist $p_k \in \mathbb{Z}$ and $m_k \in M$ ($1 \leq k \leq r$) such that

$$\mathcal{F}_p M = \sum_{k=1}^{r} (\mathcal{F}_{p-p_k} \Lambda)m_k$$

for all $p \in \mathbb{Z}$. Then the following three conditions are equivalent ([14], Chapter I, 5.2 and [20], Chapter D, IV.3)

(a) $M$ has a good filtration.
(b) $\text{gr}_\mathcal{F} M$ is a finite $\text{gr} \Lambda$-module for a filtration $\mathcal{F}$.
(c) $M$ is a finitely generated $\Lambda$-module.

Therefore, we only consider a good filtration for a finitely generated $\Lambda$-module $M$, so that $\text{gr} M$ is a finite $\text{gr} \Lambda$-module.

Let $M, N$ be filtered $\Lambda$-modules. A $\Lambda$-homomorphism $f : M \to N$ is called a filtered homomorphism, if $f(\mathcal{F}_p M) \subset \mathcal{F}_p N$ for all $p \in \mathbb{Z}$. Further, $f$ is called strict, if $f(\mathcal{F}_p M) =$
morphisms are strict filtered homomorphisms. We can always construct such a resolution

For a filtered homomorphism \( f : M \to N \), we define a map \( f_p : \mathcal{F}_pM/\mathcal{F}_{p-1}M \to \mathcal{F}_pN/\mathcal{F}_{p-1}N \) by \( f_p(\tau_p(x)) = \tau_p(f(x)) \) for \( x \in \mathcal{F}_pM \). Then we define a \( \text{gr}\Lambda\)-homomorphism

\[
\text{gr}f : \text{gr}M = \oplus \mathcal{F}_pM/\mathcal{F}_{p-1}M \to \text{gr}N = \oplus \mathcal{F}_pN/\mathcal{F}_{p-1}N
\]

by \( \text{gr}f := \oplus f_p \), so that \( \text{gr}f(\tau_p(x)) = \tau_p(f(x)) \) for \( x \in \mathcal{F}_pM \). It is easily seen that \( \text{gr}fg = (\text{gr}f)(\text{gr}g) \) for filtered homomorphisms \( f : M \to N \) and \( g : K \to M \).

For a filtered module \( M \), an exact sequence

\[
\cdots \to F_i \xrightarrow{f_i} \cdots \xrightarrow{f_1} F_0 \xrightarrow{f_0} M \to 0
\]

called a \textit{filtered free resolution} of \( M \), if all \( F_i \) are filtered free \( \Lambda \)-modules and all homomorphisms are strict filtered homomorphisms. We can always construct such a resolution with all \( F_i \) of finite rank for a finitely generated \( \Lambda \)-module (see [20], Chapter D, IV).

Let \( M, N \) be filtered \( \Lambda \)-modules. We put, for \( p \in \mathbb{Z} \),

\[
\mathcal{F}_p\text{Hom}_\Lambda(M, N) = \{ f \in \text{Hom}_\Lambda(M, N) : f(\mathcal{F}_qM) \subset \mathcal{F}_{p+q}N \text{ for all } q \in \mathbb{Z} \}
\]

Then we have an ascending chain

\[
\cdots \subset \mathcal{F}_p\text{Hom}_\Lambda(M, N) \subset \mathcal{F}_{p+1}\text{Hom}_\Lambda(M, N) \subset \cdots
\]

of additive subgroups of \( \text{Hom}_\Lambda(M, N) \). Set

\[
\text{gr}\text{Hom}_\Lambda(M, N) := \bigoplus_{p \in \mathbb{Z}} \mathcal{F}_p\text{Hom}_\Lambda(M, N)/\mathcal{F}_{p-1}\text{Hom}_\Lambda(M, N)
\]

Define an additive homomorphism

\[
\varphi = \varphi(M, N) : \text{gr}\text{Hom}_\Lambda(M, N) \to \text{Hom}_{\text{gr}\Lambda}(\text{gr}M, \text{gr}N), \quad \varphi(\tau_p(f))(\tau_q(x)) = \tau_{p+q}(f(x))
\]

for \( f \in \mathcal{F}_p\text{Hom}_\Lambda(M, N) \), \( x \in \mathcal{F}_qM \), where

\[
\tau_p : \mathcal{F}_p\text{Hom}_\Lambda(M, N) \to \mathcal{F}_p\text{Hom}_\Lambda(M, N)/\mathcal{F}_{p-1}\text{Hom}_\Lambda(M, N)
\]

is a natural homomorphism for every \( p \in \mathbb{Z} \). When \( M \) is a filtered module with a good filtration, the following facts hold (see [14], Chapter I, 6.9 or [20], Chapter D, VI.6):

1. \( \text{Hom}_\Lambda(M, N) = \bigcup_{p \in \mathbb{Z}} \mathcal{F}_p\text{Hom}_\Lambda(M, N) \).
2. \( \mathcal{F}_{-p}\text{Hom}_\Lambda(M, N) = 0 \) for \( p \gg 0 \).
3. \( \varphi \) is injective. Moreover, if \( M \) is a filtered free module, then it is bijective.
4. When \( N = \Lambda \), an additive group \( \text{Hom}_\Lambda(M, \Lambda) \) is a filtered \( \Lambda^{\text{op}} \)-module with a good filtration \( \mathcal{F} := \{ \mathcal{F}_p\text{Hom}_\Lambda(M, \Lambda) : p \in \mathbb{Z} \} \) and \( \varphi \) is a \( \text{gr}\Lambda \)-homomorphism.

Let \( \xrightarrow{f} N \xrightarrow{g} K \) be an exact sequence of filtered modules and filtered homomorphisms. Then \( \text{gr}M \xrightarrow{\text{gr}f} \text{gr}N \xrightarrow{\text{gr}g} \text{gr}K \) is exact (in mod \( \text{gr}\Lambda \)) if and only if \( f \) and \( g \) are strict (see [14], Chapter I, 4.2.4 or [20], Chapter D, III.3).

The following proposition is well-known.

2.1. **Proposition.** Let \( M \) be a filtered \( \Lambda \)-module with a good filtration. Then \( \text{gr}\text{Ext}_\Lambda^i(M, \Lambda) \) is a subfactor of \( \text{Ext}_\text{gr}\Lambda^i(\text{gr}M, \text{gr}\Lambda) \) for \( i \geq 0 \).
Proof. See [4], Chapter 2, 6.10 or [14], Chapter III, 2.2.4. □

When $\text{G-dim} \, \text{gr} M = 0$, the functor $\text{Tr}$ commutes with associated gradation.

2.2. Lemma. Let $M$ be a filtered $\Lambda$-module with a good filtration. Then there exists an epimorphism $\alpha : \text{Tr}_{\text{gr}\Lambda}(\text{gr} M) \rightarrow \text{gr} (\text{Tr}_\Lambda M)$.

Moreover, if $G\text{-dim} \, \text{gr} M = 0$ or $\text{grade} \, \text{gr} M > 1$, then $\alpha$ is an isomorphism.

Proof. Take a filtered free resolution of $M$:

$$\cdots \rightarrow F_2 \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} M \rightarrow 0.$$ 

By definition, we have an exact sequence

$$F_0^* \xrightarrow{f_1^*} F_1^* \xrightarrow{g} \text{Tr}_\Lambda M = \text{Cok} f_1^* \rightarrow 0,$$

where $g$ is a canonical epimorphism. Let $\text{Tr}_\Lambda M$ be equipped with the induced filtration by $g$. Then $g$ is a strict filtered epimorphism. Let us consider the following diagrams in $\text{mod} \, \text{gr} \Lambda$ with the commutative squares and all the $\phi$’s isomorphisms:

\begin{align*}
(1) & \quad \text{Hom}_{\text{gr}\Lambda}(\text{gr} F_0, \text{gr} \Lambda) \xrightarrow{(\text{gr} f_1)^*} \text{Hom}_{\text{gr}\Lambda}(\text{gr} F_1, \text{gr} \Lambda) \rightarrow \text{Tr}_{\text{gr}\Lambda}(\text{gr} M) \rightarrow 0 \text{(exact)} \\
& \quad \phi \uparrow \quad \phi \uparrow \quad \phi \uparrow \quad \phi \uparrow \\
& \quad \text{gr} F_0^* \xrightarrow{(\text{gr} f_1^*)} \text{gr} F_1^* \xrightarrow{\text{gr} g} \text{gr} (\text{Tr}_\Lambda M) \rightarrow 0 \\
(2) & \quad \text{Hom}_{\text{gr}\Lambda}(\text{gr} F_0, \text{gr} \Lambda) \xrightarrow{(\text{gr} f_1)^*} \text{Hom}_{\text{gr}\Lambda}(\text{gr} F_1, \text{gr} \Lambda) \xrightarrow{(\text{gr} f_2)^*} \text{Hom}_{\text{gr}\Lambda}(\text{gr} F_2, \text{gr} \Lambda) \\
& \quad \phi \uparrow \quad \phi \uparrow \quad \phi \uparrow \\
& \quad \text{gr} F_0^* \xrightarrow{(\text{gr} f_1^*)} \text{gr} F_1^* \xrightarrow{(\text{gr} f_2^*)} \text{gr} F_2^*
\end{align*}

Since the induced sequence $\cdots \rightarrow \text{gr} F_1 \xrightarrow{\text{gr} f_1} \text{gr} F_0 \rightarrow \text{gr} M \rightarrow 0$ is a free resolution of $\text{gr} M$, the first row of (1) is exact. Since $g$ is strict, $\text{gr} g$ is surjective. Hence there exists a graded epimorphism $\alpha : \text{Tr}_{\text{gr}\Lambda}(\text{gr} M) \rightarrow \text{gr} (\text{Tr}_\Lambda M)$. By assumption, we see that $\text{Ext}_{\text{gr}\Lambda}^i(\text{gr} M, \text{gr} \Lambda) = 0$, so that the first row of (2) is exact. There exists a filtered homomorphism $h : \text{Tr}_\Lambda M \rightarrow F_2^*$ such that $f_2^* = h \circ g$. Since $\text{gr} f_2^* = \text{gr} h \circ \text{gr} g$, we have $\text{Im} \, \text{gr} f_1^* \subset \text{Ker} \, \text{gr} g \subset \text{Ker} \, \text{gr} f_2^*$. The exactness of the second row of (2) implies $\text{Im} \, \text{gr} f_1^* = \text{Ker} \, \text{gr} f_2^*$. Thus $\text{Im} \, \text{gr} f_1^* = \text{Ker} \, \text{gr} g$, hence the second row of (1) is also exact, which implies that $\alpha$ is an isomorphism. □

2.3. Theorem. Let $M$ be a filtered $\Lambda$-module with a good filtration such that $\text{gr} M$ is of finite $G$-dimension. Then $G\text{-dim} \, \text{gr} M \leq G\text{-dim} \, \text{gr} M$.

Proof. We show that if $G\text{-dim} \, \text{gr} M = k < \infty$, then $G\text{-dim} \, \text{gr} M \leq k$. Let $k = 0$. Assume that $G\text{-dim} \, \text{gr} M = 0$. For $i > 0$, since $\text{gr} \text{Ext}^i_\Lambda(M, \Lambda)$ is a subfactor of $\text{Ext}^i_{\text{gr}\Lambda}(\text{gr} M, \text{gr} \Lambda)$, we have $\text{gr} \text{Ext}^i_\Lambda(M, \Lambda) = 0$. Hence $\text{Ext}^i_\Lambda(M, \Lambda) = 0$. By Lemma 2.2, $\text{Ext}^i_{\text{gr}\Lambda}(\text{gr} \text{Tr}_\Lambda M, \text{gr} \Lambda) \cong \text{Ext}^i_{\text{gr}\Lambda}(\text{Tr}_{\text{gr}\Lambda}(\text{gr} M), \text{gr} \Lambda) = 0$ for $i > 0$. Hence $\text{Ext}^i_{\text{gr}\Lambda}(\text{Tr}_\Lambda M, \Lambda) = 0$ as above. Thus $G\text{-dim} \, \text{gr} M = 0$.

Let $k > 0$. Since $\text{gr}(\Omega^k M)$ and $\Omega^k(\text{gr} M)$ are stably isomorphic (see [10], p.226 for the definition), the following holds:

$$G\text{-dim} \, \text{gr} M \leq k \Leftrightarrow G\text{-dim} \, \Omega^k(\text{gr} M) = 0 \Leftrightarrow G\text{-dim} \, \text{gr} (\Omega^k M) = 0.$$ 

Thus the statement holds by the case of $k = 0$. □
2.4. COROLLARY. Assume that $gr\Lambda$ is a *local ring with the condition (P). If $gr\Lambda$ is Gorenstein, then $id_{\Lambda}\Lambda = id_{\Lambda^P}\Lambda \leq *depth gr\Lambda$.

Proof. Let $M$ be a finitely generated $\Lambda$-module. Then $M$ is a filtered module with a good filtration. Then $G$-dim $grM < \infty$ by Theorem A.9. Hence
\[
G\text{-dim} M \leq G\text{-dim} grM = *depth gr\Lambda - *depth grM \leq *depth gr\Lambda.
\]
Therefore, $\text{Ext}^i_{\Lambda}(M, \Lambda) = 0$ for all $i > *depth gr\Lambda$, so that $id_{\Lambda}\Lambda < \infty$. Similarly, we have $id_{\Lambda^P}\Lambda \leq *depth gr\Lambda$. □

Thanks to Corollary 2.4, we call a filtered ring a "Gorenstein filtered ring", if $gr\Lambda$ is a Gorenstein local ring with the condition (P).

We give a necessary and sufficient condition when $G$-dim $grM = 0$.

2.5. THEOREM. Let $M$ be a filtered $\Lambda$-module with a good filtration. Then the following (1) and (2) are equivalent.

(1) $G$-dim $grM = 0$.

(2) (2.1) $G$-dim $M = 0$.

(2.2) Suppose that $\cdots \to F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} M \to 0$ is a filtered free resolution of $M$, then all $f_i^*(i > 0)$ are strict.

(2.2') Suppose that $\cdots \to G_1 \xrightarrow{g_1} G_0 \xrightarrow{g_0} M^* \to 0$ is a filtered free resolution of $M^*$, then all $g_i^*(i > 0)$ are strict.

(2.3) A canonical map $\theta : M \to M^{**}$ is strict.

Moreover, under the above conditions, $\varphi_M : grM^* \to (grM)^*$ and $\varphi_{M^*} : grM^{**} \to (grM^*)^*$ are isomorphisms, where $\varphi_M = \varphi(M, \Lambda)$, $\varphi_{M^*} = \varphi(M^*, \Lambda)$.

Proof. (1) $\Rightarrow$ (2): It follows from Theorem 2.3 that $G$-dim $M = 0$. From a filtered free resolution of $M$ in (2.2), we get an exact sequence
\[
0 \to M^* \xrightarrow{f_0^*} F_0^* \xrightarrow{f_1^*} F_1^* \to \cdots
\]
This exact sequence and an exact sequence in $\text{mod} gr\Lambda$:
\[
\cdots \to grF_1 \to grF_0 \to grM \to 0
\]
induced from a resolution in (2.2) give the following commutative diagram
\[
\begin{array}{cccccccc}
0 & \to & grM^* & \xrightarrow{gr(f_0^*)} & grF_0^* & \xrightarrow{gr(f_1^*)} & grF_1^* & \to \cdots \\
(f) & \varphi \downarrow & & \varphi_0 \downarrow & & \varphi_1 \downarrow & & \\
0 & \to & (grM)^* & \to & (grF_0)^* & \to & (grF_1)^* & \to \cdots
\end{array}
\]
where $\varphi = \varphi(M, \Lambda)$, $\varphi_i = \varphi(F_i, \Lambda)$. Since $G$-dim $grM = 0$, the second row is exact. For $i \geq 0$, $\varphi_i$ are isomorphisms. Thus a sequence
\[
grF_0^* \xrightarrow{gr(f_0^*)} grF_1^* \xrightarrow{gr(f_1^*)} grF_2^* \to \cdots
\]
is exact, and so $f_1^*$, $f_2^*$, $\cdots$ are strict. Hence (2.2) holds. Since $f_0$ is a strict filtered epimorphism, $f_0^*$ is a strict filtered monomorphism. Thus the first row of (f) is exact. Therefore, $\varphi : grM^* \to (grM)^*$ is an isomorphism. Since $G$-dim $(grM)^* = 0$, we have $G$-dim $grM^* = 0$. Hence (2.2') holds and $\varphi_{M^*}$ is an isomorphism.
Let $\eta : \text{gr}M \to (\text{gr}M)^{**}$ be a canonical homomorphism. Consider the commutative diagram

$$
\begin{array}{ccc}
\text{gr}M & \xrightarrow{\text{gr}\theta} & \text{gr}M^{**} \\
\downarrow \eta & & \downarrow \varphi_{M^*} \\
(\text{gr}M)^{**} & \xrightarrow{\varphi_{M^*}^{-1}} & (\text{gr}M^*). \\
\end{array}
$$

Since $\eta$, $\varphi_{M^*}^{-1}$, $\varphi_{M^*}$ are isomorphisms, $\text{gr}\theta$ is also an isomorphism. Thus $\theta$ is strict.

$(2) \Rightarrow (1)$: By $(2.1)$ and $(2.2)$, the first row of the diagram $(*)$ is exact. Thus the second row of $(*)$ is exact, so that $\text{Ext}^i_{\text{gr}\Lambda}(\text{gr}M, \text{gr}\Lambda) = 0$ for $i > 0$ and $(\text{gr}M)^* \cong \text{gr}M^*$. Since $G\text{-dim}M^* = 0$, using the diagram $(*)$ obtained from $(2.2^*)$, we can show that $\text{Ext}^i_{\text{gr}\Lambda}(\text{gr}M^*, \text{gr}\Lambda) = 0$ for $i > 0$ and $(\text{gr}M^*)^* \cong \text{gr}M^{**}$. Thus we have $\text{Ext}^i_{\text{gr}\Lambda}((\text{gr}M)^*, \text{gr}\Lambda) = 0$ for $i > 0$. By $(2.3)$ and the above argument, the maps $\text{gr}\theta$, $\varphi_{M^*}^{-1}$ and $\varphi_{M^*}$ are isomorphisms in the diagram $(**)$, so that $\eta$ is an isomorphism. Thus $G\text{-dim}grM = 0$. □

2.6. **PROPOSITION.** Assume that $\text{gr}\Lambda$ is a *local ring with the condition (P). Let $M \in \mathcal{G}_e$. Then the following equality holds.

$$G\text{-dim}M + \text{*depth gr}M = \text{*depth gr}\Lambda.$$

**Proof.** The statement follows from Theorem A.8. □

2.7. **REMARKS.** (i) It is interesting to know when $\mathcal{G}_e = \mathcal{G}$. If this is true, then we see that $G\text{-dim}M = 0$ if and only if $G\text{-dim}grM = 0$ for $M \in \mathcal{G}$. Hence the condition $(2.2)$, $(2.2^*)$, $(2.3)$ in Theorem 2.5 are superfluous.

(ii) Suppose that $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$ is a strict exact sequence of fil$\Lambda$. Then the followings are easy consequence of [9], Corollary 1.2.9 (b).

If $M', M'' \in \mathcal{G}_e$ and $G\text{-dim}M' > G\text{-dim}M''$, then $M \in \mathcal{G}_e$.

If $M, M'' \in \mathcal{G}_e$ and $G\text{-dim}M > G\text{-dim}M''$, then $M' \in \mathcal{G}_e$.

We shall study the another valuable invariant ‘grade’. Its nicest feature that an equation $\text{grade}_{\Lambda}M = \text{grade}_{\text{gr}\Lambda}\text{gr}M$ holds for a good filtered $\Lambda$-module $M$ is proved when $\text{gr}\Lambda$ is regular (see e.g. [14]). We prove this equation under ‘module-wise’ conditions by which we can apply this equation fairly wide classes of filtered rings.

2.8. **THEOREM.** Let $\Lambda$ be a filtered ring such that $\text{gr}\Lambda$ is a commutative Noetherian ring and $M$ a filtered $\Lambda$-module with a good filtration. Assume that $\text{gr}M$ has finite $G$-dimension. Then an equality $\text{grade}_{\Lambda}M = \text{grade}_{\text{gr}\Lambda}\text{gr}M$ holds.

**Proof.** Put $s = \text{grade}_{\text{gr}\Lambda}\text{gr}M$. In order to show that $\text{grade}_{\Lambda}M = s$, we must prove:

(i) $\text{Ext}_{\Lambda}^s(M, \Lambda) \neq 0$,

(ii) $\text{Ext}_{\Lambda}^i(M, \Lambda) = 0$ for $i < s$. 

2.8.1. (cf. [14], Chapter III, §1) Let \( \cdots \to F_i \overset{f_i}{\to} \cdots \to F_0 \overset{f_0}{\to} M \to 0 \) be a filtered free resolution of \( M \). Applying \((-)^*\) to it, we get a complex

\[
F_* : 0 \to F_0^* \overset{f_0^*}{\to} \cdots \to F_{i-2}^* \overset{f_{i-2}^*}{\to} F_{i-1}^* \overset{f_{i-1}^*}{\to} F_i^* \to \cdots
\]

with each \( F_i^* \) filtered free and \( f_i^* \) a filtered homomorphism. We put, for \( p, r, i \in \mathbb{N} \),

\[
Z_p^r(i) := (f_i^*)^{-1}((F_{p-r}^*) \cap F_p F_{i-1}^*), \quad Z_p^\infty(i) := \text{Ker} f_i^* \cap F_p F_{i-1}^*,
\]

\[
B_p^r(i) := f_{i-1}^*((F_{p-r-1}^*) \cap F_p F_{i-1}^*), \quad B_p^\infty(i) := \text{Im} f_{i-1}^* \cap F_p F_{i-1}^*.
\]

Then the following sequence of inclusions holds:

\[
Z_0^r(i) \supseteq Z_1^r(i) \supseteq \cdots \supseteq Z_p^\infty(i) \supseteq B_p^\infty(i) \supseteq \cdots \supseteq B_0^0(i) \supseteq B_p^0(i).
\]

We put

\[
E_p^r(i) := \frac{Z_p^r(i) + F_{p-1} F_{i-1}^*}{B_p^r(i) + F_{p-1} F_{i-1}^*}, \quad E_r^i := \bigoplus_p E_p^r(i).
\]

Then \( E_r^i \) is a grA-module for \( r, i \geq 0 \). When \( r = 0 \), we have

\[
E_0^r(i) = \bigoplus_p \frac{(f_i^*)^{-1}(F_p F_{i-1}^*) \cap F_{p-1} F_{i-1}^* + F_{p-1} F_{i-1}^*}{f_{i-1}^*(F_{p-1} F_{i-2}^*) \cap F_{p-1} F_{i-1}^* + F_{p-1} F_{i-1}^*} = \bigoplus_p \frac{F_p F_{i-1}^*}{F_{p-1} F_{i-1}^*} = \text{gr} F_i^r.
\]

Hence we get a complex

\[
E_0^r : 0 \to \text{gr} F_0^r \to \cdots \to \text{gr} F_i^r \to \cdots
\]

which is an associated graded complex of \( F_* \). We show, for \( r \geq 1 \), that \( \{E_r^i\}_{i \geq 0} \) also gives a complex \( E_r^* \). To do so, we define morphisms. By computation, it holds that

\[
E_r^0(i) = \frac{Z_r^0(i)}{B_r^0(i) + Z_{r-1}^0(i)}, \quad f_r^*(Z_r^0(i)) = F_{p-r} F_i^r \cap f_r^*(F_{p} F_{i-1}^*) = B_{p-r}^{r+1}(i + 1).
\]

Thus the following hold:

1. \( f_r^*(Z_r^0(i)) = B_{p-r}^{r+1}(i + 1) \subseteq Z_p^r(i) \),
2. \( f_r^*(B_r^0(i)) = 0 \) and \( f_r^*(Z_{r-1}^0(i)) = B_{p-r}^r(i + 1) \).

We can show that \( f_r^* \) induces a map \( \tilde{f}_r^* : E_r^0(i) \to E_{p-r}^0(i + 1) \), by

\[
\tilde{f}_r^*(x + B_r^0(i) + F_{p-1} F_{i-1}^*) = f_r^*(x) + B_{p-r}^r(i + 1) + Z_{p-r-1}^r(i + 1) \quad (x \in Z_r^0(i)).
\]

Hence \( \tilde{f}_r^*(i)(p \in \mathbb{N}) \) give a graded grA-homomorphism

\[
\tilde{f}_r^* : E_r^0(i) = \bigoplus_p E_r^0(i) \to E_{r+1}^0(i) = \bigoplus_p E_r^0(i + 1)
\]

of degree \(-r\). It is easily seen that \( E_r^* : \cdots \to E_r^0 \overset{\tilde{f}_r^*}{\to} E_{r+1}^r \to \cdots \) is a complex.

2.8.2. Lemma. (cf. [14], p.130 (6)) Under the above notation, we have \( H^r(E_r^*) \cong E_r^{r+1} \).

Proof. We show

\[
H(E_{p+r}^r(i - 1) \overset{f}{\to} E_p^r(i) \overset{g}{\to} E_{p-r}^r(i + 1)) \cong E_{p}^{r+1}(i),
\]

where we put \( f := \tilde{f}_{p+r}^*(i - 1) \), \( g := \tilde{f}_p^*(i) \). Using (1) and (2), we can show that

\[
x + B_p^0(i) + F_{p-1} F_{i-1}^* \in \text{Ker} g \iff x \in (f_r^*)^{-1}(B_{p-r}^r(i + 1) + F_{p-r-1} F_{i}^r).
\]
Thus we get
\[
\text{Kerg} = \frac{(Z_p^r(i) \cap (f_s^r)^{-1}(B_p^r(i + 1) + \mathcal{F}_{p-r+1}^r) + \mathcal{F}_{p-1}^r)}{B_p^r(i) + \mathcal{F}_{p-1}^r}.
\]

Further, we have
\[
\text{Imf} = \frac{f_{i-1}^r(Z_{p+r}^r(i - 1)) + \mathcal{F}_{p-1}^r}{B_p^r(i) + \mathcal{F}_{p-1}^r}.
\]

Hence the desired homology is
\[
\text{Kerg} = \frac{(Z_p^r(i) \cap (f_s^r)^{-1}(B_p^r(i + 1) + \mathcal{F}_{p-r+1}^r) + \mathcal{F}_{p-1}^r)}{f_{i-1}^r(Z_{p+r}^r(i - 1)) + \mathcal{F}_{p-1}^r}.
\]
\[
= \frac{Z_{p-1}^r(i) + Z_p^r(i) \cap (f_s^r)^{-1}(\mathcal{F}_{p-r+1}^r) + \mathcal{F}_{p-1}^r}{f_{i-1}^r(Z_{p+r}^r(i - 1)) + \mathcal{F}_{p-1}^r}.
\]
\[
= \frac{Z_{p-1}^r(i) + Z_{p+1}^r(i) + \mathcal{F}_{p-1}^r}{f_{i-1}^r(Z_{p+r}^r(i - 1)) + \mathcal{F}_{p-1}^r}.
\]
\[
= \frac{Z_{p+1}^r(i) + \mathcal{F}_{p-1}^r}{B_p^r(i) + \mathcal{F}_{p-1}^r}.
\]
\[
= E_{p+1}^r(i),
\]

where (2) (respectively, (1)) is used to show the second (respectively, fourth) equality. \(\square\)

2.8.3. Corollary. Assume that \(E_{i-1}^1 = 0\). Then we have \(E_{i-1}^r = 0\) for \(r \geq 1\) and there exists an exact sequence
\[
0 \rightarrow E_{i-1}^{r+1} \rightarrow E_i^r \rightarrow E_{i+1}^r
\]
of gr\(A\)-modules for each \(r \geq 1\).

Proof. The first assertion directly follows from Lemma 2.8.2. Then the complex \(E_i^r\) yields an exact sequence \(0 \rightarrow H^i(E_i^r) \rightarrow E_i^r \rightarrow E_{i+1}^r\). Since \(H^i(E_i^r) \cong E_i^{r+1}\) by lemma 2.8.2, we get the desired exact sequence. \(\square\)

2.8.4. We will show in this subsection that \(E_{s+1}^r \neq 0\).

Consider the following commutative diagram
\[
E_0^r = \text{gr}(F_i^r) : 0 \rightarrow \text{gr}F_0^r \rightarrow \cdots \rightarrow \text{gr}F_s^r \rightarrow \cdots
\]
\[
0 \rightarrow (\text{gr}F_0^r)^* \rightarrow \cdots \rightarrow (\text{gr}F_s^r)^* \rightarrow \cdots,
\]
where rows are complexes and the second row is obtained by applying Hom_{grA}(\_\_ , gr\(A\)) to a free resolution \(\cdots \rightarrow \text{gr}F_1 \rightarrow \text{gr}F_0 \rightarrow \text{gr}M \rightarrow 0\) of gr\(M\). Hence an isomorphism \(E_{i+1}^1 \cong \text{Ext}^{i}_{grA}(\text{gr}M, \text{gr}A)\) holds by Lemma 2.8.2. (Note that \(E_0^r \cong \text{gr}F_{i+1}^r\).)

By assumption, we can apply A.15 to gr\(M\) and get the fact that grade \(\text{Ext}^{i}_{grA}(\text{gr}M, \text{gr}A) = s\). Hence it holds that grade \(E_{i+1}^1 = s\) and \(E_{i+1}^1 = 0\) for \(i < s\). By Corollary 2.8.3, we get an exact sequence of gr\(A\)-modules
\[
(3) \quad 0 \rightarrow E_{s+1}^{r+1} \rightarrow E_{s+1}^r \rightarrow E_{s+2}^r.
\]

By Lemma 2.8.2, \(E_{s+2}^r\) is a subfactor of \(E_{s+1}^{r+1}\) for \(r \geq 1\). Thus every gr\(A\)-submodule \(U\) of \(E_{s+2}^r\) is also a subfactor of \(E_{s+2}^1 = \text{Ext}^{s+1}_{grA}(\text{gr}M, \text{gr}A)\), so that there exist gr\(A\)-submodules \(X, Y \subset \text{Ext}^{s+1}_{grA}(\text{gr}M, \text{gr}A)\) such that \(U \cong X/Y\). Since grade \(X \geq s+1\) and grade \(Y \geq s+1\)
by A.14, it holds that $\text{grade} U \geq s + 1$. Therefore, $\text{grade}(\text{Im} \varphi_r) \geq s + 1$ for $r \geq 1$. Consider the exact sequence induced from (3):

$$0 \rightarrow E^{r+1}_{s+1} \rightarrow E^{r+1}_r \rightarrow \text{Im} \varphi_r \rightarrow 0.$$ 

Assume that $\text{grade} E^{r+1}_{s+1} = s$. Then $\text{grade} E^{r+1}_r = s$ holds. Hence $\text{grade} E^{r}_{s+1} = s$ holds for all $r \geq 1$ by induction. Especially, $E^{r}_{s+1} \neq 0$ holds for all $r \geq 1$.

2.8.5. Lemma There is an isomorphism $E^{r}_{s+1} \cong \text{gr}(\text{Ext}^i_{\Lambda}(M, \Lambda))$ for $i \geq 0$ and $r \gg 0$.

Proof. Since the filtration $\mathcal{F}$ of $\Lambda$ is Zariskian (see [14], Chapter I, §2, 2.4; §3.3 and Chapter II, §2, 2.1, and Proposition 2.2.1), the lemma follows from [14], Chapter III, §2, Lemma 2.2.1(p. 150) and §1, Corollary 1.1.7(p. 133). \(\square\)

2.8.6. We have shown that $E^{r}_{s+1} \neq 0$. Hence $\text{Ext}^i_{\Lambda}(M, \Lambda) \neq 0$ by Lemma 2.8.5. Therefore, (i) holds.

Conversely, since $\text{grade} \, \text{gr}M = s$, we have $\text{Ext}^i_{\text{gr}A}(\text{gr}M, \text{gr}A) = 0$ for $i < s$. Since $\text{gr} \, \text{Ext}^i_{\Lambda}(M, \Lambda)$ is a subfactor of $\text{Ext}^i_{\text{gr}A}(\text{gr}M, \text{gr}A)$ by Proposition 2.1, we have $\text{gr} \, \text{Ext}^i_{\Lambda}(M, \Lambda) = 0$ for $i < s$. Therefore, $\text{Ext}^i_{\Lambda}(M, \Lambda) = 0$ for $i < s$, so that (ii) holds. This accomplishes the proof of 2.8. \(\square\)

2.9. Remarks. (i) Let $M \in \mathcal{G}$. Then it follows from 2.3 and 2.8 that

$$\text{G-dim} \, \text{gr}M \geq \text{G-dim} M \geq \text{grade} M = \text{grade} \, \text{gr}M.$$

If $\text{gr}M$ is perfect, then above inequalities are equalities. Hence $M \in \mathcal{G}_e$.

(ii) Let $M \in \mathcal{G}_e$ with $\text{G-dim} M = d$. Then every syzygy $\Omega^i M$ of $M$ is also in $\mathcal{G}_e$. For, as $\text{gr} (\Omega^i M)$ and $\Omega^i (\text{gr} M)$ are stably isomorphic, we see that $\text{G-dim} \, \Omega^i M = \text{G-dim} \, \text{gr}(\Omega^i M) = \max\{0, d - i\}$.

Applying Theorem 2.8 to the case that $\text{gr} \Lambda$ is a Gorenstein ring, we get the following.

2.10. Corollary. Let $\Lambda$ be a filtered ring such that $\text{gr} \Lambda$ is a commutative Gorenstein ring and $M$ a filtered $\Lambda$-module with a good filtration. Then the equality $\text{grade} \Lambda M = \text{grade}_{\text{gr} \Lambda} \text{gr}M$ holds.

Proof. Since all the finitely generated $\text{gr} \Lambda$-modules have finite G-dimension (see the proof of [1], Theorem 4.20), this follows from Theorem 2.8. \(\square\)

2.11. Theorem. Let $\Lambda$ be a Gorenstein filtered ring. Let $M$ be a filtered $\Lambda$-module with a good filtration. Then the following equality holds.

$$\text{grade} M + * \dim \, \text{gr} M = * \dim \, \text{gr} \Lambda = * \text{id} \, \text{gr} \Lambda.$$

Proof. This follows from A.9, A.10, A.12 and 2.8. \(\square\)

When $\Lambda$ is a Gorenstein filtered ring, due to the above equality, we can define a holonomic module. Put $* \text{id} \, \text{gr} \Lambda = n$ and $\text{id} \Lambda = d$. Let $M$ be a filtered $\Lambda$-module with a good filtration. Since $\text{grade} M \leq \text{id} \Lambda = d$, we have $n - * \dim \, \text{gr} M \leq d$, hence

$$* \dim \, \text{gr} M \geq n - d.$$

This inequality is a generalization of Bernstein's inequality for a Weyl algebra ([4]).

According to the case of Weyl algebras, we call a finitely generated filtered $\Lambda$-module $M$ a holonomic module, if $* \dim \, \text{gr} M = n - d$. 
3. Cohen-Macaulay modules and holonomic modules

Throughout this section, we assume that $\Lambda$ is a filtered ring such that $\text{gr}\Lambda$ is a Cohen-Macaulay *local* ring with the condition (P) (cf. Appendix). Let $M$ be a finitely generated filtered $\Lambda$-module such that $M \in \mathcal{G}$, i.e., $\text{G-dim} \text{gr} M < \infty$. It follows from 2.3, 2.8, A.8 and A.12 that the following holds:

(1) $\text{G-dim} M + \ast\text{depth} \text{gr} M \leq n$

(2) $\text{grade} M + \ast\text{dim} \text{gr} M = n$,

where we put $n := \ast\text{depth} \text{gr} \Lambda = \ast\text{dim} \text{gr} \Lambda$. We say that $M \in \mathcal{G}$ is a Cohen-Macaulay $\Lambda$-module of codimension $k$, if $\ast\text{depth} \text{gr} M = \ast\text{dim} \text{gr} M = n - k$. Then it is easily seen that if $M$ is Cohen-Macaulay of codimension $k$ then it is perfect of grade $k$, where, due to [1], Definition 4.34, we call $M$ perfect if $\text{G-dim} M = \text{grade} M$. Note also that $M$ is Cohen-Macaulay if and only if $\text{gr} M$ is a perfect $\text{gr}\Lambda$-module by A.8 and A.12. We put

$$C_k(\Lambda) := \{ M \in \mathcal{G} : M \text{ is a Cohen-Macaulay } \Lambda\text{-module of codimension } k \}.$$  

The following is an easy consequence of (1) and (2).

3.1. **Proposition.** Let $M \in C_k(\Lambda)$. Then $\text{Ext}^i_{\Lambda}(M, \Lambda) = 0$ for all $i \neq k$ ($i \geq 0$).

We slightly generalize [16], Lemma 2.7 and Theorem 2.8, and [15], as follows.

3.2. **Theorem.** Let $M \in \mathcal{G}$.

i) If $M \in C_k(\Lambda)$, then $\text{Ext}^k_{\Lambda}(M, \Lambda) \in C_k(\Lambda^{\text{op}})$.

ii) The functor $\text{Ext}^k_{\Lambda}(-, \Lambda)$ induces a duality between the categories $C_k(\Lambda)$ and $C_k(\Lambda^{\text{op}})$.

3.2.1. **Lemma.** Let $N$ be a finitely generated filtered $\Lambda$-module of grade $\text{gr} N = s$.

If the $G$-dimension of $\text{gr} N$ is finite, then we have an embedding $\text{gr}(\text{Ext}^s_{\Lambda}(N, \Lambda)) \hookrightarrow \text{Ext}^s_{\text{gr}\Lambda}(\text{gr} N, \text{gr} \Lambda)$. Moreover, if $\text{gr} N$ is perfect, then the embedding is an isomorphism.

**Proof.** Let $\cdots \rightarrow F_1 \rightarrow F_0 \rightarrow N \rightarrow 0$ be a filtered free resolution of $N$. We use the notation of 2.8.1. It follows from 2.8.2 and 2.8 that

$$E^s_1 \cong H^s(F^0_\ast) \cong H^{s-1}(F_\ast) = \text{Ext}^{s-1}_{\text{gr}\Lambda}(\text{gr} N, \text{gr} \Lambda) = 0,$$

where a complex $F_\ast : 0 \rightarrow F^0_\ast \rightarrow F^1_\ast \rightarrow \cdots$ is as in 2.8.1. There exists an exact sequence

$$0 \rightarrow E^{r+1}_{s+1} \rightarrow E^r_{s+1} \rightarrow E^r_{s+2}$$

for all $r \geq 1$ by 2.8.3, so that $E^r_{s+1} \subset E^1_{s+1}$ for all $r \geq 1$. It follows from Lemma 2.8.5 that, for $r \gg 0$,

$$E^r_{s+1} \cong \text{gr}(\text{Ext}^s_{\Lambda}(N, \Lambda)).$$

Thus, by 2.8.2, we get

$$\text{gr}(\text{Ext}^s_{\Lambda}(N, \Lambda)) \subset E^1_{s+1} \cong \text{Ext}^s_{\text{gr}\Lambda}(\text{gr} N, \text{gr} \Lambda).$$

Assume further that $\text{gr} N$ is perfect. Since $E^r_{s+2}$ is a subfactor of $E^1_{s+2} \cong \text{Ext}^{s+1}_{\text{gr}\Lambda}(\text{gr} N, \text{gr} \Lambda) = 0$, we see $E^r_{s+2} = 0$, which shows that the embedding is an isomorphism. □

3.2.2. **Proof of 3.2.** i) Since $\text{gr} M$ is perfect of grade $k$, it holds that $\text{Ext}^k_{\text{gr}\Lambda}(\text{gr} M, \text{gr} \Lambda)$ is perfect of grade $k$ by [1], Proposition 4.35 and its proof, and so $\text{gr} \text{Ext}^k_{\Lambda}(M, \Lambda)$ is perfect by Lemma 3.2.1. Hence $\text{Ext}^k_{\Lambda}(M, \Lambda) \in C_k(\Lambda^{\text{op}})$. ii) Consider the exact sequence

$$0 \rightarrow \text{Ext}^k_{\Lambda}(M, \Lambda) \rightarrow \text{Tr} \Omega^{k-1} M \rightarrow \Omega \text{Tr} \Omega^k M \rightarrow 0$$
A.9, where \( l \) is the category of all finitely generated filtered (left) \(-\)modules. We recall (cf. [14], Chapter III).

Hence we see that if \( M \) is called holonomic, if \( M \) is Gorenstein, and generalize the former theory which is under the assumption of regularity.

We keep to assume \( \Lambda \) to be a Gorenstein filtered ring and \( M \) is a module over a ring.

According to [6], Theorem 3.9, if \( M \) is a Gorenstein filtered ring, then \( M \) satisfies the "Auslander condition":

\[
\text{For every finitely generated } \Lambda \text{-module } M \text{ and integer } k \geq 0, \\
\text{it holds that } \text{grad}_{\Lambda^\text{op}} M \geq k \text{ for all } \Lambda^\text{op}-\text{submodules } N \subset \text{Ext}^k_{\Lambda^\text{op}}(M, \Lambda).
\]

3.3. Proposition. Let \( M \) be a finitely generated filtered \( \Lambda \)-module. Let \( M \) be holonomic and \( N \) a \( \Lambda \)-submodule of \( M \). Then \( M/N \) are holonomic.

Proof. It follows from [13], Lemma 2.11 (cf. also [6], Theorem 3.9) that \( \text{grade} M/\text{grade} N \geq d \) and \( \text{grade} M/N \geq d \), so that \( \text{grade} N = d \) and \( \text{grade} M/N = d \).

3.4. Proposition. A holonomic module is artinian. Therefore, it is of finite length.

We use the following easy lemma for a proof.

3.4.1. Lemma. Let \( M_i \) (\( i = 0, 1, \cdots \)) be a module over a ring and \( f_i : M_i \to M_{i+1} \) (\( i = 0, 1, \cdots \)) is a homomorphism. Assume that \( M_0 \) is Noetherian and \( f_i \) (\( i = 0, 1, \cdots \)) is surjective. Then there exists an integer \( m \) such that \( f_i \) is an isomorphism for all \( i \geq m \).

3.4.2. Proof of 3.4. Let \( M \) be a holonomic \( \Lambda \)-module and \( M = M_0 \supset M_1 \supset \cdots \) a descending chain of \( \Lambda \)-submodules of \( M \). Then \( M_i, M_{i-1}/M_i \) are holonomic (\( i \geq 1 \)), and so, from an exact sequence \( 0 \to M_i \to M_{i-1} \to M_{i-1}/M_i \to 0 \), we get an exact sequence

\[
0 \to \text{Ext}(M_{i-1}/M_i) \to \text{Ext} M_{i-1} \to \text{Ext} M_i \to 0,
\]

where we put \( \text{Ext}(-) = \text{Ext}_{\Lambda^\text{op}}(^{-}, \Lambda) \). By Lemma 3.4.1, there exists an integer \( m \) such that \( \text{Ext} M_i \to \text{Ext} M_i \) is an isomorphism for \( i \geq m + 1 \). Hence \( \text{Ext}(M_{i-1}/M_i) = 0 \) for \( i \geq m + 1 \). Hence \( M_{i-1}/M_i = 0 \) for \( i \geq m + 1 \) by Remark 3.2.3, that is, \( M_m = M_{m+1} = \cdots \). This completes the proof.
We generalize [14], Chapter III, 4.2.18 Theorem (p. 194), which characterizes a holonomic module by its associated graded module. We put Min(grM) = {p : p is a minimal element of Supp(grM)} for M ∈ filA.

3.5. Theorem. Let M ∈ filA. Then the following are equivalent.

(1) M is holonomic,
(2) htp = d for all p ∈ Min(grM).

A finitely generated module M over a two-sided Noetherian ring is called pure, if gradeN = gradeM for all nonzero submodules N of M.

3.5.1. Lemma. Let M ∈ filA. Then M is pure if and only if grM is a pure grA-module under a suitable filtration on M.

Proof. Let M be pure. Put s = gradeM and N := Ext^s_A(M, A). Since A satisfies Auslander condition, it follows that gradeN = s by [13], Lemma 2.8, so that grade grN = s by 2.8, hence Ext^{s+1}_{grA}(grN, grA) is pure by [13], Proposition 2.13. By 3.2.1, we have gr Ext^s_{A op}(N, A) ⊂ Ext^s_{grA}(grN, grA). Hence gr Ext^s_{A op}(N, A) is a pure grA-module. By [13], Theorem 2.3, there exists an exact sequence

0 → Ext^{s+1}_{A op}(TrΩ^sM, A) → M → Ext^s_{A op}(N, A).

Since grade Ext^{s+1}_{A op}(TrΩ^sM, A) ≥ s + 1 by Auslander condition and M is pure, we see Ext^{s+1}_{A op}(TrΩ^sM, A) = 0. Therefore, M ⊂ Ext^s_{A op}(N, A). According to a filtration on M induced from that of Ext^s_{A op}(N, A), we get an inclusion grM ⊂ gr Ext^s_{A op}(N, A), hence grM is pure. The converse is obvious by Theorem 2.8. □

3.5.2. Lemma. Let R be a commutative Gorenstein ring and M’ a pure R-module. Then gradeM’ = dimR_p for each p ∈ Min(M’).

Proof. Since R_p is a Gorenstein local ring, we have an equality gradeM’_p + dimM’_p = dimR_p (cf. [11], Proposition 4.11). Since p is minimal, dimM’_p = 0, so that, gradeM’_p = dimR_p.

Put g = gradeM’_p, g’ = gradeM’. Since Ext^g_R(M’, R)_p = Ext^g_{R_p}(M’, R_p) = 0, we have Ext^k_R(Ext^g_{R_p}(M’, R), R) = 0 for k > g’. Then there exists N ⊂ M’ such that gradeN = k > g’ by [13], Theorem 2.3, which contradicts the purity of M’. Hence Ext^k_R(Ext^g_{R_p}(M’, R), R) = 0 for all k > g’. But by A.15, grade Ext^k_{R_p}(M’, R_p) = g. Therefore, we see g ≤ g’, and so, g = g’. This completes the proof. □

3.5.3. Proof of Theorem 3.5. Put R = grA.

(1)→(2): Assume that M is holonomic. Since M is pure by Proposition 3.3, grM is pure by 3.5.1. Thus d = grade grM = dimR_p for all p ∈ Min(grM) by 3.5.2. Therefore, htp = d for all p ∈ Min(grM).

(2)→(1): Put I = [0 :_R grM]. Since R is Cohen-Macaulay, we have hti = gradeR/I by [8], Corollary 2.1.4. It follows from [8], Proposition 1.2.10(e) that gradeR/I = grade grM. By assumption, hti = d, so that, grade grM = d, that is, gradeM = d by 2.8. Hence M is holonomic. □

A module having higher grade has a good property.
3.6. Proposition. Let $M$ be a finitely generated filtered $\Lambda$-module with grade $M = \ell$, where $\ell = d - 1$ or $d - 2$. Then $M$ is a perfect $\Lambda$-module if and only if there exists a finitely generated filtered $\Lambda^{op}$-module $M'$ of grade $\ell$ with $M \cong \text{Ext}_{\Lambda^{op}}^{\ell}(M', \Lambda)$.

Proof. Assume that $M \cong \text{Ext}_{\Lambda^{op}}^{\ell}(M', \Lambda)$ with grade $M' = \ell$.

The case $\ell = d - 1$: We see grade $M = d - 1$ by assumption. It follows that grade $\text{Ext}_{\Lambda}^{d}(M, \Lambda) = \text{grade} \text{Ext}_{\Lambda}^{d}(M', \Lambda, \Lambda) \geq d + 2$ by [13], Corollary 2.10. This shows that $\text{Ext}_{\Lambda}^{d}(M, \Lambda) = 0$, that is, G-dim $M \leq d - 1$. Hence G-dim $M$ = grade $M = d - 1$, so that $M$ is perfect.

The case $\ell = d - 2$: It follows from the similar computations as the above case that grade $\text{Ext}_{\Lambda}^{d}(M, \Lambda) \geq d + 2$ and grade $\text{Ext}_{\Lambda}^{d-1}(M, \Lambda) \geq d + 1$. Hence $\text{Ext}_{\Lambda}^{d}(M, \Lambda) = \text{Ext}_{\Lambda}^{d-1}(M, \Lambda) = 0$, so that G-dim $M$ = grade $M = d - 2$, i.e., $M$ is perfect.

The converse follows from [15], Theorem 4. $\square$

3.7. Following [14], Chapter III, 4.3, we call a filtered $\Lambda$-module $M$ geometrically pure (geo-pure for short), if dim$_{\text{gr}M}$ = dim($\text{gr}\Lambda/p$) for all $p \in \text{Min}(\text{gr}M)$. Then we have the following proposition which is a generalization of [14], Chapter III, 4.3.6 Corollary.

3.7.1. Proposition. Let $M$ be a finitely generated filtered $\Lambda$-module, and put Assh $\text{gr}M := \{ p \in \text{Supp} \text{gr}M \mid \text{dim} \text{gr}\Lambda/p = \text{dim} \text{gr}M \}$. Then the following conditions are equivalent.

(1) $M$ is pure,

(2) $M$ is geo-pure and $\text{gr}M$ has no embedded prime.

(3) Ass $\text{gr}M = \text{Assh} \text{gr}M$

Proof. (1)$\Rightarrow$(2): Let $M$ be pure. Then $\text{gr}M$ is pure by 3.5.1. Take any $p \in \text{Min}(\text{gr}M)$. Since $p \in \text{Ass} \text{gr}M$, we have $\text{gr}\Lambda/p \hookrightarrow \text{gr}M$, so $\text{grade} \text{gr}\Lambda/p = \text{grade} \text{gr}M$. Using Theorem A.12, we have dim $\text{gr}\Lambda/p = \text{dim} \text{gr}M$. Hence $M$ is geo-pure. Take any $p \in \text{Ass} \text{gr}M$, then $\text{gr}\Lambda/p \hookrightarrow \text{gr}M$. Thus dim $\text{gr}\Lambda/p = \text{dim} \text{gr}M$, by A.12. Therefore, Ass $\text{gr}M = \text{Min} \text{gr}M$, i.e., $\text{gr}M$ has no embedded primes.

(2)$\Rightarrow$(3): The former condition implies Assh $\text{gr}M = \text{Min} \text{gr}M$, and the latter one implies Min $\text{gr}M = \text{Ass} \text{gr}M$.

(3)$\Rightarrow$(1): By 3.5.1, it suffices to prove that $\text{gr}M$ is pure. Let $N$ be a $\text{gr}\Lambda$-submodule of $\text{gr}M$. Take any $p \in \text{Ass}N$. Then $\text{gr}\Lambda/p \hookrightarrow N$. Thus, by A.12 and assumption, we have $\text{grade} \text{gr}\Lambda/p = \text{grade} \text{gr}M$. By [13], Lemma 2.11, we have

$$\text{grade} \text{gr}M \leq \text{grade} N \leq \text{grade} \text{gr}\Lambda/p = \text{grade} \text{gr}M.$$  

Hence grade $\text{gr}M = \text{grade} N$. This completes the proof. $\square$

3.8. Example. We provide an example of a Gorenstein filtered ring $\Lambda$.

Let $R = k[[x^2, x^3]]$ be a subring of a formal power series ring $k[[x]]$, where $k$ is a field of characteristic zero. Then $(R, m)$ is a local Gorenstein (non-regular) ring of dim $R = 1$, where $m = (x^2, x^3)$. Let a differential operator $T = x\partial$ with $\partial = d/dx$. Let $\Lambda$ be a subring of the first Weyl algebra (see [4], [14]) generated by $R$ and $T$. Then every element of $\Lambda$ is written as $\Sigma a_i T^i$, $a_i \in R$. Note that $T x^i = x^i T + i x^i$, $i \geq 2$. For $P = \Sigma a_i T^i \in \Lambda$, we put $\text{ord} P = \max \{ i : a_i \neq 0 \}$, an order of $P$. Let $\mathcal{F}_1 \Lambda := \{ P \in \Lambda : \text{ord} P \leq i \}$. Then $\{ \mathcal{F}_1 \Lambda \}$ is a filtration of $\Lambda$ and $\text{gr}\Lambda = R[t]$, where $t = \sigma_1(T)$. Thus $\text{gr}\Lambda$ is Gorenstein *local of dimension 2. Note that $m + tR[t]$ is a unique *maximal ideal.

1) id$\Lambda = 2$
Let $I := \Lambda T + \Lambda x^2$ be a left ideal of $\Lambda$. Then $I \neq \Lambda$. We put induced filtrations to $I$ and $\Lambda/I$. i.e.,

$$F_i I = I \cap F_i \Lambda, \quad F_i(\Lambda/I) = (F_i \Lambda + I)/I, \quad i \geq 0.$$  

Then $0 \to I \to \Lambda \to \Lambda/I \to 0$ is a strict exact sequence. Hence $0 \to grI \to gr\Lambda \to gr(\Lambda/I) \to 0$ is exact. Since $grI$ contains $t$ and $x^2$, $gr(\Lambda/I) = gr\Lambda/grI$ is an Artinian $gr\Lambda$-module. Hence $\dim_{gr\Lambda} gr(\Lambda/I) = 0$. Thus grade $\Lambda/I = 2$ by Theorem 2.11, and then $\id \Lambda = 2$ by Corollary 2.4. So $\Lambda/I$ is holonomic.

2) $gl\dim \Lambda = \infty$

It is easily seen that $gr(\Lambda/\Lambda m) \cong R/m[t]$, where a filtration of $\Lambda m$ is given by $F_i(\Lambda m) = (F_i \Lambda)m$. Hence $pd_{gr\Lambda} gr(\Lambda/\Lambda m) = \infty$ which implies $pd_{\Lambda} \Lambda/\Lambda m = \infty$ by Remark 2.7 (ii). Hence $gl\dim \Lambda = \infty$.

**APPENDIX**

In Appendix, we provide the fact about graded rings, especially *local rings*.

1. **SUMMARY FOR *LOCAL RINGS**

Let $R$ be a commutative Noetherian ring. We gather some facts about a graded ring. For the detail, the reader is referred to [8], [12], and [20].

A ring $R$ is called a graded ring, if

i) $R = \oplus_{i \in \mathbb{Z}} R_i$ as an additive group,

ii) $R_iR_j \subset R_{i+j}$ for all $i, j \in \mathbb{Z}$.

An $R$-module $M$ is called a graded module, if

i) $M = \oplus_{i \in \mathbb{Z}} M_i$ as an additive groups,

ii) $R_iM_j \subset M_{i+j}$ for all $i, j \in \mathbb{Z}$.

An $R$-homomorphism $f : M \to N$ of graded modules is called a graded homomorphism, if $f(M_i) \subset N_i$ for all $i \in \mathbb{Z}$. All graded modules in $\mod R$ and all graded homomorphisms form the category of graded modules, which we denote by $\mod_0 R$.

A graded submodule of a graded ring $R$ is called a graded ideal. For any ideal $I$ of $R$, we denote by $I^*$ the graded ideal generated by all homogeneous elements of $I$. A graded ideal $m$ of $R$ is called maximal, if it is a maximal element of all proper graded ideals of $R$. We say that $R$ is a *local ring, if $R$ has a unique *maximal ideal $m$. A *local ring $R$ with the *maximal ideal $m$ is denoted by $(R, m)$. The theory of *local ring is well developed and a lot of facts that hold for local rings also hold for *local rings (see [8] and [12]).

Let $M$ be a finite $R$-module. For an ideal $I$, we denote $I$-depth of $M$ by $\depth(I, M)$([18]). Let $(R, m)$ be a *local ring and $M \in \mod R$. We put *depth.$M := \depth(m, M)$. We shall use *depth as a substitute of depth for a local ring.

A graded module $M$ over a graded ring $R$ is called an injective module, if it is an injective object in $\mod_0 R$([8], §3.6). We denote by *id.$M$ the *injective dimension of $M$. By definition, *id.$M \leq k$ if and only if there exists a minimal *injective resolution

$$0 \to M \to *E^0(M) \to \cdots \to *E^k(M) \to 0.$$  

It is easily seen that *id.$M \leq k$ if and only if $\text{Ext}_i^k(N, M) = 0$ for all $i > k$ and all $N \in \mod_0 R$. 


Let \((R; m)\) be a *local ring. Consider the following condition.

(P) There exists an element of positive degree in \(R - p\) for any graded prime ideal \(p \neq m\)

A positively graded ring satisfies the condition (P). The other examples are seen in [20], Chapter B, III, 3.2.

The following is known.

A.1. Proposition. Let \((R; m)\) be a *local ring with the condition (P). Then, for every graded ideal \(a\) and every set of graded prime ideals \(p_1, \ldots, p_n\), there exists \(i\) such that \(a \subseteq p_i\), whenever all homogeneous elements of \(a\) are contained in \(\bigcup_{i=1}^n p_i\).

Proof. See [19], Lemma 2. □

Using Proposition A.1, the following is proved as the local case.

A.2. Proposition. Let \((R; m)\) be a *local ring with the condition (P). Let \(M\) be a finite graded module with \(*\text{depth}M = t\). Then there exists an \(M\)-sequence \(x_1, \ldots, x_t\) consisting of homogeneous elements in \(m\).

We note the following graded version of Nakayama’s Lemma.

A.3. Lemma. Let \((R; m)\) be a *local ring and \(M\) a finite graded \(R\)-module. If \(mM = M\), then \(M = 0\).

In the following, we assume that \((R; m)\) is a *local ring with the condition (P).

A.4. Lemma. Let \(M, N\) be the non-zero finite graded \(R\)-module with \(*\text{depth}N = 0\). Then \(\text{Hom}_R(M, N) \neq 0\).

Proof. It is well-known, so we omit the proof. □

A.5. Corollary. Assume that \(*\text{depth}R = 0\). Let \(M\) be a finite graded \(R\)-module. Then \(M^* = 0\) implies \(M = 0\).

We state the graded version of [1], 4.11-13 in the following A.6-A.8.

A.6. Proposition. Assume that \(*\text{depth}R = 0\). Let \(M\) be a finite graded \(R\)-module. Then \(\text{G-dim}M < \infty\) if and only if \(\text{G-dim}M = 0\).

Proof. It suffices to prove that \(\text{G-dim}M < \infty\) implies \(\text{G-dim}M = 0\).

Suppose that \(\text{G-dim}M \leq 1\). We have an exact sequence \(0 \rightarrow L_1 \rightarrow L_0 \rightarrow M \rightarrow 0\) with \(\text{G-dim}L_i = 0\) \((i = 0, 1)\). Hence we have an exact sequence

\[
0 \rightarrow M^* \rightarrow L_0^* \rightarrow L_1^* \rightarrow \text{Ext}^1_R(M, R) \rightarrow 0
\]

and \(\text{Ext}^i_R(M, R) = 0\) for \(i > 1\). By this sequence, we have an exact sequence

\[
0 \rightarrow \text{Ext}^1_R(M, R)^* \rightarrow L_1 \rightarrow L_0,
\]

where \(L_1 \rightarrow L_0\) is monic. Thus \(\text{Ext}^1_R(M, R)^* = 0\), and so \(\text{Ext}^1_R(M, R) = 0\) by A.5 Corollary.

Suppose that \(\text{G-dim}M \leq n\). Let \(0 \rightarrow L_n \xrightarrow{f_n} \cdots \xrightarrow{f_1} L_0 \rightarrow M \rightarrow 0\) be exact with \(\text{G-dim}L_i = 0\) \((0 \leq i \leq n)\). Since \(\text{G-dim}(\text{Im}f_{n-1}) \leq 1\), we have \(\text{G-dim}(\text{Im}f_{n-1}) = 0\) by the above argument. Repeating this process, we get \(\text{G-dim}M = 0\). □

We want to generalize [1], Theorem 4.13 (b) to the graded case. The proof of it needs a part of [1], Proposition 4.12. Thus we adapt this proposition as follows.
A.7. Proposition. Assume that \( \ast \text{depth} R = t \). Let \( M \) be a finite graded \( R \)-module with \( \text{G-dim} M < \infty \). Then the following are equivalent.

(1) \( \text{G-dim} M = 0 \).

(2) \( \ast \text{depth} M \geq \ast \text{depth} R \).

(3) \( \ast \text{depth} M = \ast \text{depth} R \).

Proof. (1) \( \Rightarrow \) (2): Let \( x_1, \cdots, x_i \) be a homogeneous regular sequence in \( m \). We show that \( x_1, \cdots, x_i \) is an \( M \)-sequence by induction on \( i \). Let \( i = 1 \). Since \( M \cong M^{**} \) is torsionfree, \( x_1 \) is \( M \)-regular.

Suppose that \( i > 1 \) and the assertion holds for \( i - 1 \). Then \( x_1, \cdots, x_{i-1} \) is an \( M \)-sequence. Put \( I = (x_1, \cdots, x_{i-1}) \), \( \overline{R} = R/I \), \( \overline{M} = M/IM \). Then \( (\overline{R}, \overline{m}) \) is an \( * \)local ring with the condition (P). By [1], Lemma 4.9, \( \text{G-dim} \overline{M} = \text{G-dim} R M = 0 \). Since \( \overline{m} \in \overline{R} \) is a regular element, \( \overline{m} \) is \( \overline{M} \)-regular, hence \( x_1, \cdots, x_i \) is an \( M \)-sequence. Therefore, \( \ast \text{depth} M \geq \ast \text{depth} R \).

(2) \( \Rightarrow \) (1): By assumption, it suffices to prove that \( \text{Ext}^i_R(M, R) = 0 \) for \( i > 0 \). We show the assertion by induction on \( t = \ast \text{depth} R \).

Let \( t = 0 \). Then \( \text{G-dim} M = 0 \) by Proposition A.6.

Let \( t > 0 \). Then \( \ast \text{depth} M \geq \ast \text{depth} R \geq 1 \). We take a homogeneous element \( x \in m \) which is \( R \) and \( M \)-regular. Then, by [8], 1.2.10 (d),

\[
\ast \text{depth} R M/x M = \ast \text{depth} R M - 1 \geq \ast \text{depth} R - 1 = \ast \text{depth} R/x R.
\]

Hence we have \( \text{Ext}^i_R(M/x M, R/x R) = 0 \) for \( i > 0 \) by induction. This gives \( \text{Ext}^i_R(M, R/x R) = 0 \) for \( i > 0 \). From an exact sequence \( 0 \to R \xrightarrow{\partial} R \to R/x R \to 0 \), we get an exact sequence

\[
\text{Ext}^i_R(M, R) \xrightarrow{\partial} \text{Ext}^i_R(M, R) \to \text{Ext}^i_R(M, R/x R) = 0.
\]

By Nakayama’s Lemma, it holds that \( \text{Ext}^i_R(M, R) = 0 \) for \( i > 0 \).

(2) \( \Rightarrow \) (3): When \( \ast \text{depth} R = 0 \), we have \( \text{G-dim} M = 0 \) by Proposition A.6. Since \( \ast \text{depth} R = 0 \), we have an exact sequence \( 0 \to \text{Hom}_R(M^*, R/m) \to M^{**} \cong M \).

Since \( M^* \neq 0 \), we have \( \text{Hom}_R(M^*, R/m) \neq 0 \). Since \( m \text{Hom}_R(M^*, R/m) = 0 \), we see that \( m \) has no \( M \)-regular element, so that \( \ast \text{depth} M = 0 \). Thus (3) holds.

Let \( \ast \text{depth} R > 0 \). We have \( \ast \text{depth} M \geq \ast \text{depth} R \geq 1 \), so that there is a homogeneous element \( x \in m \) which is \( R \) and \( M \)-regular. By [1], Lemma 4.9, we have \( \text{G-dim} R M/x M < \infty \). We have

\[
\ast \text{depth} R M/x M = \ast \text{depth} R M - 1 \geq \ast \text{depth} R - 1 = \ast \text{depth} R/x R.
\]

Hence, by induction on \( \ast \text{depth} R \), we have \( \ast \text{depth} R M/x M = \ast \text{depth} R/x R \), and then \( \ast \text{depth} M = \ast \text{depth} R \).

Since (3) \( \Rightarrow \) (2) is obvious, we accomplish the proof. \( \square \)

A.8. Theorem. Let \( M \) be a finite graded \( R \)-module with \( \text{G-dim} M < \infty \). Then we have an equality

\[
\text{G-dim} M + \ast \text{depth} M = \ast \text{depth} R.
\]

Proof. We state the proof which is an adaptation of [1]. If \( \text{G-dim} M = 0 \), we are done by the previous proposition. Suppose that \( \text{G-dim} M = n > 0 \) and the equation holds for \( n - 1 \). Let \( 0 \to K \to F \to M \to 0 \) be exact with \( F \) graded free and \( K \) a
graded module. Since $\text{G-dim} K = n - 1$, we have $\text{G-dim} K + \ast \text{depth} K = \ast \text{depth} R$ by induction. Suppose that $\ast \text{depth} M \geq \ast \text{depth} F = \ast \text{depth} R$. Then $\text{G-dim} M = 0$ holds by the previous proposition. This contradicts to $\text{G-dim} M > 0$. Hence $\ast \text{depth} M < \ast \text{depth} F$, so $\ast \text{depth} K = \ast \text{depth} M + 1$ by, e.g., [8], 1.2.9. Therefore, $n + \ast \text{depth} M = \ast \text{depth} R$. □

Let $M$ be a finite graded $R$-module. Then the similar argument to [1], 4.14 and 4.15 shows that $\text{G-dim} M \leq n$ if and only if G-dim$_p M \leq n$ for all graded prime (respectively, graded maximal) ideals $p$ of $R$. Note that all the prime ideals in Ass$M$ are graded ideals (e.g. [8], Lemma 1.5.6). Thus, in *local case, we have that $\text{G-dim} M \leq n$ if and only if G-dim$_M M \leq n$. Thus we give the following characterization of Gorensteiness.

A.9. Theorem. Let $(R, m)$ be a *local ring with the condition (P). Then the following are equivalent.

(1) $R$ is Gorenstein.

(2) Every finite graded $R$-module has finite $G$-dimension.

Under these equivalent conditions, the equality $\ast \text{id} R = \ast \text{depth} R$ holds.

Proof. (1) $\Rightarrow$ (2): Since $R_m$ is Gorenstein, we have $\text{G-dim} M_m < \infty$, hence $\text{G-dim} M < \infty$ by above.

(2) $\Rightarrow$ (1): Let $t = \ast \text{depth} R$. Take any finite graded $R$-module $M$. Since $\text{G-dim} M = t - \ast \text{depth} M \leq t$ by Theorem A.8, we have that $\text{Ext}_R^i(M, R) = 0$ for all $i > t$. Hence $\ast \text{id} R \leq t$. It holds from [8], Theorem 3.6.5 or [20], Chapter B, III.1.7 that $\text{id} R \leq \ast \text{id} R + 1 \leq t + 1$. Hence $R$ is Gorenstein.

The second statement follows from the similar argument to the local case (cf. [8], Theorem 3.1.17). We note that ‘the residue field’ in the local case should be replaced by ‘the unique graded simple module $R/\mathfrak{m}$’ in *local case and the use of the graded version of Bass’s Lemma (see e.g. [20], Chapter B, III.1.9) is effective. □

Let $(R, m)$ be a *local ring. Then one of the following cases occurs ([12], §1 or [8], §1.5):

A. $R/m$ is a field,

B. $R/m \cong k[t, t^{-1}]$, where $k$ is a field and $t$ is a homogeneous element of positive degree and transcendental over $k$.

We put $\ast \text{dim} R := \text{ht} \mathfrak{m}$ the *dimension of a *local ring $(R, m)$. Note that $\ast \text{dim} R$ equals the supremum of all numbers $h$ such that there exists a chain of graded prime ideals $\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_h$ in $R$ [8]. Let $M$ be a finite graded $R$-module. It is easily seen that $[0 : R M]$ is a graded ideal. Thus we put $\ast \text{dim} M := \ast \text{dim} R/[0 : R M]$.

A.10. Lemma. Let $(R, m)$ be a Cohen-Macaulay *local ring with the condition (P) and $\text{dim} R = n$, and $M$ a finite graded $R$-module. Then we have

$$\ast \text{dim} R = \ast \text{depth} R = \begin{cases} n & \text{for Case A}, \\ n - 1 & \text{for Case B}. \end{cases}$$

$$\ast \text{dim} M = \begin{cases} \text{dim} M & \text{for Case A}, \\ \text{dim} M - 1 & \text{for Case B}. \end{cases}$$

Moreover, assume that $R$ is Gorenstein, then $\text{id} R = \text{dim} R = n$, where $\text{id} R$ stands for the injective dimension of $R$. 
Proof. Case A. Let \( \mathfrak{n} \) be a maximal ideal with \( \text{ht} \mathfrak{n} = n \). If \( \mathfrak{n} = \mathfrak{m} \), then \( \text{ht} \mathfrak{m} = n \). Suppose that \( \mathfrak{n} \) is not equal to \( \mathfrak{m} \). Then \( \mathfrak{n} \) is not graded, so \( \text{ht} \mathfrak{n}/\mathfrak{n}^* = 1 \). Since \( R_\mathfrak{n} \) is Cohen-Macaulay,

\[
\text{ht} \mathfrak{n}^* R_\mathfrak{n} + \dim R_\mathfrak{n}/\mathfrak{n}^* R_\mathfrak{n} = \dim R_\mathfrak{n} = n
\]

([18], Theorem 17.4). Hence \( \text{ht} \mathfrak{n}^* R_\mathfrak{n} = n - 1 \), so \( \text{ht} \mathfrak{n}^* = n - 1 \). Thus \( \mathfrak{m} \geq \text{ht} \mathfrak{n}^* + 1 = n \), so that \( \text{ht} \mathfrak{m} = n \). Therefore,

\[
*\text{depth} R = \text{depth} R_\mathfrak{m} = \dim R_\mathfrak{m} = \text{ht} \mathfrak{m} = n.
\]

Case B. Let \( \mathfrak{n} \) be the same as in Case A. Since \( \mathfrak{n} \) is not graded, we have \( \text{ht} \mathfrak{n}^* = n - 1 \) by the similar way to Case A. By assumption, we have that \( \mathfrak{m} \supset \mathfrak{n}^* \) and \( \mathfrak{m} \) is not maximal, so \( \mathfrak{m} = \mathfrak{n}^* \). Therefore, \( \text{ht} \mathfrak{m} = n - 1 \), hence we get \( *\text{depth} R = n - 1 \) by the similar way to Case A.

The equality concerning \( *\dim M \) follows from the fact that cases A and B are preserved modulo \([0 : R M]\).

The latter statement is proved in [3] more generally. \( \square \)

A.11. Lemma Let \((R, m)\) be a Cohen-Macaulay \(*\)local ring with the condition (P) and \(x\) a homogeneous element in \(m\). If \(x\) is regular, then \(\dim R/xR = \dim R - 1\).

Proof. The well-known induction argument works due to A.10 Lemma. \( \square \)

A.12. Theorem Let \((R, m)\) be a Cohen-Macaulay \(*\)local ring with the condition (P) and \(M\) a finite graded \(R\)-module. Then

\[
\text{grade} M + \dim M = \dim R
\]

Proof. We follow the proof of [11], Proposition 4.11. Put \(n = \dim R\). We prove the statement by induction on \(n\). Suppose that \(\dim M = n\) and take \(p \in \text{Supp} M \) with \(\dim R/p = n\). Then \(\dim R_p = 0\), so that \(\text{depth} R_p = 0\). Thus \(p R_p \subseteq \text{Ass} R_p\). Hence \(\text{Hom}_{R_p}(M_p, R_p/p R_p) \neq 0\) implies \(\text{Hom}_{R_p}(M_p, R_p) \neq 0\). Thus \(\text{Hom}_{R}(M, R) \neq 0\), i.e., \(\text{grade} M = 0\).

When \(n = 0\), we have \(\dim M = 0\). Then the equality holds by above. Let \(n > 0\). Then we can assume \(\dim M < n\). Since \(\dim R/p = n\) for any minimal prime ideal \(p\) of \(R\), it holds from the assumption that \([0 : R M] \nsubseteq p\) for any minimal prime ideal \(p\) of \(R\). Thus \([0 : R M] \nsubseteq p\) for any \(p \in \text{Ass} R\). Since \([0 : R M] \) is a graded ideal, \([0 : R M] \) contains a homogeneous regular element \(x\) by A.1 Proposition. We have that \(\text{Ext}^i_{R}(M, R) \cong \text{Ext}^{i-1}_{R/(xR)}(M, R/xR)\) for \(i \geq 0\). Thus \(\text{grade}_{R/(xR)} M = \text{grade}_{R} M - 1\). By Lemma A.11 and induction, we get \(\dim R_{/xR} M + \text{grade}_{R/(xR)} M = n - 1\), hence \(\dim R M + \text{grade}_{R} M - 1 = n - 1\), which gives the desired equality. \( \square \)

We state a characterization of a Cohen-Macaulay graded module over a \(*\)local ring by means of the \(*\text{depth} \) and \(*\text{dimension} \).

A.13. Theorem Let \((R, m)\) be a \(*\)local ring with the condition (P) and \(M \in \text{mod}_0 R\). Then \(M\) is Cohen-Macaulay if and only if \(*\text{depth} M = *\text{dim} M\).

Proof. Put \(I = [0 : R M] \) and \(\overline{R} = R/I\), \(\overline{m} = m/I\). Then we have that \(*\text{dim} M = \dim \overline{R} m = \dim R_m/[0 : R_m M_m] = *\text{dim} M_m\). It holds from [19] or [20], Chapter B, Theorem III.2.1 that \(M\) is Cohen-Macaulay if and only if \(M_m\) is Cohen-Macaulay. Look at the following inequalities

\[
*\text{depth} M = \text{depth}(m, M) \leq \text{depth} M_m \leq \dim M_m = *\text{dim} M.
\]
If \( \ast \text{depth} M = \ast \text{dim} M \), then \( M_\mathfrak{m} \) is Cohen-Macaulay by above. Conversely, suppose \( M \) to be Cohen-Macaulay. Then \( \text{depth}(\mathfrak{m}, M) = \text{depth} M_\mathfrak{m} \) holds by [18], Theorem 17.3. Thus we get \( \ast \text{depth} M = \ast \text{dim} M \) from the above inequalities. □

A.14. Lemma. ([1], Proposition 4.16) Let \( R \) be a commutative Noetherian ring and \( X \) a finite \( R \)-module with \( \text{G-dim} X < \infty \). Then \( \text{grade} U \geq i \) for all \( i > 0 \) and all \( R \)-submodules \( U \) of \( \text{Ext}_R^i(X, R) \).

Proof. Let \( p \in \text{Supp} U \). Then \( \text{Ext}_R^i(X_p, R_p) \neq 0 \). Hence \( \text{G-dim}_R X_p \geq i \). By Auslander-Bridger formula ([1], Theorem 4.13 (b) or [9], Theorem 1.4.8), it follows that

\[
\text{depth} R_p = \text{depth} X_p + \text{G-dim}_R X_p \geq \text{G-dim}_R X_p \geq i.
\]

Hence \( \text{grade} U \geq \min \{ \text{depth} R_p : p \in \text{Supp} U \} \geq i \) by [1], Corollary 4.6. □

A.15. Lemma. Let \( R \) be a commutative Noetherian ring and \( X \) a finite \( R \)-module of grade \( s \). Assume \( \text{G-dim} X \) to be finite. Then the equality \( \text{grade} \text{Ext}_R^s(X, R) = s \) holds true.

Proof. When \( s = 0 \), that is, \( X^* \neq 0 \), then \( X^{***} \neq 0 \). Hence \( X^{**} \neq 0 \).

We assume that \( s > 0 \). By A.14, it holds that \( \text{grade} \text{Ext}_R^s(X, R) \geq s \). The converse inequality follows from [13], Lemma 4.4 (Its proof contains trivial misprints : in the last line of p.182, \( X_n \) should be read \( (\Omega^n X)^* \) and three places in line 3-5 of p.183 should be read similarly). Hence we get the desired equality. □

References


Department of Mathematical Sciences, Shinshu University, Matsumoto, 390-8621, Japan

E-mail address: (*) miyahara shinshu@yahoo.co.jp, (**)kenisida@math.shinshu-u.ac.jp