微分積分学 I 演習問題 7

問題 1. 以下の 2 変数関数 $f(x,y)$ の 2 階偏導関数 $\frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial x \partial y}, \frac{\partial^2 f}{\partial y^2}$ を求めよ.

(1) $f(x,y) = 0.$

(2) $f(x,y) = 1.$

(3) $f(x,y) = x.$

(4) $f(x,y) = y^2.$

(5) $f(x,y) = 2x + y.$

(6) $f(x,y) = x + xy.$

(7) $f(x,y) = x^3 y^2.$

(8) $f(x,y) = x^2 y + xy^3 + y^2.$

(9) $f(x,y) = e^{x-2y}.$

(10) $f(x,y) = e^{xy}.$

(11) $f(x,y) = \log(xy).$

(12) $f(x,y) = \sin(xy).$

(13) $f(x,y) = \dfrac{y}{x}.$

(14) $f(x,y) = \log(1 - xy).$

(15) $f(x,y) = e^{x^2+2y^2}.$

(16) $f(x,y) = \cosh(2x^2 - y^2).$

(17) $f(x,y) = \dfrac{x-2y}{2x+y}.$

(18) $f(x,y) = \log(2x^2 + y^2).$

(19) $f(x,y) = \dfrac{1}{x^2+2y^2}.$

(20) $f(x,y) = \dfrac{xy}{x^2+2y^2}.$

問題 2. 微分可能な関数 $g(z)$ が与えられたとする. このとき, g の導関数 g' を用いて, 以下で与えられる関数 $f(x,y)$ の偏導関数 $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ を書き下せ.

(1) $f(x,y) = g(x+y).$ (2) $f(x,y) = g(x^2 - y^2).$

(3) $f(x,y) = g(y/x).$ (4) $f(x,y) = g(\sqrt{x^2+y^2}).$

問題 3. 微分可能な関数 $f(x,y)$ が与えられたとする. このとき, f の偏導関数 $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ 用いて, 以下で与えられる関数 $F(t)$ の導関数 $F'(t)$ を書き下せ.

(1) $F(t) = f(1,t).$ (2) $F(t) = f(t^2, t^3).$

(3) $F(t) = f(\cos t, \sin t).$ (4) $F(t) = f(\cosh t, \sinh t).$

問題 1 の解答 ($f_{xx} = \frac{\partial^2 f}{\partial x^2}, f_{xy} = \frac{\partial^2 f}{\partial x \partial y}, f_{yy} = \frac{\partial^2 f}{\partial y^2}$ とおく):

(1) $f_{xx} = 0,$ $f_{xy} = 0,$ $f_{yy} = 0.$

(2) $f_{xx} = 0,$ $f_{xy} = 0,$ $f_{yy} = 0.$

(3) $f_{xx} = 0,$ $f_{xy} = 0,$ $f_{yy} = 0.$

(4) $f_{xx} = 0,$ $f_{xy} = 0,$ $f_{yy} = 2.$

(5) $f_{xx} = 0,$ $f_{xy} = 0,$ $f_{yy} = 0.$

(6) $f_{xx} = 0,$ $f_{xy} = 1,$ $f_{yy} = 0.$

(7) $f_{xx} = 6xy^2,$ $f_{xy} = 6x^2y,$ $f_{yy} = 2x^3.$

(8) $f_{xx} = 2y,$ $f_{xy} = 2x + 3y^2,$ $f_{yy} = 6xy + 2.$

(9) $f_{xx} = e^{x-2y},$ $f_{xy} = -2e^{x-2y},$ $f_{yy} = 4e^{x-2y}.$

(10) $f_{xx} = y^2 e^{xy},$ $f_{xy} = (1 + xy)e^{xy},$ $f_{yy} = x^2 e^{xy}.$

(11) $f_{xx} = -\dfrac{1}{x^2},$ $f_{xy} = 0,$ $f_{yy} = -\dfrac{1}{y^2}.$

(12) $f_{xx} = -y^2 \sin xy,$ $f_{xy} = \cos xy - xy \sin xy,$ $f_{yy} = -x^2 \sin xy.$

(13) $f_{xx} = \dfrac{2y}{x^3},$ $f_{xy} = -\dfrac{1}{x^2},$ $f_{yy} = 0.$

(14) $f_{xx} = \dfrac{-y^2}{(1 - xy)^2},$ $f_{xy} = \dfrac{-1}{(1 - xy)^2},$ $f_{yy} = \dfrac{-x^2}{(1 - xy)^2}.$

(15) $f_{xx} = 2(1 + 2x^2)e^{x^2+2y^2},$ $f_{xy} = 8xye^{x^2+2y^2},$ $f_{yy} = 4(1 + 4y^2)e^{x^2+2y^2}.$

(16) $f_{xx} = 4\sinh(2x^2 - y^2) + 16x^2 \cosh(2x^2 - y^2),$ $f_{xy} = -8xy \cosh(2x^2 - y^2),$ $f_{yy} = -2\sinh(2x^2 - y^2) + 4y^2 \cosh(2x^2 - y^2).$

(17) $f_{xx} = \dfrac{-20y}{(2x + y)^3},$ $f_{xy} = \dfrac{10x - 5y}{(2x + y)^3},$ $f_{yy} = \dfrac{10x}{(2x + y)^3}.$

(18) $f_{xx} = \dfrac{-8x^2 + 4y^2}{(2x^2 + y^2)^2},$ $f_{xy} = \dfrac{-8xy}{(2x^2 + y^2)^2},$ $f_{yy} = \dfrac{4x^2 - 2y^2}{(2x^2 + y^2)^2}.$

(19) $f_{xx} = \dfrac{2x^2 - 4y^2}{(x^2 + 2y^2)^3},$ $f_{xy} = \dfrac{16xy}{(x^2 + 2y^2)^3},$ $f_{yy} = \dfrac{-4x^2 + 8y^2}{(x^2 + 2y^2)^3}.$

(20) $f_{xx} = \dfrac{2x^3y - 12xy^3}{(x^2 + 2y^2)^3},$ $f_{xy} = \dfrac{-x^4 + 12x^2y^2 - 4y^4}{(x^2 + 2y^2)^3},$ $f_{yy} = \dfrac{-8x^3y}{(x^2 + 2y^2)^3}.$

問題 2 の解答:

(1) $\dfrac{\partial f}{\partial x} = g'(x + y),$ $\dfrac{\partial f}{\partial y} = g'(x + y).$

(2) $\dfrac{\partial f}{\partial x} = 2xg'(x^2 + y^2),$ $\dfrac{\partial f}{\partial y} = -2yg'(x^2 + y^2).$

(3) $\dfrac{\partial f}{\partial x} = \dfrac{-y}{x^2} g'(y/x),$ $\dfrac{\partial f}{\partial y} = \dfrac{1}{x} g'(y/x).$

(4) $\dfrac{\partial f}{\partial x} = \dfrac{x}{\sqrt{x^2 + y^2}} g'(x^2 + y^2),$ $\dfrac{\partial f}{\partial y} = \dfrac{y}{\sqrt{x^2 + y^2}} g'(x^2 + y^2).$

問題 3 の解答:

(1) $$F'(t) = \frac{\partial f}{\partial y}(1, t).$$

(2) $$F'(t) = 2t\frac{\partial f}{\partial x}(t^2, t^3) + 3t^2\frac{\partial f}{\partial y}(t^2, t^3).$$

(3) $$F'(t) = -\sin t\frac{\partial f}{\partial x}(\cos t, \sin t) + \cos t\frac{\partial f}{\partial y}(\cos t, \sin t).$$

(4) $$F'(t) = \sinh t\frac{\partial f}{\partial x}(\cosh t, \sinh t) + \cosh t\frac{\partial f}{\partial y}(\cosh t, \sinh t).$$