CLIFFORD MODULES, FINITE-DIMENSIONAL APPROXIMATION AND TWISTED $K$-THEORY

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Abstract. A twisted version of Furuta’s generalized vector bundle provides a finite-dimensional model of twisted $K$-theory. We generalize this fact involving actions of Clifford algebras. As an application, we show that an analogy of the Atiyah-Singer map for the generalized vector bundles is bijective. Also, a finite-dimensional model of twisted $K$-theory with coefficients $\mathbb{Z}/p$ is given.

1. Introduction

Furuta’s generalized vector bundle [10], which we call a vectorial bundle in this paper, arises naturally as a geometric object approximating a family of Fredholm operators. This means that there is a natural homomorphism of groups

$$\alpha : [X, \mathcal{F}(\mathcal{H})] \to KF(X),$$

where $[X, \mathcal{F}(\mathcal{H})]$ is the group of homotopy classes of continuous maps from a topological space $X$ to the space $\mathcal{F}(\mathcal{H})$ of Fredholm operators on a separable Hilbert space $\mathcal{H}$, and $KF(X)$ is the group of homotopy classes of $(\mathbb{Z}/2$-graded) vectorial bundles on $X$. Usual vector bundles are examples of vectorial bundles, so that there exists a natural homomorphism from the $K$-group $K(X)$ to $KF(X)$. It is shown [10] that this homomorphism is an isomorphism on a compact Hausdorff space $X$. In this case, the $K$-group of $X$ is also realized as $[X, \mathcal{F}(\mathcal{H})]$, as is well-known [1]. Hence the homomorphism $\alpha$, coming from a “finite-dimensional approximation”, turns out to be bijective.

In [11], the construction above is generalized to

$$\alpha : K^\tau(X) \to KF^\tau(X),$$

where $K^\tau(X)$ stands for the twisted $K$-group [6, 8] twisted by a principal bundle $\tau$ over $X$ whose structure group is the projective unitary group of $\mathcal{H}$, and $KF^\tau(X)$ consists of homotopy classes of $\tau$-twisted vectorial bundles on $X$. The homomorphism $\alpha$ again comes from an idea of finite-dimensional approximation of a family of Fredholm operators, and turns out to be bijective for any CW complex $X$. It should be noticed that a general description of a class in $K^\tau(X)$ usually involves some infinite-dimensional objects. The isomorphism above provides a way to describe $K^\tau(X)$ in terms of finite-dimensional objects.

The aim of this paper is to generalize the isomorphisms $\alpha$ involving actions of Clifford algebras: let $Cl(n) = Cl(\mathbb{R}^n)$ be the Clifford algebra associated to $\mathbb{R}^n$ equipped with the standard metric, $\mathcal{H}_n$ a separable infinite-dimensional $\mathbb{Z}/2$-graded Hilbert space which contains each irreducible $\mathbb{Z}/2$-graded module of $Cl(n)$ infinitely many, and $\mathcal{F}_n$ the non-contractible connected component of the space of self-adjoint
Fredholm operators on $\mathcal{H}_n$ which are degree 1 (i.e. switching the gradings) and anti-commute with the actions of generators of $Cl(n)$. As is known [7], $\mathcal{F}_n$ classifies the $K$-cohomology $K^{-n}$, so that $[X, \mathcal{F}_n] \cong K^{-n}(X)$. On the other hand, vectorial bundles with $Cl(n)$-actions are also introduced in [10]. Their homotopy classes constitute a group $KF_{Cl(n)}(X)$, providing a model of the $K$-cohomology $K^{-n}(X)$. As before, we can construct a natural homomorphism

$$\alpha : [X, \mathcal{F}_n] \to KF_{Cl(n)}(X).$$

Taking a “twist” into account, we also have a natural homomorphism

$$K^{\tau-n}(X) \to KF_{Cl(n)}^\tau(X).$$

Then we will prove:

**Theorem 1.** For any twist $\tau$ on a CW complex $X$, the homomorphism $K^{\tau-n}(X) \to KF_{Cl(n)}^\tau(X)$ is bijective.

The idea of the proof of Theorem 1 is parallel to that in [11]: we lift $K^{\tau-n}(X)$ and $KF_{Cl(n)}^\tau(X)$ to certain generalized cohomology theories, and compare these theories by using a natural transformation induced from $\alpha$. Then the problems reduce to the case of a single point: The key fact that the natural transformation is bijective in this case again relies on a result of Furuta [10].

The main result in [11] allows us to describe classes in $K^{\tau-n}(X)$ by using ordinary twisted vectorial bundles on $X \times [0, 1]^n$, whereas Theorem 1 provides a different way to describe classes in $K^{\tau-n}(X)$. The equivalence of these two options is useful in studying $K^{\tau-n}(X)$, and will be applied to a construction of twisted differential $K$-cohomology in a forthcoming paper.

A more simple application of Theorem 1 is the bijectivity of a homomorphism

$$AS : KF_{Cl(n)}^\tau(X) \to KF_{Cl(n-1)}^{\tau-1}(X),$$

whose construction is similar to that of the homotopy equivalence $\mathcal{F}_n \to \Omega \mathcal{F}_{n-1}$ of Atiyah-Singer [7]. Another application of Theorem 1 is an introduction of a finite-dimensional model of twisted mod $p$ $K$-theory, or twisted $K$-theory with coefficients in $\mathbb{Z}/p$, based on twisted vectorial bundles with Clifford action.

The organization of this paper is as follows: In Section 2, we recall Clifford modules [2, 9], and the classifying space $\mathcal{F}_n$ of the $K$-cohomology constructed out of the space of Fredholm operators [7]. In Section 3, we briefly review a definition of twisted $K$-theory, and summarize axioms of the induced cohomology theory. In Section 4, we introduce twisted vectorial bundles with Clifford action, generalizing an idea in [10]. The definition is quite parallel to that of twisted vectorial bundles without Clifford action [11]. In this section, we also summarize axioms of certain cohomology theory induced from $KF_{Cl(n)}^\tau(X)$: its proof is skipped, because the argument in [11] is straightly generalized to the present case. Then, in Section 5, we introduce the homomorphisms $\alpha$ and prove our main theorem (Theorem 5.2), from which Theorem 1 is derived as a corollary. In the proof of the main theorem, we refrain from reproducing the same argument as that in [11], and only details a proof of a key proposition. Finally, in Section 6, we introduce the counterpart of the Atiyah-Singer map to twisted vectorial bundles with Clifford action, and prove its bijectivity. Our finite-dimensional model of twisted mod $p$ $K$-theory is also provided in this section.
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2. Review of Clifford modules and Fredholm operators

2.1. Clifford modules. For $n > 0$, we let $Cl(n) = Cl(\mathbb{R}^n)$ be the Clifford algebra associated to the standard $\mathbb{R}^n$, that is, the algebra over $\mathbb{R}$ generated by the generators $e_i$, $(i = 1, \ldots, n)$ subject to the relation $e_ie_j + e_je_i = -2\delta_{ij}$.

By a (unitary) module of $Cl(n)$, we mean a $\mathbb{Z}/2\mathbb{Z}$-graded Hermitian vector space $V = V^0 \oplus V^1$ over $\mathbb{C}$ equipped with an algebra homomorphism $\rho : Cl(n) \to \text{End}_\mathbb{C}(V)$ such that $\rho(e_i) : V \to V$, $(i = 1, \ldots, n)$, are skew-Hermitian maps of degree 1. (As a convention of this paper, we put a hat on the symbol of the direct sum to distinguish the grading of a $\mathbb{Z}/2\mathbb{Z}$-graded vector space $V$: the even part appears on the left of $\oplus$ and the odd part on the right.)

Finite-rank irreducible modules of $Cl(n)$ are classified as follows: if $n$ is odd, then $Cl(n)$ has essentially a unique irreducible module $\Delta_n$; if $n$ is even, then $Cl(n)$ has essentially two distinct irreducible modules $\Delta^\pm_n$. One irreducible module is obtained by switching the grading of the other. These irreducible modules are distinguished by the action of the volume element, that is,

$$\rho_{\Delta^\pm_n}(e_1 \cdots e_n) = \pm(\sqrt{-1})^{n/2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

with respect to the decomposition $\Delta^\pm_n = (\Delta^+_n)^0 \oplus (\Delta^-_n)^1$. For convenience, we put $\Delta_n = \Delta^+_n \oplus \Delta^-_n$.

Under the natural isomorphism $Cl(n) \otimes Cl(n') \cong Cl(n + n')$, a $Cl(n)$-module $V$ and a $Cl(n')$-module $V'$ give a $Cl(n + n')$-module $V \otimes V'$, where the tensor product is taken in the $\mathbb{Z}/2\mathbb{Z}$-graded sense. If $n$ or $n'$ is even, and both $V$ and $V'$ are irreducible, then $V \otimes V'$ is also irreducible. In particular, $\Delta^+_m \otimes \Delta^+_m \cong \Delta^+_n$.

The above behaviour of irreducible modules under tensor products implies:

Lemma 2.1 ([9, 10]). Let $n$ and $m$ be positive integers.

1. The category of $Cl(n)$-modules and that of $Cl(n + 2m)$-modules are equivalent under the functor assigning $V \otimes \Delta^+_m$ to a $Cl(n)$-module $V$ and $f \otimes \text{id}$ to a homomorphism $f$ of $Cl(n)$-modules.

2. The functor induces an isomorphism $H_{\mathbb{Z}/2}(V) \cong H_{\mathbb{Z}/2}(V \otimes \Delta^+_m)$, where $H_{\mathbb{Z}/2}(V)$ is the following vector space introduced to any $Cl(n)$-module $V$:

$$H_{\mathbb{Z}/2}(V) = \left\{ \gamma : V \to V \mid \text{degree 1, Hermitian,} \quad \rho_V(e_i)\gamma + \gamma \rho_V(e_i) = 0 \text{ for } i = 1, \ldots, n \right\},$$

and $H_{\mathbb{Z}/2}(V \otimes \Delta^+_m)$ is defined similarly.

Notice that this lemma also makes sense in the case of $n = 0$. (In this case, we forget Clifford actions, and regard a $Cl(0)$-module $V$ as just a $\mathbb{Z}/2\mathbb{Z}$-graded Hermitian vector space, and $H_{\mathbb{Z}/2}(V)$ as the space of degree 1 Hermitian maps on $V$.)

For $n = 1, 2$, we describe the irreducible $Cl(n)$-modules explicitly. In the case of $n = 1$, the irreducible module is $\Delta_1 = \mathbb{C} \oplus \mathbb{C}$ and $\rho_{\Delta_1}(e_1) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. In the
case of \( n = 2 \), the irreducible \( Cl(2) \)-module \( \Delta^+\) is \( \Delta^+_2 = \mathbb{C} \oplus \mathbb{C} \) and
\[
\rho_{\Delta^+_2}(e_1) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \rho_{\Delta^+_2}(e_2) = \begin{pmatrix} 0 & -\sqrt{-1} \\ -\sqrt{-1} & 0 \end{pmatrix}.
\]
We easily see \( H_{\mathbb{Z}/2}(\Delta_4) = \mathbb{C} \), with its basis \( \gamma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \), and \( H_{\mathbb{Z}/2}(\Delta^+_2) = 0 \).

2.2. Fredholm operators. For \( n > 0 \), let \( \mathcal{H}_n \) be a separable infinite-dimensional \( \mathbb{Z}/2 \)-graded Hilbert space which contains each irreducible \( Cl(n) \)-module. A particular construction of \( \mathcal{H}_n \) is \( \mathcal{H}_n = H \otimes \Delta_n \), where \( H \) is an ungraded separable Hilbert space of infinite-dimension. We also let \( \mathcal{F}_n \) be the space of degree 1 self-adjoint Fredholm operators on \( \mathcal{H}_n \) anti-commuting with the actions of \( e_i \in Cl(n), (i = 1, \ldots, n) \):
\[
\mathcal{F}_n = \left\{ A : \mathcal{H}_n \to \mathcal{H}_n \mid \begin{array}{c}
\text{degree 1, Fredholm, } A^* = A \\
AE_i + e_iA = 0 \text{ for } i = 1, \ldots, n
\end{array} \right\}.
\]
We topologize this space by the operator norm. In the case that \( n \) is odd, \( \mathcal{F}_n \) has three connected components \([7]\). Two of them are contractible, and we will denote the remaining non-trivial component by \( \mathcal{F}_n(\mathcal{H}_n) = \mathcal{F}_n \). In the case that \( n \) is even, we put \( \mathcal{F}_n(\mathcal{H}_n) = \mathcal{F}_n = \mathcal{F}_n \). In the case of \( n = 0 \), we also define \( \mathcal{F}_0 = \mathcal{F}_0 \) to be the space of degree 1 self-adjoint Fredholm operators on a separable infinite-dimensional \( \mathbb{Z}/2 \)-graded Hilbert space.

Notice that there exists a homotopy equivalence \([7]\):
\[
AS : \mathcal{F}_n(\mathcal{H}_n) \longrightarrow \Omega \mathcal{F}_{n-1}(\mathcal{H}_n),
\]
where \( \Omega \mathcal{F}_{n-1}(\mathcal{H}_n) \) stands for the space of maps \( \hat{A} : [-1,1] \to \mathcal{F}_{n-1}(\mathcal{H}_n) \) such that \( \hat{A}(\pm 1) \) is invertible. For \( A \in \mathcal{F}_n(\mathcal{H}_n) \), an explicit description of the map \( AS(A) : [-1,1] \to \mathcal{F}_{n-1}(\mathcal{H}_n) \) is
\[
AS(A)(t) = A + \sqrt{-1}te_n.
\]
Notice also that there is a homeomorphism \( \mathcal{F}_n \cong \mathcal{F}_{n+2m} \) \(([7]\)). This homeomorphism \( \mathcal{F}_n(\mathcal{H}_n) \to \mathcal{F}_{n+2m}(\mathcal{H}_{n+2m}) \) is given by \( A \mapsto A \otimes \text{id} \) under the identification \( \mathcal{H}_{n+2m} \cong \mathcal{H}_n \otimes \Delta^+_m \).

Because of the homotopy equivalence \( \mathcal{F}_n \to \Omega \mathcal{F}_{n-1} \), the space \( \mathcal{F}_n \) provides a model of the classifying space of the \( K \)-theory of degree \(-n \). Put differently, we may define the \( K \)-group \( K^{-n}(X) \) of a CW complex \( X \) to be the homotopy classes of continuous maps from \( X \) to \( \mathcal{F}_n \). Under this realization of \( K^{-n} \), the homeomorphism \( \mathcal{F}_n \cong \mathcal{F}_{n+2m} \) induces the Bott periodicity.

Remark 1. As a model of the classifying space of \( K^{-n} \), the space of Fredholm operators \( \mathcal{F}_n \) is chosen in this paper. We can also choose the model provided in \([6]\). With this choice, the subsequent argument is still valid.

3. Twisted \( K \)-theory

3.1. Twisted \( K \)-theory. To twist usual topological \( K \)-theory, we will use a principal bundle whose structure group is a projective unitary group: For a separable infinite-dimensional Hilbert space \( H \), the projective unitary group \( PU(H) \) is defined by the quotient \( PU(H) = U(H)/U(1) \). We topologize \( PU(H) \) by using the the operator norm topology on \( U(H) \). Then, for \( n \geq 0 \), the group \( PU(H) \) acts on \( \mathcal{F}_n(\mathcal{H}_n) = \mathcal{F}_n(H \otimes \Delta_n) \) by conjugation, and we can associate a fiber bundle
For a closed subspace \( P \) in a space \( X \), give \( \pi_1(P) \) the group of \( C^0 \)-sections of \( X \) at \( x \) in \( P \) where \( \pi_1(P) \) is the set of points \( x \in X \) at which \( \pi_1(P) \) is not invertible:

\[
\text{Supp}(\pi_1(P)) = \{ x \in X \mid \pi_1(P) \text{ is not invertible} \}.
\]

For a closed subspace \( Y \subset X \), we denote by \( \pi_0(X, Y, \pi_1(P)) \) the set of sections \( \pi_1(P) \), such that \( \text{Supp}(\pi_1(P)) \cap Y = \emptyset \).

Now, we define \( K_{\text{Cl}}(n)(X, Y) \) to be the homotopy classes of \( \pi_1(P) \), including \( \pi_1(P) \) as a homotopy if there exists a section \( \pi_1(P) \) such that \( \pi_1(P) \mid_{\pi_1(P)} = \pi_1(P) \), \((i = 0, 1)\). (We denote by \( I = [0, 1] \) the unit interval.) A choice of an identification \( \pi_1(P) \mid_{\pi_1(P)} = \pi_1(P) \) such that \( \pi_1(P) \mid_{\pi_1(P)} = \pi_1(P) \) makes \( K_{\text{Cl}}(n)(X, Y) \) into an abelian group. In view of the homotopy equivalence \( \pi_1(P) \rightarrow \pi_1(P) \), the group \( K_{\text{Cl}}(n)(X, Y) \) is isomorphic to \( K_{\text{Cl}}(n)(X, Y) = K_{\text{Cl}}(n)(X \times I^n, Y \times I^n \cup X \times \partial I^n) \), the \( \tau \)-twisted \( K \)-group \([6, 8]\) of degree \( n \).

### 3.2. Axioms of twisted \( K \)-theory.

To lift the group \( K_{\text{Cl}}(n)(X, Y) \) into a generalized cohomology theory, we introduce a category \( \hat{\mathcal{C}} \) as follows: an object in \( \hat{\mathcal{C}} \) is a triple \((X, Y, \tau)\) consisting of a CW pair \((X, Y)\) and a principal \( PU(H) \)-bundle \( \tau \rightarrow X \). A morphism \((f, F) : (X', Y', \tau') \rightarrow (X, Y, \tau)\) in \( \hat{\mathcal{C}} \) consists of a continuous map \( f : X' \rightarrow X \) such that \( f(Y') \subset Y \) and a bundle isomorphism \( F : \tau' \rightarrow f^*\tau \) covering the identity of \( X' \).

For \((X, Y, \tau) \in \hat{\mathcal{C}}\), we define the group \( K_{\text{Cl}}(n)(X, Y) \) by

\[
K_{\text{Cl}}(n)(X, Y) = \begin{cases} \text{Ker}(\tau \times P_0 : X \times I^j \rightarrow X \times X), & (j \geq 0) \\ K_{\text{Cl}}(n)(X, Y), & (j < 0) \end{cases},
\]

A morphism \((f, F) : (X', Y', \tau') \rightarrow (X, Y, \tau)\) clearly induces a homomorphism \((f, F)^* : K_{\text{Cl}}(n)(X, Y) \rightarrow K_{\text{Cl}}(n)(X', Y')\). Thus, the assignment \((X, Y, \tau) \rightarrow K_{\text{Cl}}(n)(X, Y)\) gives rise to a functor from \( \hat{\mathcal{C}} \) to the category of abelian groups. Since \( K_{\text{Cl}}(n)(X, Y) \rightarrow K_{\text{Cl}}(n)(X', Y')\), we see the following properties from \([8]\):

**Proposition 3.1.** The functors assigning \( K_{\text{Cl}}(n)(X, Y) \) to \((X, Y, \tau) \in \hat{\mathcal{C}}\), \((j \in \mathbb{Z})\) have the following properties:

1. **(Homotopy axiom)** If \((f_i, F_i) : (X', Y', \tau') \rightarrow (X, Y, \tau), \ (i = 0, 1)\) are homotopic, then the induced homomorphisms coincide: \((f_0, F_0)^* = (f_1, F_1)^*\).
2. **(Excision axiom)** For subcomplexes \( A, B \subset X \), the inclusion map induces the isomorphism:

\[
K_{\text{Cl}}(n)(A \cup B, B) \cong K_{\text{Cl}}(n)(A, A \cap B).
\]
3. **(Exactness axiom)** There is the natural long exact sequence:

\[
\cdots \rightarrow K_{\text{Cl}}(n)(Y, X) \rightarrow K_{\text{Cl}}(n)(X, Y) \rightarrow K_{\text{Cl}}(n)(X, Y) \rightarrow K_{\text{Cl}}(n)(X) \rightarrow \cdots.
\]
This isomorphism is given by multiplying a generator of sphere. In general, there exists a multiplication \( Z \) that is induced from the map \( F \). Let Definition 4.1.

\[ \text{We notice that the proof of the exactness axiom uses the Bott periodicity} \]

\[ K^{n-1}_{\text{Cl}(n)}(X,Y) \cong K^{n-2}_{\text{Cl}(n)}(X,Y). \]

This isomorphism is given by multiplying a generator of \( K^{-2}(pt) = K^0(D^2,S^1) \cong \mathbb{Z} \). (For \( k > 0 \), we denote by \( D^k \) the unit disk in \( \mathbb{R}^k \), and by \( S^{k-1} = \partial D^k \) the unit sphere.) In general, there exists a multiplication

\[ K^{n-1}_{\text{Cl}(n)}(X,Y) \times K^{k}_{\text{Cl}(n)}(X,Y) \rightarrow K^{n-k}_{\text{Cl}(n+k)}(X,Y). \]

This is induced from the map \( \mathcal{F}_n(H_n) \times \mathcal{F}_m(H_m) \rightarrow \mathcal{F}_{n+m}(H_n \otimes H_m) \) given by \((A,A') \mapsto A \otimes 1 + 1 \otimes A'\), where the tensor products are taken in the graded sense.

4. Vectorial bundles with Clifford actions

4.1. Definitions.

**Definition 4.1.** Let \( n \) be a positive integer and \( X \) a topological space. For a subset \( U \subset X \), we define the category \( \mathcal{H}(\text{Cl}(n))(U) \) as follows. An object in \( \mathcal{H}(\text{Cl}(n))(U) \) is a pair \((E,h)\) consisting of a finite-rank \( \mathbb{Z}/2 \)-graded Hermitian vector bundle \( E \rightarrow U \) equipped with bundle maps \( e_i : E \rightarrow E \) (i = 1, ..., n) of degree 1 satisfying \( e_i e_j + e_j e_i = -2h_{i,j} \) and of a Hermitian map \( h : E \rightarrow E \) of degree 1 satisfying \( h e_i + e_i h = 0 \) (i = 1, ..., n). The homomorphisms in \( \mathcal{H}(\text{Cl}(n))(U) \) are

\[ \text{Hom}_{\mathcal{H}(\text{Cl}(n))(U)}((E,h),(E',h')) = \left\{ \phi : E \rightarrow E' \mid \begin{array}{l} \text{degree 0}, \ \phi h = h' \phi, \\
e_i \phi = \phi e_i \text{ for } i = 1, \ldots, n \end{array} \right\} \cong, \]

where the meaning of the equivalence relation \( \phi \cong \phi' \) is as follows:

For each point \( x \in U \), there are a positive number \( \mu > 0 \) and an open subset \( V \subset U \) containing \( x \) such that: for all \( y \in V \) and \( \xi \in (E,h)_y, \mu \), we have \( \phi(\xi) = \phi'(\xi) \).

In the above, we put

\[ (E,h)_y,\mu = \bigoplus_{\lambda < \mu} \text{Ker}(h^2_\lambda - \lambda) = \bigoplus_{\lambda < \mu} \{ \xi \in E_y \mid h^2_\lambda \xi = \lambda \xi \}. \]

We will just write \( \phi \) to mean the homomorphism in the category \( \mathcal{H}(\text{Cl}(n))(U) \) represented by \( \phi : (E,h) \rightarrow (E',h') \).

**Definition 4.2.** Let \( X \) be a topological space, \( \tau \rightarrow X \) a principal \( PU(H) \)-bundle, and \( U \subset X \) a subset.

(a) We define the category \( \mathcal{P}^\tau(U) \) as follows. The objects in \( \mathcal{P}^\tau(U) \) consist of sections \( s : U \rightarrow \tau|_U \). The morphisms in \( \mathcal{P}^\tau(U) \) are defined by

\[ \text{Hom}_{\mathcal{P}^\tau(U)}(s,s') = \{ g : U \rightarrow \tau(H) \mid s' \pi(g) = s \}, \]

where \( \pi : PU(H) \rightarrow U(H) \) is the projection. The composition of morphisms is defined by the pointwise multiplication.
(b) We define the category $\mathcal{H}^\tau_{Cl(n)}(U)$ as follows. The objects in $\mathcal{H}^\tau_{Cl(n)}(U)$ are the same as those in $P^\tau(U) \times \mathcal{H}^\tau_{Cl(n)}(U)$:

$$\text{Obj}(\mathcal{H}^\tau_{Cl(n)}(U)) = \text{Obj}(P^\tau(U)) \times \text{Obj}(\mathcal{H}^\tau_{Cl(n)}(U)).$$

The homomorphisms in $\mathcal{H}^\tau_{Cl(n)}(U)$ are defined by:

$$\text{Hom}_{\mathcal{H}^\tau_{Cl(n)}(U)}((s, (E, h)), (s', (E', h'))) = \text{Hom}_{P^\tau(U)}(s, s') \times \text{Hom}_{\mathcal{H}^\tau_{Cl(n)}(U)}((E, h), (E', h'))/ \sim,$$

where the equivalence relation $\sim$ identifies $(g, \phi)$ with $(g\zeta, \phi\zeta)$ for any $U(1)$-valued map $\zeta : U \to U(1)$.

**Definition 4.3.** For a positive integer $n$ and a principal $PU(H)$-bundle $\tau$ over a topological space $X$, we define the category $K^\tau_{Cl(n)}(X)$ as follows.

1. An object $(\mathcal{U}, \mathcal{E}_\alpha, \Phi_{\alpha\beta})$ in $K^\tau_{Cl(n)}(X)$ consists of an open cover $\mathcal{U} = \{U_\alpha\}_{\alpha \in \mathcal{A}}$ of $X$, objects $\mathcal{E}_\alpha$ in $\mathcal{H}^\tau_{Cl(n)}(U_\alpha)$, and homomorphisms $\Phi_{\alpha\beta} : \mathcal{E}_\alpha \to \mathcal{E}_\beta$ in $\mathcal{H}^\tau_{Cl(n)}(U_{\alpha\beta})$ such that:

$$\Phi_{\alpha\beta}\Phi_{\beta\gamma} = 1 \quad \text{in} \quad \mathcal{H}^\tau_{Cl(n)}(U_{\alpha\beta});$$

$$\Phi_{\alpha\beta}\Phi_{\beta\gamma} = \Phi_{\alpha\gamma} \quad \text{in} \quad \mathcal{H}^\tau_{Cl(n)}(U_{\alpha\beta\gamma}),$$

where $U_{\alpha\beta} = U_\alpha \cap U_\beta$ and $U_{\alpha\beta\gamma} = U_\alpha \cap U_\beta \cap U_\gamma$ as usual. We call an object in the category $K^\tau_{Cl(n)}(X)$ a $\tau$-twisted $Cl(n)$-vectorial bundle over $X$.

2. A homomorphism $((\{U_\alpha\}, \mathcal{E}_\alpha', \Phi_{\alpha\beta}') \to (\{U_\alpha\}, \mathcal{E}_\alpha, \Phi_{\alpha\beta}))$ consists of homomorphisms $\Psi_{\alpha\beta} : \mathcal{E}_\alpha' \to \mathcal{E}_\alpha$ in $\mathcal{H}^\tau_{Cl(n)}(U_{\alpha\beta})$ such that the following diagrams commute in $\mathcal{H}^\tau_{Cl(n)}(U_{\alpha\beta} \cap U_{\alpha'} \cap U_{\beta'})$ and $\mathcal{H}^\tau_{Cl(n)}(U_{\alpha} \cap U_{\beta} \cap U_{\beta'}')$, respectively.

In the case of $n = 0$, we can identify $K^\tau_{Cl(0)}(X) = K^\tau(X)$ with the category of $\tau$-twisted vectorial bundles ([11]) on $X$. Also, in the case that $\tau$ is the trivial $PU(H)$-bundle $\tau = X \times PU(H)$, we can identify $K^\tau_{Cl(n)}(X) = K^\tau_{Cl(n)}(X)$ with the category of $(\mathbb{Z}/2$-graded) $Cl(n)$-vectorial bundles ([10]) on $X$.

By definition, we can specify an object $\mathbb{E} \in K^\tau_{Cl(n)}(X)$ by the data

$$(\mathcal{U}, s_\alpha, g_{\alpha\beta}, (E_\alpha, h_\alpha, \phi_{\alpha\beta}))$$

consisting of:

- an open cover $\mathcal{U} = \{U_\alpha\}$ of $X$;
- local sections $s_\alpha : U_\alpha \to \tau|_{U_\alpha}$, which define the transition functions $\check{g}_{\alpha\beta} : U_{\alpha\beta} \to PU(H)$ by $s_\alpha \check{g}_{\alpha\beta} = s_\beta$;
- functions $g_{\alpha\beta} : U_{\alpha\beta} \to U(H)$ such that $\pi \circ g_{\alpha\beta} = \check{g}_{\alpha\beta}$, which define $z_{\alpha\beta\gamma} : U_{\alpha\beta\gamma} \to U(1)$ by $g_{\alpha\beta\gamma} g_{\beta\gamma} = z_{\alpha\beta\gamma} g_{\alpha\beta\gamma}$;
- $\mathbb{Z}/2$-graded Hermitian vector bundles $E_\alpha \to U_\alpha$ of finite rank whose fibers are $Cl(n)$-modules by means of bundle maps $e_i : E_\alpha \to E_\alpha, (i = 1, \ldots, n)$ of degree 1 satisfying $e_i e_j + e_j e_i = -2\delta_{i,j}$. 

• Homotopic maps \( h_\alpha : E_\alpha \to E_\alpha \) of degree 1 such that \( h_\alpha e_i + e_i h_\alpha = 0 \) for all \( i = 1, \ldots, n \);

• Maps \( \phi_{\alpha \beta} : E_\beta|_{U_{\alpha \beta}} \to E_\alpha|_{U_{\alpha \beta}} \) of degree 0 such that \( h_\alpha \phi_{\alpha \beta} = \phi_{\alpha \beta} h_\beta \)

\( e_i \phi_{\alpha \beta} = \phi_{\alpha \beta} e_i \) for \( i = 1, \ldots, n \) and:

\[
\begin{align*}
\phi_{\alpha \beta} \phi_{\beta \alpha} & \equiv 1 & \text{on } U_{\alpha \beta}; \\
\phi_{\alpha \beta} \phi_{\beta \gamma} & \equiv z_{\alpha \beta \gamma} \phi_{\alpha \gamma} & \text{on } U_{\alpha \beta}.
\end{align*}
\]

The support of \( E \) is defined by

\[ \text{Supp}(E) = \{ x \in X \mid (h_\alpha)_x \text{ is not invertible for some } \alpha \}. \]

For a subspace \( Y \subset X \), we define \( KF_{\text{Cl}(n)}(X, Y) \) to be the full subcategory consisting of the objects \( E \in KF_{\text{Cl}(n)}(X) \) such that \( \text{Supp}(E) \cap Y = \emptyset \).

Now, for \((X, Y, \tau) \in \hat{\mathcal{C}}\), we define \( KF_{\text{Cl}(n)}(X, Y) \) to be the homotopy classes of \( \tau \)-twisted \( \text{Cl}(n) \)-vectorial bundles \( E \in KF_{\text{Cl}(n)}(X, Y) \): we say \( E_0 \) and \( E_1 \) are homotopic if there exists \( E \in KF_{\text{Cl}(n)}(X \times I, Y \times I) \) such that \( E|_{X \times \{i\}} \) is isomorphic to \( E_i \) in \( KF_{\text{Cl}(n)}(X, Y) \) for each \( i = 0, 1 \). In the same way as in the case without \( \text{Cl}(n) \)-actions [10, 11], \( KF_{\text{Cl}(n)}(X, Y) \) gives rise to an abelian group.

### 4.2. Axioms

For \((X, Y, \tau) \in \hat{\mathcal{C}} \) and \( j \geq 0 \), we put:

\[ KF_{\text{Cl}(n)}^{\tau+j}(X, Y) = KF_{\text{Cl}(n)}^{\tau+j}(X \times P, Y \times P \cup U \times \partial P). \]

We also put \( KF_{\text{Cl}(n)}^{\tau+j}(X, Y) = KF_{\text{Cl}(n)}^{\tau+j}(X, Y) \). Then, the argument in [11] applies to \( \text{Cl}(n) \)-vectorial bundles, and we have a “cohomology theory”:

**Proposition 4.4.** The functors assigning \( KF_{\text{Cl}(n)}^{\tau+j}(X, Y) \) to \((X, Y, \tau) \in \hat{\mathcal{C}}, \ (j \leq 1)\) have the following properties:

1. **(Homotopy axiom)** If \((f_i, F_i) : (X', Y', \tau') \to (X, Y, \tau), \ (i = 0, 1)\) are homotopic, then the induced homomorphisms coincide: \((f_0, F_0)^* = (f_1, F_1)^*\).

2. **(Excision axiom)** For subcomplexes \( A, B \subset X \), the inclusion map induces the isomorphism:

\[ KF_{\text{Cl}(n)}^{\tau+j}(A \cup B, B) \cong KF_{\text{Cl}(n)}^{\tau+j}(A, A \cap B). \]

3. **("Exactness" axiom)** There is the natural complex of groups:

\[
\cdots \xrightarrow{\delta_{-1}} KF_{\text{Cl}(n)}^{\tau+0}(X, Y) \to KF_{\text{Cl}(n)}^{\tau+0}(X) \to KF_{\text{Cl}(n)}^{\tau+0}(Y) \xrightarrow{\delta_0} KF_{\text{Cl}(n)}^{\tau+1}(X, Y).
\]

This complex is exact except at the term \( KF_{\text{Cl}(n)}^{\tau+0}(Y) \).

4. **(Additivity axiom)** For a family \( \{(X_\lambda, Y_\lambda, \tau_\lambda)\}_{\lambda \in \Lambda} \) in \( \hat{\mathcal{C}} \), the inclusion maps \( X_\lambda \to \coprod_\lambda X_\lambda \) induce the natural isomorphism:

\[ KF_{\text{Cl}(n)}^{\tau+j}(\coprod_\lambda X_\lambda, \coprod_\lambda Y_\lambda) \cong \prod_\lambda KF_{\text{Cl}(n)}^{\tau+j}(X_\lambda, Y_\lambda). \]

Notice that, in constructing \( \delta_0 \) above, we use the multiplication

\[ KF_{\text{Cl}(n)}^{\tau}(X, Y) \times KF(D^2, S^1) \to KF_{\text{Cl}(n)}^{\tau}(X \times D^2, Y \times D^2 \cup X \times S^1). \]

In general, we can define a multiplication

\[ \otimes : KF_{\text{Cl}(n)}^{\tau}(X, Y) \times KF_{\text{Cl}(n)}^{m}(X, Y) \to KF_{\text{Cl}(n+m)}^{\tau}(X, Y). \]
This is induced from the functor $\otimes : \mathcal{H}F_{Cl(n)}(U) \times \mathcal{H}F_{Cl(m)}(U) \to \mathcal{H}F_{Cl(n+m)}(U)$ given by $(E, h) \otimes (E', h') = (E \otimes E', h \otimes 1 + 1 \otimes h')$, where the tensor products are taken in the $\mathbb{Z}/2$-graded sense.

5. Main theorem

5.1. Finite-dimensional approximation. To begin with, we construct the following homomorphism via a “finite-dimensional approximation”:

$$\alpha : K^\tau_{Cl(n)}(X) \to KF^\tau_{Cl(n)}(X).$$

The construction is exactly the same as that performed in [11]: let $A \in \Gamma(X, \mathcal{F}_n(\tau))$ be a section given. We then make the following choice:

- an open cover $\{U_\alpha\}$ of $X$;
- local sections $s_\alpha : U_\alpha \to \tau|_{U_\alpha}$ of $\tau$, which define the transition functions $g_{\alpha \beta} : U_{\alpha \beta} \to PU(H)$ by $s_\alpha g_{\alpha \beta} = s_\beta$;
- lifts $g_{\alpha \beta} : U_{\alpha \beta} \to U(H)$ of the transition functions $\tilde{g}_{\alpha \beta} : U_{\alpha \beta} \to PU(H)$;
- positive numbers $\mu_\alpha$ such that the family of vector spaces $E_\alpha = \bigcup_{x \in U_\alpha} (\mathcal{H}_n, (A_\alpha)_x)_{< \mu_\alpha} = \bigcup_{x \in U_\alpha} \bigoplus_{\lambda < \mu_\alpha} \{\xi \in \mathcal{H}_n| (A_\alpha)_x^2 \xi = \lambda \xi\}

becomes a vector bundle of finite rank.

By means of the trivializations $s_\alpha$, the section $A$ induces maps $A_\alpha : U_\alpha \to \mathcal{F}_n(\mathcal{H}_n)$ such that $g_{\alpha \beta} A_\beta g_{\alpha \beta}^{-1} = A_\alpha$. Now, for $i = 1, \ldots, n$, the action of $e_i \in Cl(n)$ on $\mathcal{H}_n$ induces a vector bundle map $e_i : E_\alpha \to E_\alpha$ of degree 1 satisfying $e_i e_j + e_j e_i = -2 \delta_{i,j}$. The restriction of $A_\alpha$ defines a Hermitian map $h_\alpha : E_\alpha \to E_\alpha$ of degree 1 anti-commuting with $e_i$. Finally, we define a map $\phi_{\alpha \beta} : E_{\alpha \beta}|_{U_{\alpha \beta}} \to E_\alpha|_{U_{\alpha \beta}}$ by the composition of the following maps:

$$E_{\alpha \beta}|_{U_{\alpha \beta}} \xrightarrow{\text{inclusion}} U_{\alpha \beta} \times \mathcal{H}_n \xrightarrow{\text{id} \times g_{\alpha \beta}} U_{\alpha \beta} \times \mathcal{H}_n \xrightarrow{\text{projection}} E_\alpha|_{U_{\alpha \beta}}.$$

Then $E = (\{U_\alpha\}, s_\alpha, g_{\alpha \beta}, (E_\alpha, h_\alpha), \phi_{\alpha \beta})$ is a $\tau$-twisted $Cl(n)$-vectorial bundle on $X$, and a well-defined homomorphism $\alpha : K^\tau_{Cl(n)}(X) \to KF^\tau_{Cl(n)}(X)$ is induced from the assignment $A \mapsto E$.

The construction above also induces $\alpha : K^\tau_{Cl(n)}(X, Y) \to KF^\tau_{Cl(n)}(X, Y)$ as well as $\alpha_j : K^{\tau+j}_{Cl(n)}(X, Y) \to KF^{\tau+j}_{Cl(n)}(X, Y)$ for any $(X, Y, \tau) \in \mathcal{C}$ and $j \leq 1$. Then, in the same way as in [11], we get:

Proposition 5.1. The homomorphisms $\alpha_j : K^{\tau+j}_{Cl(n)}(X, Y) \to KF^{\tau+j}_{Cl(n)}(X, Y)$, $(j \leq 1)$ give rise to natural transformations from the functors in Proposition 3.1 to those in Proposition 4.4.

5.2. Main theorem and its corollary. Theorem 1 in Section 1 is a corollary (Corollary 5.4) to:

Theorem 5.2. Let $\tau$ be any principal $PU(H)$-bundle over a CW complex $X$. For any $n, j \geq 0$, the homomorphism $\alpha_{-j} : K^{\tau-j}_{Cl(n)}(X) \to KF^{\tau-j}_{Cl(n)}(X)$ is bijective.

The key to this theorem is the following proposition, which we will prove in the next subsection:
Proposition 5.3. For any $k, j \geq 0$, the following homomorphism is bijective:

$$\alpha_{-j} : K_{\text{Cl}}^{r-j}(D^k, S^{k-1}) \longrightarrow KF_{\text{Cl}}^{r-j}(D^k, S^{k-1}),$$

where $(D^k, S^{k-1})$ means $(pt, \emptyset)$ when $k = 0$.

Proof of Theorem 5.2. In view of Proposition 3.1, 4.4, 5.1 and 5.3, the proof is exactly the same as that of the main result of [11]: First, in the case that $X$ is a finite CW complex, we prove the bijectivity of $\alpha_{-j}$ by an induction on the number of cells in $X$. Then, the bijectivity of $\alpha_{-j}$ in the general case follows from that in the finite case through an argument by using the telescope of $X$. \hfill \Box

Corollary 5.4. Suppose $(X, Y, \tau) \in \hat{C}$ and $j \geq 0$ are given.

(a) The finite-dimensional approximation induces the bijection:

$$\alpha_{-j} : K_{\text{Cl}}^{r-j}(X, Y) \longrightarrow KF_{\text{Cl}}^{r-j}(X, Y).$$

(b) The multiplication of a generator of $K(D^2, S^1) \cong KF(D^2, S^1) \cong \mathbb{Z}$ induces the bijection:

$$KF_{\text{Cl}}^{r-j}(X, Y) \longrightarrow KF_{\text{Cl}}^{r-j-2}(X, Y).$$

(c) There exists a natural isomorphism

$$K^{r-j-n}(X, Y) \cong KF_{\text{Cl}}^{r-j}(X, Y).$$

5.3. Key proposition. This subsection is devoted to the proof of Proposition 5.3, which is clearly equivalent to:

Proposition 5.5. For any $n, k \geq 0$, the following homomorphism is bijective:

$$\alpha : K_{\text{Cl}}(D^k, S^{k-1}) \longrightarrow KF_{\text{Cl}}(D^k, S^{k-1}).$$

Notice that the principal $PU(H)$-bundle $\tau$ is absent (or trivial) in the present case. Therefore $K_{\text{Cl}}(D^k, S^{k-1})$ is identified with the homotopy classes of maps from the $k$-dimensional disk $D^k$ to $\mathcal{F}_n$ which carry all points in the sphere $S^{k-1} = \partial D^k$ into the subspace $\mathcal{F}_n \subset \mathcal{F}_n$ consisting of invertible operators:

$$K_{\text{Cl}}(D^k, S^{k-1}) = [(D^k, S^{k-1}), (\mathcal{F}_n, \mathcal{F}_n)].$$

To prove Proposition 5.5, recall the homeomorphism $\mathcal{F}_n(\mathcal{H}_n) \rightarrow \mathcal{F}_{n+2m}(\mathcal{H}_{n+2m})$ given by $A \mapsto A \otimes 1$ under the identification $\mathcal{H}_n \otimes \Delta^+_{2m} \cong \mathcal{H}_{n+2m}$. Consequently, for any CW pair $(X, Y)$, we have a natural isomorphism

$$K_{\text{Cl}}(X, Y) \longrightarrow K_{\text{Cl}}(X, Y).$$

There is a similar “periodicity” for vectorial bundles:

Lemma 5.6. Let $n$ be a non-negative integer. For any CW pair $(X, Y)$ and $m > 0$, the tensor product of the irreducible $\text{Cl}(2m)$-module $\Delta^+_{2m}$ induces a natural isomorphism $KF_{\text{Cl}}(X, Y) \rightarrow KF_{\text{Cl}}(X, Y)$ fitting in the commutative diagram:

$$\begin{array}{ccc}
K_{\text{Cl}}(X, Y) & \longrightarrow & K_{\text{Cl}}(X, Y) \\
\alpha \downarrow & & \downarrow \alpha \\
KF_{\text{Cl}}(X, Y) & \longrightarrow & KF_{\text{Cl}}(X, Y).
\end{array}$$

Proof. The first part of this lemma, which is shown in [10], follows from Lemma 2.1. The second part is clear by construction. \hfill \Box
As a consequence of this lemma, it suffices to consider the case of \( n = 0 \) and \( n = 1 \) only in Proposition 5.5. In the case of \( n = 0 \), the proposition is established in [11]. Hence we are left with the case of \( n = 1 \). To deal with this case, we use the following fact (Remark 10.29 (2), [10]):

**Proposition 5.7** ([10]). For \( n, k > 0 \), there is a natural isomorphism

\[
KF_{Cl(n)}(pt) \rightarrow KF_{Cl(k+n)}(D^k, S^{k-1})
\]

given by the multiplication of the “symbol of the \( k \)-dimensional supersymmetric harmonic oscillator”.

The symbol of the 1-dimensional supersymmetric harmonic oscillator ([9]) is the \( Cl(1) \)-vectorial bundle \( (F, h) \in KF_{Cl(1)}(I, \partial I) \) defined by:

\[
F = I \times \Delta_1 = I \times (\mathbb{C} \oplus \mathbb{C}), \quad h = \begin{pmatrix} 0 & t \\ t & 0 \end{pmatrix}, \quad (t \in I = [-1, 1]).
\]

The symbol of the \( k \)-dimensional supersymmetric harmonic oscillator is the \( Cl(k) \)-vectorial bundle \( \otimes_{i=1}^k \pi_i^*(F, h) \in KF_{Cl(k)}(I^k, \partial I^k) \), where \( \pi_i : I^k \rightarrow I \) is the projection onto the \( i \)th factor.

Proposition 5.7 leads to the following computational result:

**Corollary 5.8.** For \( k \geq 0 \), we have:

\[
KF_{Cl(1)}(D^k, S^{k-1}) \cong \begin{cases} \mathbb{Z}, & (k : odd) \\ 0, & (k : even) \end{cases}
\]

**Proof.** First, we consider the case that \( k \) is an odd integer \( k = 2m + 1 \). By means of Lemma 5.6 and Proposition 5.7, we have

\[
KF_{Cl(1)}(D^{2m+1}, S^{2m}) \cong KF_{Cl(2m+1)}(D^{2m+1}, S^{2m}) \cong KF(pt) \cong \mathbb{Z}.
\]

In the even case \( k = 2m \), we use Lemma 5.6 and Proposition 5.7 again to have

\[
KF_{Cl(1)}(D^{2m}, S^{2m-1}) \cong KF_{Cl(2m+1)}(D^{2m}, S^{2m-1}) \cong KF_{Cl(1)}(pt).
\]

That \( KF_{Cl(1)}(pt) = 0 \) is shown as follows: any element in \( KF_{Cl(1)}(pt) \) can be represented by a pair \((E, h)\) of a \( Cl(1) \)-module \( E \) and a Hermitian map \( h : E \rightarrow E \) of degree 1 anti-commuting with the action of \( e_1 \in Cl(1) \). Since the irreducible \( Cl(1) \)-module is unique up to an equivalence, we can express \( E \) as \( E = V \otimes \Delta_1 \), where \( V \) is a vector space of finite rank. Now we define \((\tilde{E}, \tilde{h}) \in KF_{Cl(1)}([0, 1])\) by setting \( \tilde{E} = I \times E \) and \( \tilde{h}_t = (\cos \frac{\pi}{2} t) h + \sqrt{-1} (\sin \frac{\pi}{2} t) 1 \otimes \gamma \), where \( \gamma \) is a basis of \( H_{\mathbb{Z}/2}(\Delta_1) = \mathbb{C} \) such that \( \gamma^2 = 1 \). Then \((\tilde{E}, \tilde{h})\) is a homotopy between \((E, h)\) and a \( Cl(1) \)-vectorial bundle representing \( 0 \in KF_{Cl(1)}(pt) \).

As is well-known, we have

\[
K_{Cl(1)}(D^k, S^{k-1}) = [(D^k, S^{k-1}), (\mathcal{F}_1, \mathcal{F}_1^*)] = \pi_k(\mathcal{F}_1) = \begin{cases} \mathbb{Z}, & (k : odd) \\ 0, & (k : even) \end{cases}
\]

Therefore \( \alpha : K_{Cl(1)}(D^k, S^{k-1}) \rightarrow KF_{Cl(1)}(D^k, S^{k-1}) \) is apparently bijective in the case of \( k \) even.

Now, it remains the case of \( k \) odd. Since we have \( K_{Cl(1)}(D^k, S^{k-1}) \cong \mathbb{Z} \) and \( KF_{Cl(1)}(D^k, S^{k-1}) \cong \mathbb{Z} \) in this case, it suffices to see the correspondence of generators through \( \alpha \). As is well-known [5], a self-adjoint Fredholm operator whose spectral flow is \( 1 \) generates \([I, \partial I], (\mathcal{F}_1, \mathcal{F}_1^*)] \cong \pi_1(\mathcal{F}_1) \cong \mathbb{Z} \). Hence the bijectivity of \( \alpha \) in the case of \( k = 1 \) (and \( n = 1 \)) follows from:
Lemma 5.9. There is a continuous map $A : (I, \partial I) \to (\mathcal{F}_1, \mathcal{F}_1^*)$ such that:

1. its spectral flow is 1;
2. $\alpha([A]) = ([F, h])$ in $K_{CI(1)}(I, \partial I)$.

Proof. Let $H$ be the Hilbert space with its complete orthonormal basis $\{e_l\}_{l \in \mathbb{Z}}$. For $t \in \mathbb{R}$, we define a bounded self-adjoint operator $a_t : H \to H$ by $a_t e_l = (t + l)/\sqrt{(t + l)^2 + 1}$. A computation shows

$$
\|(a_t - a_{t'})e_l\| \leq \frac{t + \ell - (t' + \ell)}{\sqrt{(t + l)^2 + 1}} + \frac{\sqrt{(t' + l)^2 + 1}}{t' + \ell - (t + l)}
$$

$$
\leq \frac{1}{\sqrt{(t' + l)^2 + 1}} |t - t'| \leq |t - t'|.
$$

Thus, we get $\|(a_t - a_{t'})u\| \leq |t - t'| \|u\|$ for $u \in H$, so that $\|a_t - a_{t'}\| \leq |t - t'|$. This means the map $a : \mathbb{R} \to B(H)$ is continuous. (Here $B(H)$ is topologized by the operator norm. In the case where the topology of $B(H)$ is the compact-open topology in the sense of [6], the map $a : \mathbb{R} \to B(H)$ is still continuous, since $\mathbb{R} \times H \to H$, $(t, u) \mapsto a_t u$ is.) Now, we choose $\epsilon > 0$ sufficiently small.

Then, setting $\mathcal{H}_1 = H \otimes \Delta_1 = H \oplus H$, $A_t = \left( \begin{array}{cc} 0 & a_t \\ a_t & 0 \end{array} \right)$ and $I = [-\epsilon, \epsilon]$, we get $A : (I, \partial I) \to (\mathcal{F}_1, \mathcal{F}_1^*)$ such that its spectral flow is 1 and its finite-dimensional approximation is a $Cl(1)$-vectorial bundle on $(I, \partial I)$ homotopic to $(F, h)$. \hfill \Box

To establish the bijectivity of $\alpha$ in the case of general odd number $k = 2m + 1$, we recall the map $\mathcal{F}_p(\mathcal{H}_p) \times \mathcal{F}_q(\mathcal{H}_q) \to \mathcal{F}_{p+q}(\mathcal{H}_p \otimes \mathcal{H}_q)$ inducing the ring structure on the $K$-cohomology theory: the explicit description of the map is $(A, B) \mapsto A \otimes 1 + 1 \otimes B$. From this description and that of vectorial bundles, we see the commutative diagram

$$
\begin{array}{ccc}
K_{CI(p)}(X, Y) \times K_{CI(q)}(X, Y) & \longrightarrow & K_{CI(p+q)}(X, Y) \\
\left( \begin{array}{c} \alpha \\ \alpha \end{array} \right) & & \left( \begin{array}{c} \alpha \end{array} \right) \\
K_{FCI(p)}(X, Y) \times K_{FCI(q)}(X, Y) & \longrightarrow & K_{FCI(p+q)}(X, Y).
\end{array}
$$

This induces the following commutative diagram:

$$
\begin{array}{ccc}
\prod_{i=1}^{2m+1} K_{CI(1)}(I, \partial I) & \longrightarrow & K_{CI(2m+1)}(I^{2m+1}, \partial I^{2m+1}) \\
\left( \begin{array}{c} \alpha \\ \alpha \end{array} \right) & & \left( \begin{array}{c} \alpha \end{array} \right) \\
\prod_{i=1}^{2m+1} K_{FCI(1)}(I, \partial I) & \longrightarrow & K_{FCI(2m+1)}(I^{2m+1}, \partial I^{2m+1}).
\end{array}
$$

By Proposition 5.7, $K_{FCI(2m+1)}(I^{2m+1}, \partial I^{2m+1}) \cong \mathbb{Z}$ is generated by the symbol of the $(2m + 1)$-dimensional supersymmetric harmonic oscillator, which is the product of $2m + 1$ copies of $(F, h) \in K_{FCI(1)}(I, \partial I)$. Thus, by Lemma 5.9 and the commutative diagram above, the homomorphism

$$
\alpha : K_{CI(2m+1)}(I^{2m+1}, \partial I^{2m+1}) \to K_{FCI(2m+1)}(I^{2m+1}, \partial I^{2m+1})
$$

is surjective. Since any surjective homomorphism $\mathbb{Z} \to \mathbb{Z}$ is bijective, we conclude that $\alpha$ above is bijective. Therefore the following homomorphism is also bijective.
by Lemma 5.6:
\[ \alpha : K_{Cl(1)}(I^{2m+1}, \partial I^{2m+1}) \to K_{Cl(1)}(I^{2m+1}, \partial I^{2m+1}), \]
which completes the proof of Proposition 5.5.

6. Applications

6.1. The Atiyah-Singer map. As is mentioned, the map of Atiyah-Singer [7]
\[ \text{AS} : \mathcal{F}_n(H_n) \to \Omega \mathcal{F}_{n-1}(H_n) \]
is a homotopy equivalence for \( n > 0 \), and induces the natural isomorphism
\[ \text{AS} : K_{Cl(n)}(X, Y) \to K_{Cl(n-1)}^j(X, Y). \]
The aim of this subsection is to introduce a counterpart of this construction to twisted \( Cl(n) \)-vectorial bundles: For any space \( U \) and \((E, h) \in \mathcal{H}F_{Cl(n)}(U)\), we can define an object \((\tilde{E}, \tilde{h}) \in \mathcal{H}F_{Cl(n-1)}(U \times I)\) by setting
\[ \tilde{E} = E \times I, \quad \tilde{h}(x, t) = h(x) + \sqrt{-1}te_n, \]
where \( I = [-1, 1] \). The assignment \((E, h) \mapsto (\tilde{E}, \tilde{h})\) gives rise to a functor
\[ \text{AS} : \mathcal{H}F_{Cl(n)}(U) \to \mathcal{H}F_{Cl(n-1)}(U \times I). \]
It is easy to globalize this construction to get the following functor for any principal \( PU(H) \)-bundle \( \tau \) over a space \( X \) and its subspace \( Y \subset X \):
\[ \text{AS} : \mathcal{K}F^\tau_{Cl(n)}(X, Y) \to \mathcal{K}F^\tau_{Cl(n-1)}(X \times I, Y \times I \cup X \times \partial I). \]
This then induces a natural homomorphism for any \( j \geq 0 \)
\[ \text{AS} : K_{Cl(n)}^\tau(X, Y) \to K_{Cl(n-1)}^{\tau-j}(X, Y). \]

Lemma 6.1. For any positive integer \( n > 0 \) and any principal \( PU(H) \)-bundle \( \tau \) over a space \( X \) and its subspace \( Y \subset X \), the following diagram is commutative:
\[
\begin{array}{ccc}
K_{Cl(n)}^\tau(X, Y) & \xrightarrow{\text{AS}} & K_{Cl(n-1)}^{\tau-1}(X, Y) \\
\downarrow \alpha & & \downarrow \alpha \\
K_{Cl(n)}^\tau(Y, X) & \xrightarrow{\text{AS}} & K_{Cl(n-1)}^{\tau-1}(Y, X).
\end{array}
\]

Proof. Let \( A \in \Gamma(X, Y, \mathcal{F}_n(\tau)) \) represent an element in \( K_{Cl(n)}^\tau(X, Y) \). Suppose that we apply the construction in Subsection 5.1 to \( A \) to have a vectorial bundle
\[ E = \{(U_\alpha)_{\alpha \in \mathfrak{A}}, \sigma_\alpha g_{\alpha \beta} : (E_\alpha, h_\alpha, \phi_{\alpha \beta}) \in \mathcal{K}F^\tau_{Cl(n)}(X, Y) \}. \]
Hence we have \( E_\alpha = \bigcup_{x \in U_\alpha} (H_n, (A_\alpha)_x)_{< \mu_\alpha} \) under a choice of a positive number \( \mu_\alpha \). Without loss of generality, we can assume that there is \( \epsilon_\alpha > 0 \) satisfying
\[ \lambda_1(x) \leq \lambda_2(x) \leq \cdots \leq \lambda_r(x) < \mu_\alpha - \epsilon_\alpha < \mu_\alpha + \epsilon_\alpha < \lambda_{r+1}(x) \]
for all \( x \in U_\alpha \), where \( r \) is the rank of the vector bundle \( E_\alpha \), and \( \lambda_1(x) \) is the ith eigenvalue of \( (A_\alpha)_x^2 \). Then the twisted \( Cl(n-1) \)-vectorial bundle
\[ \text{AS}(E) = \{(\tilde{U}_\alpha)_{\alpha \in \mathfrak{A}}, \tilde{\sigma}_\alpha g_{\alpha \beta} : (\tilde{E}_\alpha, \tilde{h}_\alpha, \tilde{\phi}_{\alpha \beta}) \in \mathcal{K}F^\tau_{Cl(n-1)}(X \times I, Y \times I \cup X \times \partial I) \}
\]
is given by setting \( \tilde{U}_\alpha = \pi^{-1}(U_\alpha) = U_\alpha \times I, \tilde{\sigma}_\alpha = \pi^\ast \sigma_\alpha, \tilde{g}_{\alpha \beta} = \pi^\ast g_{\alpha \beta}, \tilde{E} = \pi^\ast E_\alpha, \tilde{h}_\alpha(x, t) = h_\alpha(x) + \sqrt{-1}te_n \) and \( \tilde{\phi}_{\alpha \beta} = \pi^\ast \phi_{\alpha \beta} \), where \( \pi : X \times I \to X \) is the projection. Then \( \text{AS}(E) \) represents the image \( \text{AS}(\alpha([A])) \).
Next, we describe the image \( \alpha(\text{AS}(\mathbb{A})) \) applying the construction in Subsection 5.1 to \( \text{AS}(\mathbb{A}) \in \Gamma_0(X \times I, Y \times I, X \times \partial I, \pi^* \mathcal{F}_n(\tau)) \). By means of the local trivialization \( \pi^* s_\alpha \) of \( \pi^* \tau = \tau \times I \), the section \( \text{AS}(\mathbb{A}) \) defines a map \( \tilde{A}_\alpha : U_\alpha \to \mathcal{F}_{n-1}(\mathcal{H}_n) \). By our definition of the Atiyah-Singer map, we have \( (\tilde{A}_\alpha)_{(x,t)} = (A_\alpha)_x + \sqrt{-1} \text{te}_n \). We define an open cover \( \{V(s; \varepsilon_\alpha)\}_{s \in I} \) of \( I = [-1, 1] \) by

\[
V(s; \varepsilon_\alpha) = \{ t \in I | s - \varepsilon_\alpha < t^2 < s + \varepsilon_\alpha \}.
\]

Then, for any \((x,t) \in U_\alpha \times V(s; \varepsilon_\alpha)\), the eigenvalues \( \tilde{\lambda}_1(x,t) \leq \tilde{\lambda}_2(x,t) \leq \cdots \leq \tilde{\lambda}_r(x,t) < \mu_\alpha + s < \tilde{\lambda}_{r+1}(x,t) \), since \( \tilde{\lambda}_i(x,t) = \lambda_i(x) + t^2 \). This implies

\[
\bigcup_{(x,t) \in U_\alpha \times V(s; \varepsilon_\alpha)} (\mathcal{H}_n, \tilde{A}(x,t)) < \mu_\alpha + s = \tilde{E}_\alpha|_{U_\alpha \times V(s; \varepsilon_\alpha)}.
\]

Thus, \( \alpha(\text{AS}(\mathbb{A})) \) is represented by the twisted cotriple \( Cl(n-1) \)-vectorial bundle obtained from \( \text{AS}(\mathbb{E}) \) through the refinement \( \{U_\alpha \times V(s; \varepsilon_\alpha)\} \) of the open cover \( \{U_\alpha\} \), which is isomorphic to \( \text{AS}(\mathbb{E}) \) itself. Hence \( \alpha(\text{AS}(\mathbb{A})) = \alpha(\text{AS}(\mathbb{A})) \). \( \square \)

**Theorem 6.2.** For any \((X,Y,\tau) \in \hat{C}, j \in \mathbb{Z} \) and \( n > 0 \), the homomorphism

\[
\text{AS} : KF_{Cl(n)}^{s-j}(X,Y) \longrightarrow KF_{Cl(n-1)}^{s-j}(X,Y)
\]

is bijective.

**Proof.** Lemma 6.1 provides us the commutative diagram

\[
\begin{array}{ccc}
K_{Cl(n)}^{s-j}(X,Y) & \xrightarrow{\text{AS}} & K_{Cl(n-1)}^{s-j}(X,Y) \\
\alpha \downarrow & & \alpha \downarrow \\
KF_{Cl(n)}^{s-j}(X,Y) & \xrightarrow{\text{AS}} & KF_{Cl(n-1)}^{s-j}(X,Y).
\end{array}
\]

Since \( \text{AS} \) in the upper row is bijective by [7], Theorem 5.2 implies the conclusion. \( \square \)

Lemma 5.6 is generalized to the twisted case, so that we have a natural isomorphism \( KF_{Cl(n)}^{s-j}(X,Y) \to KF_{Cl(n+2m)}^{s-j}(X,Y) \). The composition of maps

\[
KF_{Cl(n)}^{s-j}(X,Y) \longrightarrow KF_{Cl(n+2m)}^{s-j}(X,Y)^{\text{AS}} \longrightarrow KF_{Cl(n)}^{s-j}(X,Y)
\]

is readily identified with the multiplication of a generator of \( K(D^2, S^1) \). Thus, Theorem 6.2 reproduces Corollary 5.4 (b).

6.2. **Twisted K-theory with coefficients \( \mathbb{Z}/p \).** Let \( p \) be a positive integer. The aim of this subsection is to provide a model of twisted \( K \)-theory with its coefficients \( \mathbb{Z}/p \), or twisted mod \( p \) \( K \)-theory by using twisted vectorial bundles. For this aim, we begin with a formulation of twisted mod \( p \) \( K \)-theory based on an idea in [4].

**Definition 6.3.** Let \( \tau \) be a principal \( PU(H) \)-bundle over a space \( X \).

(a) For a non-negative integer \( n \), we define a \( \tau \)-twisted mod \( p \) \( K \)-cocycle of degree \(-n+1\) on \( X \) to be a pair \((A, T)\) consisting of \( A \in \Gamma(X, \tau \times PU(H) \mathcal{F}_n(\mathcal{H}_n)) \) and \( T \in \Gamma(X \times [0,1], (\tau \times [0,1]) \times PU(H) \mathcal{F}_n(\mathcal{H}_n^{OP})) \) such that \( T|_{t=0} = A^{OP} \) and \( \text{Supp}(T|_{t=1}) = \emptyset \).
Lemma 6.4. There exists a natural exact sequence:
\[
K_{\text{Cl}(n)}^{-1}(X) \xrightarrow{m_p} K_{\text{Cl}(n)}^{-1}(X) \xrightarrow{\rho_p} K_{\text{Cl}(n)}^{-1}(X; \mathbb{Z}/p) \xrightarrow{\delta_p} K^\tau(X) \xrightarrow{m_p} K^\tau(X).
\]

Proof. We define \( \delta_p \) by \( \delta_p([A,T]) = [A] \) and \( m_p \) by \( m_p([A]) = [A^{\otimes p}] = p[A] \). To define \( \rho_p \), we represent an element in \( K_{\text{Cl}(n)}^{-1}(X) \) by a section \( B \in \Gamma(X \times I, X \times \partial I, (\tau \times I) \times PU(H) \mathcal{F}_n(\mathcal{H}_n^{\otimes p})) \), where \( I = [0,1] \). The section \( B_{|t=0} \) takes values in the space of invertible operators in \( \mathcal{F}_n(\mathcal{H}_n^{\otimes p}) \). Hence we can assume \( B_{|t=0} = J^{\otimes p} \) for some invertible operator \( J \in \mathcal{F}_n^*(\mathcal{H}_n) \). If we put \( \rho_p([B]) = ([J,B]) \), then \( \rho_p \) gives rise to a well-defined homomorphism.

Now, if \([B] \in K_{\text{Cl}(n)}^{-1}(X) \) is such that \( \rho_p([B]) = 0 \), then there exists a homotopy \((\hat{A}, \hat{T})\) between \((J,B)\) and \((J,J^{\otimes p})\). By a reparametrization of \( \hat{T} \), we can construct a homotopy connecting \( B \) and \( \hat{A}^{\otimes p} \), so that the exactness at the second term \( K_{\text{Cl}(n)}^{-1}(X) \) holds. To see the exactness at the third term \( K_{\text{Cl}(n)}^{-1}(X; \mathbb{Z}/p) \), let \((A,T)\) be such that \([A] = 0 \in K^\tau(X) \). Then there is a homotopy \( H \) between \( A \in \mathcal{F}_n(\mathcal{H}_n) \) and an invertible operator \( J \in \mathcal{F}_n^*(\mathcal{H}_n) \). Concatenating \( H^{\otimes p} \) and \( T \), we have \( B \) such that \( \rho_p([B]) = ([A,T]) \). The exactness at the forth term \( K^\tau(X) \) directly follows from the definitions of \( \delta_p \) and \( m_p \). \( \square \)

Since \( K_{\text{Cl}(n)}^{-1}(X) \cong K_{\text{Cl}(n)}^{-n-1}(X) \), the group \( K_{\text{Cl}(n)}^{-n-1}(X; \mathbb{Z}/p) \) fits into
\[
K_{\text{Cl}(n)}^{-n-1}(X) \xrightarrow{m_p} K_{\text{Cl}(n)}^{-n-1}(X) \xrightarrow{\rho_p} K_{\text{Cl}(n)}^{-n-1}(X; \mathbb{Z}/p) \xrightarrow{\delta_p} K^\tau(X) \xrightarrow{m_p} K^\tau(X).
\]
Thus, the \( \tau \)-twisted mod \( p \) \( K \)-theory \( K_{\text{Cl}(n)}^{-n-1}(X; \mathbb{Z}/p) \) of \( X \) of degree \(-n-1\) can be defined as \( K_{\text{Cl}(n)}^{-n-1}(X; \mathbb{Z}/p) = K_{\text{Cl}(n)}^{-n-1}(X; \mathbb{Z}/p) \). (By the help of the Bott periodicity, we can actually give an isomorphism between \( K_{\text{Cl}(n)}^{-n-1}(X; \mathbb{Z}/p) \) and the group \( K^\tau(X; \mathbb{Z}/p) \) constructed out of the so-called Moore space.)

Now, we introduce our finite-dimensional model of \( K_{\text{Cl}(n)}^{-1}(X; \mathbb{Z}/p) \).

Definition 6.5. Let \( \tau \) be a principal \( PU(H) \)-bundle over a space \( X \).

(a) For a non-negative integer \( n \), we define a \( \tau \)-twisted mod \( p \) \( \text{Cl}(n) \)-vectorial bundle on \( X \) to be a pair \((E, H)\) consisting of \( E \in K\mathcal{F}_{\text{Cl}(n)}(X) \) and \( H \in K\mathcal{F}^{\chi}_{\text{Cl}(n)}(X \times I) \) such that \( \mathbb{H}_{|t=0} \) is isomorphic to \( E \) and \( \text{Supp}(\mathbb{H}_{|t=1}) = \emptyset \).

(b) We define a homotopy between \( \tau \)-twisted mod \( p \) \( \text{Cl}(n) \)-vectorial bundles \((E_0, H_0)\) and \((E_1, H_1)\) on \( X \) to be a \((\tau \times I)\)-twisted mod \( p \) \( \text{Cl}(n) \)-vectorial bundle \((E, H)\) on \( X \times I \) such that \( E_{|t=i} \) and \( H_{|t=i} \) are isomorphic to \( E_i \) and \( H_i \), respectively, for \( i = 0,1 \).

(c) We define \( K\mathcal{F}^{\chi}_{\text{Cl}(n)}(X) \) to be the group of homotopy classes of \( \tau \)-twisted mod \( p \) \( \text{Cl}(n) \)-vectorial bundles on \( X \).
Lemma 6.6. There exists a natural exact sequence:

\[ KF_{Cl(n)}^{-1}(X) \xrightarrow{m_p} KF_{Cl(n)}^{-1}(X) \xrightarrow{\rho_p} KF_{Cl(n)}^{-1}(X; \mathbb{Z}/p) \xrightarrow{\delta_p} KF^{\tau}(X) \xrightarrow{m_p} KF^{\tau}(X). \]

Proof. We define \( \delta_p \) by \( \delta_p([E;H]) = [E] \) and \( m_p([F]) = [F] \). To define \( \rho_p \), let \( F \in KF_{Cl(n)}^{r \times \ell}(X \times I, X \times \partial I) \) represent an element in \( KF_{Cl(n)}^{-1}(X) \). Then \( \text{Supp}([F]|_{t=0}) = \emptyset \), so that \( F|_{t=0} \) is isomorphic to \( O^{p \times p} \), where \( O \in KF_{Cl(n)}^{-1}(X) \) is such that \( \text{Supp}(O) = \emptyset \), or equivalently \( [O] = 0 \) in \( KF_{Cl(n)}^{-1}(X) \). If we put \( \rho_p([F]) = ([0, F]) \), then \( \rho_p \) is a well-defined homomorphism. Now, the exactness of the sequence can be shown by using the argument in the proof of Lemma 6.4: The only thing to notice is that we apply a Mayer-Vietoris construction (Lemma 4.2, [11]) to a “concatenation” of twisted \( Cl(n) \)-vectorial bundles.

\[ \square \]

Lemma 6.7. There exists a natural homomorphism

\[ \alpha : KF_{Cl(n)}^{-1}(X; \mathbb{Z}/p) \longrightarrow KF_{Cl(n)}^{-1}(X; \mathbb{Z}/p) \]

making the following diagram commutative:

\[ \begin{array}{ccc}
KF_{Cl(n)}^{-1}(X) & \xrightarrow{\rho_p} & KF_{Cl(n)}^{-1}(X; \mathbb{Z}/p) \\
\downarrow & & \downarrow \alpha \\
KF_{Cl(n)}^{-1}(X) & \xrightarrow{\rho_p} & KF_{Cl(n)}^{-1}(X; \mathbb{Z}/p) \\
\end{array} \]

where the vertical maps other than \( \alpha \) are those constructed in Subsection 5.1.

Proof. We define \( \alpha \) in question based on the construction in Subsection 5.1: Suppose that a \( \tau \)-twisted mod \( p \) K-cocycle \( (A, T) \) of degree \(-n-1\) on \( X \) is given. By definition, \( A_x = T(x,0) \) holds for all \( x \in X \). To have a finite-dimensional approximation of \( A \), we choose an open cover \( \{U_{\alpha} \} \) of \( X \), local trivializations \( s_{\alpha} \) of \( \tau \), lifts of transition functions \( g_{\alpha \beta} \) and positive numbers \( \mu_{\alpha} \) so that \( \bigcup_{x \in U_{\alpha}} (\mathcal{H}_{\alpha}, (A_{\alpha})_x)_{<\mu_{\alpha}} \) gives rise to a vector bundle. Also, to have a finite-dimensional approximation of \( T \), we choose an open cover \( \{U_{\alpha} \} \) of \( X \times I \), local trivializations \( s_{\alpha} \) of \( \tau \times I \), lifts of transition functions \( g_{\alpha \beta} \), and positive numbers \( \mu_{\alpha} \) so that \( \bigcup_{(z, 0) \in U_{\alpha}} (\mathcal{H}_{\alpha}^{p \times p}, (T_{\alpha}(z,0))_{<\mu_{\alpha}} \) gives rise to a vector bundle. We can choose these data for \( T \) in a way compatible with the data for \( A \), that is,

- the open cover \( \{U_{\alpha} \} \) agrees with the open cover \( \{U_{\alpha}|_{t=0} \} \) of \( X \times \{0\} \);
- if \( U_{\alpha} = U_{\beta}|_{t=0} \), then \( s_{\alpha} = s_{\beta}|_{t=0} \), \( g_{\alpha \beta} = g_{\alpha \beta}|_{t=0} \) and \( \mu_{\alpha} = \mu_{\beta} \).

Such a choice is possible because the eigenvalues of \( (T_{\alpha})^2_{(x,t)} \) are continuous in \((x,t)\). Under the choice above, we get a \( \tau \)-twisted mod \( p \) \( Cl(n) \)-vectorial bundle \((\mathbb{E}, \mathbb{H})\) as a finite-dimensional approximation of \((A, T)\). We put \( \alpha([(A,T)]) = ([E, H]) \) and define the homomorphism \( \alpha \). Once \( \alpha \) is defined, the commutativity of the diagram is obvious from the construction.

\[ \square \]

Theorem 6.8. For any \((X, \emptyset, \tau) \in \mathcal{C}\), the homomorphism in Lemma 6.6

\[ \alpha : KF_{Cl(n)}^{-1}(X; \mathbb{Z}/p) \longrightarrow KF_{Cl(n)}^{-1}(X; \mathbb{Z}/p) \]

is bijective, so that there is an isomorphism \( KF^{-n-1}(X; \mathbb{Z}/p) \cong KF_{Cl(n)}^{-1}(X; \mathbb{Z}/p) \).

Proof. The theorem follows from Lemma 6.4, 6.6, 6.7 and Theorem 5.2.
Though will not be detailed anymore, we can take into account additional support conditions to define the relative versions $K_{\text{Cl}(n)}^{-1}(X,Y;\mathbb{Z}/p)$ as well as $KF_{\text{Cl}(n)}^{-1}(X,Y;\mathbb{Z}/p)$ for any $(X,Y,\tau) \in \hat{\mathcal{C}}$. Then, in the same way as above, we get isomorphisms $K_{\text{Cl}(n)}^{-1}(X,Y;\mathbb{Z}/p) \cong KF_{\text{Cl}(n)}^{-1}(X,Y;\mathbb{Z}/p)$ and

$$K^{\tau-j-n-1}(X,Y;\mathbb{Z}/p) \cong KF_{\text{Cl}(n)}^{-1}(X \times I^j, Y \times I^j \cup X \times \partial I^j;\mathbb{Z}/p).$$

References


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