Multiplication in differential cohomology and cohomology operation

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Feb 17, 2009

Talk about

a relationship between

- **multiplications in differential cohomology theories**
- **classical cohomology operations**

Introduction

- **In general, a differential cohomology is a refinment of a generalized cohomology theory involving information of differential forms on smooth manifolds.**

ordinary cohomology	$\longrightarrow$	**differential ordinary cohomology**
K**-cohomology**	$\longrightarrow$	**differential** K**-cohomology**
$\vdots$		$\vdots$

- **A differential cohomology theory is also called a "smooth cohomology theory".**
- **For the ordinary cohomology, its differential version (differential ordinary cohomology) has been known as:**
  - **the group of Cheeger-Simons' differential characters**
  - **the smooth Deligne cohomology**
- **The differential version of any generalized cohomology was introduced in a work of Hopkins and Singer [JDG, math/0211216].**

- For some generalized cohomology theory h^*, its differential version $\check{h}^*$ admits a multiplication:

$$\cup : \check{h}^m(X) \otimes \check{h}^n(X) \longrightarrow \check{h}^{m+n}(X)$$

compatible with the multiplication in the underlying cohomology theory h^*. (e.g. $\check{H}^*$ and $\check{K}^*$)

- If the multiplication in $\check{h}^*$ is graded-commutative, then the squaring map on odd classes

$$\begin{array}{cccc} \check{q} : & \check{h}^{2k+1}(X) & \longrightarrow & \check{h}^{4k+2}(X) \\ & x & \longmapsto & x^2 \end{array}$$

is a homomorphism.

$$\check{q}(x+y) = x^2 + xy + yx + y^2 = \check{q}(x) + \check{q}(y)$$

- **Moreover, the map reduces to a homomorphism**

$$\check{q}: \ h^{2k+1}(X) \longrightarrow h^{4k+1}(X; \mathbb{R}/\mathbb{Z}).$$

"Main Theorem"

In the case of $\check{H}^*$ and $\check{K}^*$, the homomorphisms $\check{q}$ are related to the Steenrod operations and the Adams operations.

- **The map $\check{q}$ appeas in two contexts of physics**
  - **Chern-Simons theory in 5-dimensions [Witten]**
  - **Hamiltonian quantization of self-dual generalized abelian gauge fields [Freed-Moore-Segal]**

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Definition of differential cohomology

Definition of differential ordinary cohomology

- The **differential ordinary cohomology** $\check{H}^n(X)$ of a smooth manifold X consists of the equivalence classes of differential cocycles of degree n.
- A **differential cocycle** of degree n is a triple:

$$(c, h, \omega) \in C^n(X;\mathbb{Z}) \times C^{n-1}(X;\mathbb{R}) \times \Omega^n(X)$$
$$\delta c = 0, \quad d\omega = 0, \quad \omega = c + \delta h.$$

- (c, h, ω) and (c', h', ω') are **equivalent** if:

$$\exists (b, k) \in C^{n-1}(X;\mathbb{Z}) \times C^{n-2}(X;\mathbb{R})$$
$$c' - c = \delta b, \quad \omega' = \omega, \quad h - h' = b + \delta k.$$

Examples

$$\check{H}^0(X) = \{(c,\omega) \in C^0_{\mathbb{Z}} \times \Omega^0 |\ \delta c = 0, d\omega = 0, c = \omega\}$$
$$\cong H^0(X;\mathbb{Z})$$

$$\check{H}^1(X) \cong C^\infty(X, U(1))$$

$$\check{H}^2(X) \cong \{U(1)\text{-bundle with connection}/X\}/\text{isom}$$

$$\check{H}^3(X) \cong \{\text{abelian gerbe with connection}/X\}/\text{isom}$$

Addition and Multiplication

- **The differential cohomology $\check{H}^*(X)$ is an additive group:**

$$(c, h, \omega) + (c', h', \omega') = (c + c', h + h', \omega + \omega').$$

- **$\check{H}^*(X)$ is a graded-commutative ring:**

$$\begin{aligned}&(c, h, \omega) \cup (c', h', \omega') \\ &= (c \cup c', (-1)^{|c|} c \cup h' + h \cup \omega' + B(\omega \otimes \omega'), \omega \wedge \omega'),\end{aligned}$$

where $B : \Omega^*(X) \otimes \Omega^*(X) \to C^*(X; \mathbb{R})$ is a functorial homomorphism satisfying

$$\omega \wedge \omega' - \omega \cup \omega' = Bd(\omega \otimes \omega') - \delta B(\omega \otimes \omega').$$

Example

$$\cup : \ \check{H}^1(S^1) \times \check{H}^1(S^1) \longrightarrow \check{H}^2(S^1)$$

$$\begin{cases} \check{H}^1(S^1) = C^\infty(S^1, U(1)) \ (= LU(1)) \\ \check{H}^2(S^1) = \mathbb{R}/\mathbb{Z} \ (= \text{holonomy around } S^1) \end{cases}$$

$$f : S^1 \to U(1) \Rightarrow \begin{cases} F : S^1 \to \mathbb{R} & \text{lift of } f \\ \Delta_f = F(\theta + 2\pi) - F(\theta) & \text{winding } \sharp \end{cases}$$

$$f \cup g = \Delta_f G(0) - \int_0^{2\pi} F \frac{dG}{d\theta} d\theta \mod \mathbb{Z}$$

Example

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Remark $c(f, g) = \exp 2\pi\sqrt{-1}(f \cup g)$ gives a 2-cocycle defining the cetral extension of $LU(1)$ of level 2.

The 1st exact sequence

$$0 \longrightarrow \Omega^{n-1}(X)/\Omega^{n-1}_{\mathbb{Z}}(X) \xrightarrow{i} \check{H}^n(X) \xrightarrow{\chi} H^n(X;\mathbb{Z}) \longrightarrow 0,$$
$$(c, h, \omega) \mapsto c$$

where $\Omega^p_{\mathbb{Z}}(X)$ means the group of closed integral p-forms.

Example

$$\check{H}^2(X) \cong \{U(1)\text{-bundle with connection}/X\}/\text{isom}$$
$$\chi[(P, A)] = \text{Chern class of } P$$
$$\Omega^1(X)/\Omega^1_{\mathbb{Z}}(X) = \{\text{connection on } P\}/\text{gauge equivalence}$$

The 2nd exact sequence

$$0 \longrightarrow H^{n-1}(X;\mathbb{R}/\mathbb{Z}) \longrightarrow \check{H}^n(X) \xrightarrow{\delta} \Omega^n_{\mathbb{Z}}(X) \longrightarrow 0.$$
$$(c, h, \omega) \mapsto \quad \omega$$

Example

$$\check{H}^2(X) \cong \{U(1)\text{-bundles with connection}/X\}/\text{isom}$$
$$\delta[(P, A)] = \frac{-1}{2\pi\sqrt{-1}} F(A)$$
$$H^2(X;\mathbb{R}/\mathbb{Z}) = \{\text{flat } U(1)\text{-bundle}/X\}/\text{isom}$$

The multiplication is compatible with the exact sequences

$$0 \longrightarrow \Omega^{n-1}(X)/\Omega_{\mathbb{Z}}^{n-1}(X) \xrightarrow{i} \check{H}^n(X) \xrightarrow{\chi} H^n(X;\mathbb{Z}) \longrightarrow 0,$$

$$0 \longrightarrow H^{n-1}(X;\mathbb{R}/\mathbb{Z}) \longrightarrow \check{H}^n(X) \xrightarrow{\delta} \Omega_{\mathbb{Z}}^n(X) \longrightarrow 0.$$

- The cup product in $H^*(X;\mathbb{Z})$;
- The wedge product in $\Omega_{\mathbb{Z}}^*(X)$;
- The product in $\Omega^*(X)/\Omega_{\mathbb{Z}}^*(X)$.

$$\begin{array}{ccc} \Omega^{m-1}/\Omega_{\mathbb{Z}}^{m-1} \otimes \Omega^{n-1}/\Omega_{\mathbb{Z}}^{n-1} & \longrightarrow & \Omega^{m+n-1}/\Omega_{\mathbb{Z}}^{m+n-1} \\ \eta \otimes \eta' & \mapsto & \eta \wedge d\eta' \end{array}$$

Preliminary to the definition of differential K-cohomology

- **For $n \in \mathbb{Z}$, define $C^{|n|}(X;\mathbb{R})$ and $\Omega^{|n|}(X)$ by**

$$C^{|n|}(X;\mathbb{R}) = \begin{cases} \prod_{m\geq 0} C^{2m}(X;\mathbb{R}), & (n : \text{even}) \\ \prod_{m\geq 0} C^{2m+1}(X;\mathbb{R}). & (n : \text{odd}) \end{cases}$$

$$\Omega^{|n|}(X) = \begin{cases} \prod_{m\geq 0} \Omega^{2m}(X), & (n : \text{even}) \\ \prod_{m\geq 0} \Omega^{2m+1}(X). & (n : \text{odd}) \end{cases}$$

- **Let $\mathbb{K}^n$ be the classifying space of K^n.**
  (i.e. $K^n(X) = [X, \mathbb{K}^n]$ for any CW complex.)
- **Let $\iota_n \in C^{|n|}(\mathbb{K}^n;\mathbb{R})$ be a cocycle representing the universal Chern character class.**
  (i.e. $\mathrm{ch}([c]) = [c^*\iota_n]$ for any $c : X \to \mathbb{K}^n$.)

The definition of differential K-cohomology

- The **differential K-cohomology $\check{K}^n(X)$** of a smooth manifold X consists of the equivalence classes of differential K-cocycles of degree n.
- A **differential K-cocycle** of degree n is a triple:

$$(c, h, \omega) \in \mathrm{Map}(X, \mathbb{K}^n) \times C^{|n-1|}(X; \mathbb{R}) \times \Omega^{|n|}(X)$$
$$d\omega = 0, \quad \omega = c^* \iota_n + \delta h.$$

- $x = (c, h, \omega)$ and $x' = (c', h', \omega')$ are **equivalent** if there is a differentila K-cocycle $\tilde{x} = (\tilde{c}, \tilde{h}, \tilde{\omega})$ on $X \times I$ s.t.

$$\tilde{x}|_{t=0} = x, \quad \tilde{x}|_{t=1} = x', \quad \tilde{\omega}(\tfrac{\partial}{\partial t}) = 0.$$

Property of the differential K-cohomology

- $\check{K}^*(X)$ gives rise to a graded-commutative ring.
- $\check{K}^n(X)$ fits into the natural exact sequences:

$$0 \longrightarrow \Omega^{|n-1|}(X)/\Omega_K^{|n-1|}(X) \longrightarrow \check{K}^n(X) \longrightarrow K^n(X) \longrightarrow 0,$$
$$0 \longrightarrow K^{n-1}(X;\mathbb{R}/\mathbb{Z}) \longrightarrow \check{K}^n(X) \longrightarrow \Omega_K^{|n|}(X) \longrightarrow 0,$$

where $\Omega_K^{|n|}(X)$ is the group of closed forms representing the image of the Chern character $\mathrm{ch} : K^n(X) \to H^{|n|}(X;\mathbb{R})$.

- The multiplication is compatible with these sequences.

Comments on the general case

- For any generalized cohomology theory h^*, its differential version $\check{h}^*$ is defined in a way similar to the case of the K-cohomology, by using classifying spaces.
- $\check{h}^*$ also fits into two natural exact sequences.
- It is unclear whether the differential cohomology of a multiplicative cohomology theory admits a compatible multiplication.

  (All the cohomology theories obtained by the Landweber exact functor theorem have compatible multiplications. [Bunke-Schick-Schröder-Wiethaup])

Differential cohomology in Physics

- **Relationship between $\check{q}$ and:**
  - **Chern-Simons theory in 5-dimensions**
  - **Hamiltonian quantization of self-dual abelian generalized gauge fields**
- **The key is interpretation of differential cocycles.**

Interpretation of differential cocycles

- **Differential (ordinary) cocycles are abelian gauge fields and their generalization.**

$$\check{H}^1(X) \cong C^\infty(X, U(1))$$

$$\check{H}^2(X) \cong \{U(1)\text{-bundle with connection}/X\}/\text{isom}$$

$$\check{H}^3(X) \cong \{\text{abelian gerbe with connection}/X\}/\text{isom}$$

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- Differential (ordinary) cocycles are abelian gauge fields and their generalization.

$\check{H}^1(X) \cong C^\infty(X, U(1))$

periodic scalar field

$\check{H}^2(X) \cong \{U(1)$-bundle with connection$/X\}/$isom

$U(1)$-gauge field

$\check{H}^3(X) \cong \{$abelian gerbe with connection$/X\}/$isom

B-field (2-form gauge field)

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B-field (2-form gauge field)

- Differential K-cocycles can be thought of as Ramond-Ramond fields in type II string theory. (degree 0 for IIA, and degree 1 for IIB)

Witten's 5-dimensional Chern-Simons theory

- X : closed oriented 5-dimensional manifold
- the space of fields (modulo gauge transformation)

$$\mathcal{B}_{\mathrm{RR}} \times \mathcal{B}_{\mathrm{NS}} = \check{H}^3(X) \times \check{H}^3(X)$$

- the action functional $I_{\mathrm{CS}} : \mathcal{B}_{\mathrm{RR}} \times \mathcal{B}_{\mathrm{NS}} \to \mathbb{R}/\mathbb{Z}$

$$I_{\mathrm{CS}}(B_{\mathrm{RR}}, B_{\mathrm{NS}}) = \int_X^{\check{H}} B_{\mathrm{RR}} \cup B_{\mathrm{NS}},$$

where $\int_X^{\check{H}}$ is the integration in $\check{H}^*$:

$$\int_X^{\check{H}} : \ \check{H}^6(X) \xrightarrow{\cong} \check{H}^1(\mathrm{pt}) \cong \mathbb{R}/\mathbb{Z}.$$

- **On $\mathcal{B}_{\mathrm{RR}} \times \mathcal{B}_{\mathrm{NS}}$, $SL(2,\mathbb{Z})$ acts:**

$$(B_{\mathrm{RR}}, B_{\mathrm{NS}}) \begin{pmatrix} a & b \\ c & d \end{pmatrix} = (aB_{\mathrm{RR}} + cB_{\mathrm{NS}}, bB_{\mathrm{RR}} + dB_{\mathrm{NS}})$$

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- **But, I_{CS} is not generally invariant under the action:**

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- But, I_{CS} is not generally invariant under the action:

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

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- But, I_{CS} is not generally invariant under the action:

$$I_{\mathrm{CS}}((B_{\mathrm{RR}}, B_{\mathrm{NS}}) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}) = I_{\mathrm{CS}}(B_{\mathrm{RR}}, B_{\mathrm{NS}}),$$

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

- On $\mathcal{B}_{\mathrm{RR}} \times \mathcal{B}_{\mathrm{NS}}$, $SL(2,\mathbb{Z})$ acts:

$$(B_{\mathrm{RR}}, B_{\mathrm{NS}}) \begin{pmatrix} a & b \\ c & d \end{pmatrix} = (aB_{\mathrm{RR}} + cB_{\mathrm{NS}}, bB_{\mathrm{RR}} + dB_{\mathrm{NS}})$$

- But, I_{CS} is not generally invariant under the action:

$$I_{\mathrm{CS}}((B_{\mathrm{RR}}, B_{\mathrm{NS}}) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}) = I_{\mathrm{CS}}(B_{\mathrm{RR}}, B_{\mathrm{NS}}),$$

$$I_{\mathrm{CS}}((B_{\mathrm{RR}}, B_{\mathrm{NS}}) \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}) = I_{\mathrm{CS}}(B_{\mathrm{RR}}, B_{\mathrm{NS}}) + \int_X^{\check{H}} B_{\mathrm{RR}} \cup B_{\mathrm{RR}}.$$

- $\check{q} : \check{H}^3(X) \to \check{H}^6(X)$ is the obstruction to the $SL(2,\mathbb{Z})$-invariance of I_{CS}.
- In the case that $X = 4$-dimensional spin manifold $\times\, S^1$, Witten showed the $SL(2,\mathbb{Z})$-invariance on a subgroup in $\mathcal{B}_{\mathrm{RR}} \times \mathcal{B}_{\mathrm{NS}}$.
- Main result implies that I_{CS} is generally $SL(2,\mathbb{Z})$-invariant for any 5-dimensional spin manifold.

Hamiltonian quantization of Freed-Moore-Segal

- **Freed-Moore-Segal studied Hamiltonian quantization of self-dual generalized abelian gauge theories.**
- **In particular, they defined the quantum Hilbert space to be a unique representation space of a central extension $\hat{A}$ of an abelian group A.**
  1. **self-dual $2k$-form fields in $(4k+2)$-dimensions:**

     $$A = \check{H}^{2k+1}(X), \quad \dim X = 4k+1$$

     ($k = 0$: theory of chiral scalar fields or $U(1)$ WZW)
  2. **RR fields in type II string:**

     $$A = \check{K}^{n}(X), \quad \dim X = 9$$

- **In the construction of $\hat{A}$, the squaring map $\check{q}$ appears.**

Classification of central extensions

A : an abelian group with a resonable condition.

$$\left\{\text{central extension } \hat{A} \text{ of } A \text{ by } U(1)\right\} /\text{isom}$$
$$\cong H^2_{\text{group}}(A; \mathbb{R}/\mathbb{Z})$$
$$= \{c : A \times A \to \mathbb{R}/\mathbb{Z} | \text{ cocycle condition}\} /\text{coboundary}$$

$$\begin{array}{ccccccccc} 0 & \longrightarrow & U(1) & \longrightarrow & \hat{A} & \longrightarrow & A & \longrightarrow & 0 \\ & & & & \| & & & & \\ & & & & A \times U(1) & & & & \end{array}$$

$$(a, z) \cdot (a', z') = (a + a', zz' \exp 2\pi\sqrt{-1}c(a, a'))$$

$$\Big\{\text{central extension } \hat{A} \text{ of } A \text{ by } U(1)\Big\} / \text{isom}$$

$$\cong \{c : A \times A \to \mathbb{R}/\mathbb{Z} \,|\, \text{cocycle condition}\} / \text{coboundary}$$

$$\cong \{s : A \times A \to \mathbb{R}/\mathbb{Z} \,|\, \text{biadditive, skew, alternating}\}$$

$$\begin{aligned} \text{biadditive} &: \begin{cases} s(a+a', b) = s(a,b) + s(a',b) \\ s(a, b+b') = s(a,b) + s(a,b') \end{cases} \\ \text{skew} &: s(a,b) = -s(b,a) \\ \text{alternating} &: s(a,a) = 0 \end{aligned}$$

$$\begin{aligned} c(a,b) &\mapsto s(a,b) = c(a,b) - c(b,a) \\ \text{coboundary} &\mapsto 0 \end{aligned}$$

- **Apply the classification to:**

$$A = \check{H}^{2k+1}(X), \qquad \dim X = 4k+1.$$

To specify a central extension of A, it suffices to choose $s : A \times A \to \mathbb{R}/\mathbb{Z}$ which is biadditive, skew and alternating.

- **A naive construction:**

$$s_{\text{trial}}(x, y) = \int_X^{\check{H}} x \cup y.$$

This is biadditive and skew, but not alternating:

$$s_{\text{trial}}(x, x) = \int_X^{\check{H}} \check{q}(x) \overset{?}{=} 0$$

- **Freed-Moore-Segal:**

$$s(x,y) = \int_X^{\check{H}} x \cup y - \frac{1}{2}\epsilon(x)\epsilon(y),$$

$$\epsilon(x) = \begin{cases} 0, & (\check{q}(x) = 0 \in \mathbb{R}/\mathbb{Z}) \\ 1. & (\check{q}(x) = 1/2 \in \mathbb{R}/\mathbb{Z}) \end{cases}$$

This is biadditive, skew and alternating. $(\Rightarrow \hat{A})$

- **In the case that $k = 0$ and $X = S^1$, the resulting central extension $\hat{A}$ is the basic central extension $\widetilde{LU(1)}$ of $A = \check{H}^1(S^1) = C^\infty(S^1, U(1)) = LU(1)$.**

Main result in the case of oridnary cohomology

Lemma

The squaring map $\check{q} : \check{H}^{2k+1}(X) \to \check{H}^{4k+2}(X)$ reduces to a homomorphism $\check{q} : H^{2k+1}(X;\mathbb{Z}) \to H^{4k+1}(X;\mathbb{R}/\mathbb{Z})$.

Proof

$$\begin{array}{ccccc}
\Omega^{2k}(X)/\Omega^{2k}_{\mathbb{Z}}(X) & \xrightarrow{i} & \check{H}^{2k+1}(X) & \longrightarrow & H^{2k+1}(X;\mathbb{Z}) \\
 & & \downarrow & & \\
H^{4k+1}(X;\mathbb{R}/\mathbb{Z}) & \longrightarrow & \check{H}^{4k+2}(X) & \xrightarrow[\delta]{} & \Omega^{4k+2}(X)_{\mathbb{Z}}
\end{array}$$

- $\delta(\check{x} \cup \check{x}) = (\text{odd form})^2 = 0$

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Proof

$$\Omega^{2k}(X)/\Omega^{2k}_{\mathbb{Z}}(X) \xrightarrow{\ i\ } \check{H}^{2k+1}(X) \longrightarrow H^{2k+1}(X;\mathbb{Z})$$

$$\swarrow$$

$$H^{4k+1}(X;\mathbb{R}/\mathbb{Z}) \longrightarrow \check{H}^{4k+2}(X) \xrightarrow[\delta]{} \Omega^{4k+2}(X)_{\mathbb{Z}}$$

- $\delta(\check{x} \cup \check{x}) = (\text{odd form})^2 = 0$
- $i([\eta]) \cup i([\eta]) = i([\eta \wedge d\eta]) = i([\frac{1}{2}d(\eta^2)]) = 0$

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Proof

$$\Omega^{2k}(X)/\Omega^{2k}_{\mathbb{Z}}(X) \xrightarrow{\ i\ } \check{H}^{2k+1}(X) \longrightarrow H^{2k+1}(X;\mathbb{Z})$$

$$H^{4k+1}(X;\mathbb{R}/\mathbb{Z}) \longrightarrow \check{H}^{4k+2}(X) \xrightarrow[\delta]{} \Omega^{4k+2}(X)_{\mathbb{Z}}$$

- $\delta(\check{x} \cup \check{x}) = (\text{odd form})^2 = 0$
- $i([\eta]) \cup i([\eta]) = i([\eta \wedge d\eta]) = i([\frac{1}{2}d(\eta^2)]) = 0$ $\square$

Theorem (case of ordinary cohomology)

The homomorphism $\check{q} : H^{2k+1}(X;\mathbb{Z}) \to H^{4k+1}(X;\mathbb{R}/\mathbb{Z})$ factors through the Steenrod squaring operation Sq^{2k}:

$$\begin{array}{ccc} H^{2k+1}(X;\mathbb{Z}) \xrightarrow{\rho} H^{2k+1}(X;\mathbb{Z}/2) & & \\ \downarrow \mathrm{Sq}^{2k} & & \\ H^{4k+1}(X;\mathbb{Z}/2) \longrightarrow H^{4k+1}(X;\mathbb{R}/\mathbb{Z}) & & \end{array}$$

Proof The theorem follows from a description of Sq^{2k}. □

Cororally

- For any closed 5-dimensional spin manifold X, the map $\check{q} : \check{H}^3(X) \to \check{H}^5(X)$ is trivial: $\check{q} = 0$.
- Thus, $I_{\mathrm{CS}} : \mathcal{B}_{\mathrm{RR}} \times \mathcal{B}_{\mathrm{NS}} \to \mathbb{R}/\mathbb{Z}$ is $SL(2,\mathbb{Z})$-invariant.

$$\mathrm{Sq}^2(c) = w_2(X) \cup c, \quad c \in H^2(X;\mathbb{Z}/2)$$

Main result in the case of K-cohomology

Lemma

The map $\check{q}$ is compatible with the Bott periodicity:

$$\begin{array}{ccc} \check{K}^{2k+1}(X) & \xrightarrow{\check{q}} & \check{K}^{4k+2}(X) \\ \cong\downarrow & & \downarrow\cong \\ \check{K}^{2k+2\ell+1}(X) & \xrightarrow{\check{q}} & \check{K}^{4k+4\ell+2}(X) \end{array}$$

Consider the case that $k = 0$ only.

Lemma

The map $\check{q}$ reduces to $\check{q} : K^1(X) \to K^1(X; \mathbb{R}/\mathbb{Z})$.

The proof is the same as that in the ordinary case.

Proposition

The map $\check{q} : K^1(X) \to K^1(X;\mathbb{R}/\mathbb{Z})$ factors as follows:

$$K^1(X) \xrightarrow{q} K^1(X;\mathbb{Z}/2) \longrightarrow K^1(X;\mathbb{R}/\mathbb{Z}).$$

The map q is a lift of the squaring in K^*, induced from the graded-commutativity of the multiplication in K^*:

$$\begin{array}{ccccccccc} & & K^1(X) & & & & & & \\ & & \downarrow q & \searrow^{\text{square}} & & & & & \\ \cdots & \xrightarrow[\rho]{} & K^1(X;\mathbb{Z}/2) & \xrightarrow[\text{Bock}]{} & K^2(X) & \xrightarrow[2(\cdot)]{} & K^2(X) & \xrightarrow[\rho]{} & \cdots \end{array}$$

Comments

- $\mathrm{Sq}^{2k} \circ \rho$ has the same property as q in K^*:

$$\begin{array}{ccccc}
H^{2k+1}(X;\mathbb{Z}) & & & & \\
\downarrow{\scriptstyle \mathrm{Sq}^{2k}\circ\rho} & \searrow{\scriptstyle \text{square}} & & & \\
H^{4k+1}(X;\mathbb{Z}/2) & \xrightarrow[\mathrm{Bock}]{} & H^{4k+2}(X;\mathbb{Z}) & \xrightarrow[2(\cdot)]{} & H^{4k+2}(X;\mathbb{Z})
\end{array}$$

- It is unclear whether q in K^* factors through ρ:

$$\begin{array}{ccc}
K^1(X) & \xrightarrow{\rho} & K^1(X;\mathbb{Z}/2) \\
 & \searrow{\scriptstyle q} & \downarrow{\scriptstyle ?} \\
 & & K^1(X;\mathbb{Z}/2)
\end{array}$$

(Reasonable multiplications in $K^*(X;\mathbb{Z}/2)$ are not graded-commutative. [Araki-Toda])

Theorem (case of K-cohomology)

The map $q : \tilde{K}^1(X) \to \tilde{K}^1(X;\mathbb{Z}/2)$ is the delooping of the composition of the Adams operation ψ^2 and the mod 2 reduction ρ:

$$\begin{array}{ccccc}
\tilde{K}^0(X) & \xrightarrow{\psi^2} & \tilde{K}^0(X) & \xrightarrow{\rho} & \tilde{K}^0(X;\mathbb{Z}/2) \\
\cong\downarrow & & & & \downarrow\cong \\
\tilde{K}^1(S^1 \wedge X) & & \xrightarrow{q} & & \tilde{K}^1(S^1 \wedge X;\mathbb{Z}/2)
\end{array}$$

Proof Theorem follows from:

$$q(xy) = q(x) \cup \rho(y^2), \qquad (x \in K^1(X), y \in K^0(X))$$
$$\rho(y^2) = \rho(\psi^2(y)), \qquad (y \in K^0(X))$$
$$q(x) = \rho(x). \qquad (x \in K^1(S^1)).$$

Computation of $q : K^1(S^1) \to K^1(S^1; \mathbb{Z}/2)$

1. Realize $1 \in K^1(S^1) = \mathbb{Z}$ by a family of self-adjoint Fredholm operators parametrized by $(I, \partial I)$.
2. Rewrite $q(1)$ through:

$$\begin{aligned}
&K^1(I, \partial I; \mathbb{Z}/2) \\
&\quad \cong K^{-1}(I, \partial I; \mathbb{Z}/2) && \text{(periodicity)} \\
&\quad \cong K^{-1}(I, \partial I)/2K^{-1}(I, \partial I) && (\tilde{K}^0(S^1) = 0) \\
&\quad = K^0(I^2, \partial I^2)/2K^0(I^2, \partial I^2) && \text{(definition)}
\end{aligned}$$

3. Formulations of $K^0(D^2, S^1) = \mathbb{Z}$: ∞-dim $\to$ finite-dim
4. Compute a winding number to conclude $q(1) \neq 0$. □

- **As a simple corollary, we can compute**

$$q : K^1(X) \to K^1(X; \mathbb{Z}/2)$$

in the case of $X = S^{2m+1}$.

$$m = 0 \quad q : \mathbb{Z} \to \mathbb{Z}/2 \text{ is non-trivial.}$$
$$m > 0 \quad q : \mathbb{Z} \to \mathbb{Z}/2 \text{ is trivial.}$$

Problem/Conjecture [Freed]

The map q factors as follows:

$$\begin{array}{ccccccccc}
 & & & & \tilde{K}^1(X) & & & & \\
 & & & \swarrow^{\Sigma^{-1}\psi^2} & \downarrow q & \searrow^{\text{square}}\ (\text{dashed}) & & & \\
\cdots & \xrightarrow[2(\cdot)]{} & \tilde{K}^1(X) & \xrightarrow[\rho]{} & \tilde{K}^1(X;\mathbb{Z}/2) & \xrightarrow[\mathrm{Bock}]{} & \tilde{K}^2(X) & \xrightarrow[2(\cdot)]{} & \cdots,
\end{array}$$

where $\Sigma^{-1}\psi^2$ is a delooping of the Adams square ψ^2 defined by Freed-Hopkins:

$$\begin{array}{ccc}
\tilde{K}^0(X) & \xrightarrow{\psi^2} & \tilde{K}^0(X) \\
\cong \downarrow & & \downarrow \cong \\
\tilde{K}^1(S^1 \wedge X) & \xrightarrow{\Sigma^{-1}\psi^2} & \tilde{K}^1(S^1 \wedge X).
\end{array}$$

Problem/Conjecture [Freed]

The map q factors as follows:

$$
\begin{array}{ccccccccc}
 & & & & \tilde{K}^1(X) & & & & \\
 & & & \swarrow^{\Sigma^{-1}\psi^2} & \downarrow q & \searrow^{\text{square}} (\text{dashed}) & & & \\
\cdots & \xrightarrow[2(\cdot)]{} & \tilde{K}^1(X) & \xrightarrow[\rho]{} & \tilde{K}^1(X;\mathbb{Z}/2) & \xrightarrow[\text{Bock}]{} & \tilde{K}^2(X) & \xrightarrow[2(\cdot)]{} & \cdots,
\end{array}
$$

where $\Sigma^{-1}\psi^2$ is a delooping of the Adams square ψ^2 defined by Freed-Hopkins:

$$
\begin{array}{ccc}
\tilde{K}^0(X) & \xrightarrow{\psi^2} & \tilde{K}^0(X) \\
\cong \downarrow & & \downarrow \cong \\
\tilde{K}^1(S^1 \wedge X) & \xrightarrow{\Sigma^{-1}\psi^2} & \tilde{K}^1(S^1 \wedge X).
\end{array}
$$

<u>Remark</u> $x^2 = 0$ for $\forall x \in K^1(X)$. ($K^*(U_\infty)$ is torsion free.)