An approach to finite-dimensional realizations of twisted K-theory

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Problem in twisted K-theory

Realize twisted K-theory generally by means of finite dimensional geometric objects.

Main theorem

We can define a group by means of "*twisted $\mathbb{Z}_2$-graded Hermitian general vector bundles*", into which there exists a monomorphism from twisted K-theory.

Plan

§1 Twisted K-theory

§2 Hermitian general vector bundle

§1 Twisted K-theory

Origin

P. Donovan and M. Karoubi (1970)

J. Rosenberg (1989)

Application

D-brane charge
[Witten, Kapustin, ...]

The Verlinde algebra
[Freed-Hopkins-Teleman]

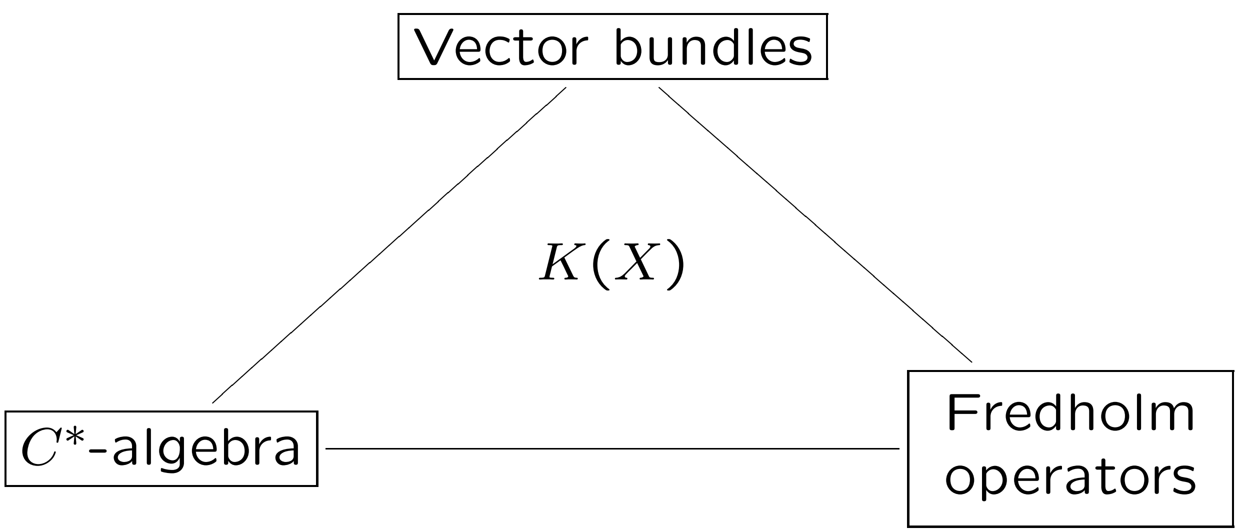
The quantum Hall effect
[Carey-Hannabuss-Mathai-McCann]

K-theory X : compact

$\mathrm{Vect}(X)$ = the isomorphism classes of finite dimensional vector bundles over X

Definition

$$K(X) = K(\mathrm{Vect}(X))$$
$$= \mathrm{Vect}(X) \times \mathrm{Vect}(X) / \Delta(\mathrm{Vect}(X))$$



Fredholm operators

$\mathcal{H}$: separable Hilbert space ($\dim\mathcal{H} = \infty$)

A Fredholm operator $A : \mathcal{H} \to \mathcal{H}$

$$\overset{\text{def}}{\Longleftrightarrow} \begin{cases} \text{bounded linear,} \\ \text{Image}(A) \subset \mathcal{H} : \text{closed,} \\ \dim\text{Ker}(A), \dim\text{Coker}(A) < \infty. \end{cases}$$

$\mathcal{F}(\mathcal{H}) = \{\text{Fredholm operators } A : \mathcal{H} \to \mathcal{H}\}$

Fact [Atiyah, Jänich]

X : compact

$$C(X, \mathcal{F}(\mathcal{H}))/\text{htpy} \xrightarrow{\text{iso}} K(X)$$

Twisted K-theory

$$PU(\mathcal{H}) = U(\mathcal{H})/U(1) \overset{\text{Ad}}{\curvearrowright} \mathcal{F}(\mathcal{H})$$

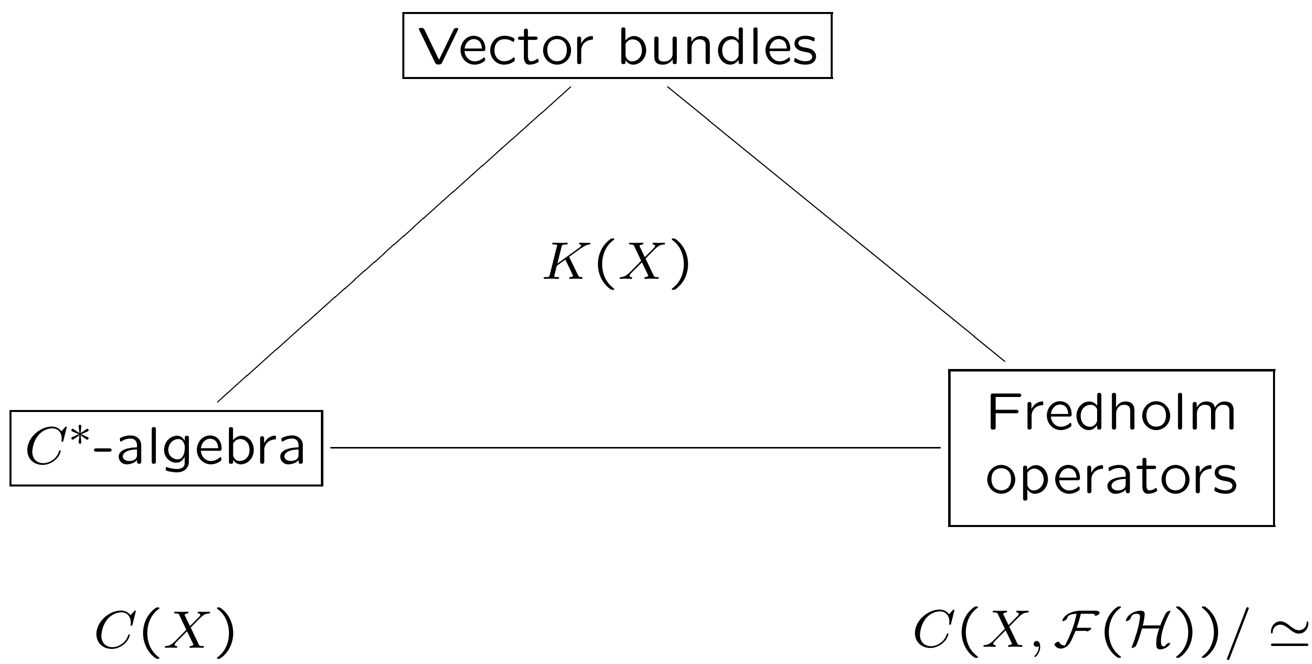
Definition

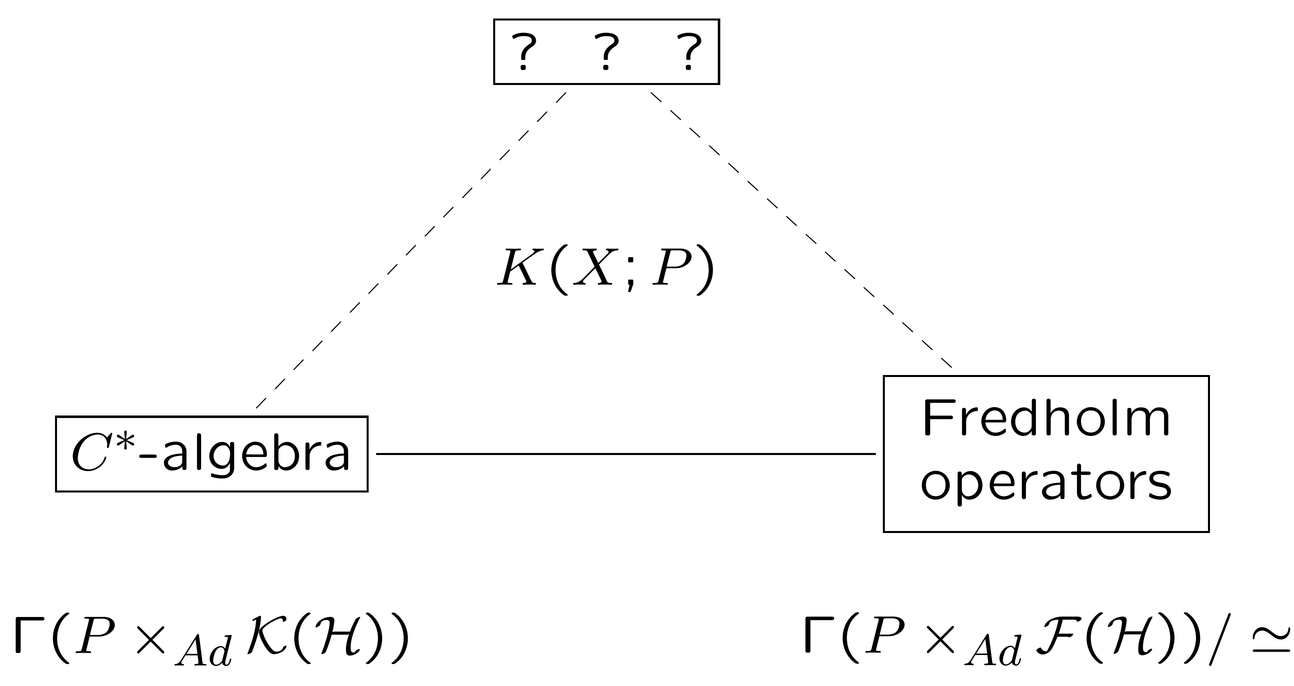
$P \to X$: principal $PU(\mathcal{H})$-bundle

$$K(X;P) = \Gamma(X, P \times_{Ad} \mathcal{F}(\mathcal{H}))/\text{htpy}$$

- $P \cong X \times PU(\mathcal{H}) \Rightarrow K(X;P) \cong K(X)$.
- Principal $PU(\mathcal{H})$-bundles P are classified by their Dixmier-Douady classes:

$$\delta(P) \in H^3(X;\mathbb{Z}).$$


Vector bundles
$K(X)$
C^*-algebra
Fredholm operators
$C(X)$
$C(X, \mathcal{F}(\mathcal{H}))/\simeq$


? ? ?
$K(X; P)$
C^*-algebra
Fredholm operators
$\Gamma(P \times_{Ad} \mathcal{K}(\mathcal{H}))$
$\Gamma(P \times_{Ad} \mathcal{F}(\mathcal{H}))/\simeq$

Problem

Realize twisted K-theory $K(X;P)$ generally by means of finite dimensional geometric objects.

$\delta(P)$: finite order $\Rightarrow$ ${}^{\exists}$answer

Fact

$$\begin{cases} X & : \text{ compact manifold} \\ P & : \ \delta(P) \text{ is finite order} \end{cases}$$

We can define a group by means of "*twisted vector bundles*", to which there exists an isomorphism from $K(X;P)$.

$$K(X;P) \xrightarrow{\text{iso}} K(\{\text{twisted vector bundles}\}/\cong)$$

Twisted vector bundle

$$(\mathcal{U}, E_\alpha, \phi_{\alpha\beta})$$

$$\begin{cases} \mathcal{U} = \{U_\alpha\} \quad \text{open cover of } X; \\ E_\alpha \to U_\alpha \quad \text{finite rank vector bundle;} \\ \phi_{\alpha\beta} : E_\beta|_{U_{\alpha\beta}} \to E_\alpha|_{U_{\alpha\beta}} \quad \text{isomorphism;} \end{cases}$$

$$\phi_{\alpha\beta}\phi_{\beta\gamma} = c_{\alpha\beta\gamma}\phi_{\alpha\gamma}.$$

$$\left(\begin{array}{ccccc} (c_{\alpha\beta\gamma}) & \in & \check{Z}^2(\mathcal{U}; \underline{U(1)}) & & \\ & & \downarrow & & \\ \delta(P) & \in & H^2(X; \underline{U(1)}) & \cong & H^3(X; \mathbb{Z}) \end{array} \right)$$

<u>Remark</u> $(\mathcal{U}, E_\alpha, \phi_{\alpha\beta})$: rank $r \Rightarrow r \cdot \delta(P) = 0$.

$$(\det\phi_{\alpha\beta})(\det\phi_{\beta\gamma}) = (c_{\alpha\beta\gamma})^r(\det\phi_{\alpha\gamma})$$

§2 Hermitian general vector bundle

M. Furuta, "*Index theorem, II*". (Japanese)
Iwanami Series in Modern Mathematics.
Iwanami Shoten, Publishers, Tokyo, 2002.

- to define $K(X)$;

Theorem[Furuta]

X : compact

We can define a group by means of $\mathbb{Z}_2$-graded Hermitian general vector bundles, which is isomorphic to $K(X)$.

- to approximate Dirac-type operators.

  a linear version of the finite dimensional approximation of the Seiberg-Witten equations

Hermitian general vector bundle on X

$$(\mathcal{U}, (E_\alpha, h_\alpha), \phi_{\alpha\beta})$$

$$\begin{cases} \mathcal{U} = \{U_\alpha\} & \text{open cover of } X; \\ E_\alpha \to U_\alpha & \mathbb{Z}_2\text{-gr. Hermitian vector bundle;} \\ h_\alpha : E_\alpha \to E_\alpha & \text{Hermitian map of degree 1;} \\ \phi_{\alpha\beta} : E_\beta|_{U_{\alpha\beta}} \to E_\alpha|_{U_{\alpha\beta}} & \text{map of degree 0} \\ & \text{s.t. } h_\alpha \phi_{\alpha\beta} = \phi_{\alpha\beta} h_\beta; \end{cases}$$

1. "$\phi_{\alpha\beta}\phi_{\beta\alpha} = 1$",

$$\left(\begin{array}{c} {}^\forall x \in U_{\alpha\beta};\ \begin{cases} x \in {}^\exists V \subset U_{\alpha\beta}, \\ {}^\exists \mu > 0, \end{cases} \text{such that :} \\ \begin{cases} {}^\forall y \in V, \\ {}^\forall v \in \bigoplus_{\lambda < \mu} \{v \in (E_\alpha)_y |\ h_\alpha^2 v = \lambda v\}, \end{cases} \\ \\ \phi_{\alpha\beta}\phi_{\beta\alpha}(v) = v. \end{array} \right)$$

2. "$\phi_{\alpha\beta}\phi_{\beta\gamma} = \phi_{\alpha\gamma}$".

Fredholm operator $A : \mathcal{H} \to \mathcal{H}$

$$\rightsquigarrow \text{approximate}$$

(E, h)

$$\begin{cases} E = E^0 \oplus E^1 & \mathbb{Z}_2\text{-gr. Herm. vector space} \\ h : E \to E & \text{Hermitian map of degree 1} \end{cases}$$

<u>Step 1</u>
$$\begin{cases} \widehat{\mathcal{H}} = \mathcal{H} \oplus \mathcal{H} & \mathbb{Z}_2\text{-graded} \\ \widehat{A} = \begin{pmatrix} 0 & A^* \\ A & 0 \end{pmatrix} & \text{self-adjoint, degree 1} \end{cases}$$

<u>Step 2</u> $\sigma(\widehat{A}^2) \ni 0$: discrete $\Rightarrow$ $^\exists\mu > 0$ s.t.

- $\mu \notin \sigma(\widehat{A}^2)$;
- $\sigma(\widehat{A}^2) \cap [0, \mu)$ consists of a finite number of eigenvalues: $0 = \lambda_1 < \lambda_2 < \cdots < \lambda_n < \mu$;
- $(\mathcal{H}, \widehat{A})_{\lambda_i} = \{v \in \widehat{\mathcal{H}} | \ \widehat{A}^2 v = \lambda_i v\}$: finite dim.

($(\mathcal{H}, \widehat{A})_0 = \mathrm{Ker}\widehat{A}^2 \cong \mathrm{Ker}A \oplus \mathrm{Coker}A$)

$$
\begin{array}{ccc}
\widehat{\mathcal{H}} & \xrightarrow{\widehat{A}} & \widehat{\mathcal{H}} \\
\| & & \| \\
(\widehat{\mathcal{H}}, \widehat{A})_0 & \xrightarrow{0} & (\widehat{\mathcal{H}}, \widehat{A})_0 \\
\oplus & & \oplus \\
(\widehat{\mathcal{H}}, \widehat{A})_{\lambda_2} & \cong & (\widehat{\mathcal{H}}, \widehat{A})_{\lambda_2} \\
\oplus & & \oplus \\
(\widehat{\mathcal{H}}, \widehat{A})_{\lambda_3} & \cong & (\widehat{\mathcal{H}}, \widehat{A})_{\lambda_3} \\
\oplus & & \oplus \\
(\widehat{\mathcal{H}}, \widehat{A})_{\lambda_4} & \cong & (\widehat{\mathcal{H}}, \widehat{A})_{\lambda_4} \\
\oplus & & \oplus \\
\vdots & & \vdots \\
\oplus & & \oplus \\
(\widehat{\mathcal{H}}, \widehat{A})_{\lambda_n} & \cong & (\widehat{\mathcal{H}}, \widehat{A})_{\lambda_n} \\
\oplus & & \oplus \\
\text{complement} & \cong & \text{complement}
\end{array}
$$

Step 3 Put $\begin{cases} E = \oplus_{\lambda<\mu}(\widehat{\mathcal{H}}, \widehat{A})_\lambda, \\ h = \widehat{A}|_E. \end{cases}$

Remark family $\{A_x : \mathcal{H} \to \mathcal{H}\}_{x \in X}$

$\rightsquigarrow$ approximate

$\mathbb{Z}_2$-graded Hermitian general vector bundle over X

Main theorem

$$\begin{cases} X: & \text{compact manifold} \\ P: & PU(\mathcal{H})\text{-bundle} \end{cases}$$

We can define a group by means of *twisted* $\mathbb{Z}_2$-graded Hermitian general vector bundles, into which there exists a *monomorphism* from $K(X;P) = \Gamma(P \times_{Ad} \mathcal{F}(\mathcal{H}))/\simeq$.

- twisting $\Leftarrow$ "$\phi_{\alpha\beta}\phi_{\beta\gamma} = c_{\alpha\beta\gamma}\phi_{\alpha\gamma}$"

- monomorphism $\Leftarrow$ finite dimensional approximation