Note on a construction of TQFT from cohomology

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Abstract

This is a note about a simple construction of even dimensional topological quantum field theories, based on cohomology with its coefficients in a finite field, the cup product, fundamental classes and Poincare-Lefschetz duality. A variation of the construction is also given, which only uses the axioms of cohomology theory and produces TQFT's of any dimension.

1 Introduction

A d-dimensional topological quantum field theory (TQFT) in the sense of Atiyah is a functor from a coboridism category of manifolds to the category of vector spaces. More precisely, it assigns:

- a finite-rank vector space H_X over $\mathbb{C}$ to a compact oriented $(d-1)$-dimensional manifold X without boundary;
- a vector $Z_W \in H_{\partial W}$ to a compact oriented d-dimensional manifold W with its boundary ∂W,

satisfying the following axioms:

- (Functorial) Any orientation preserving diffeomorphism $X_1 \to X_2$ of $(d-1)$-dimensional manifolds induces an isomorphism $H_{X_1} \to H_{X_2}$. Moreover, the isomorphism $H_{\partial W_1} \to H_{\partial W_2}$ induced from any orientation preserving diffeomorphism of d-dimensional manifolds $W_1 \to W_2$ carries Z_{W_1} to Z_{W_2}.
- (Involutory) For any $(d-1)$-dimensional manifold X, there is a natural isomorphism $H_X^* \cong H_{X^*}$, where X^* is the $(d-1)$-dimensional manifold whose orientation is opposite to that on X.
- (Multiplicative) There is a natural isomorphism $H_{X_1 \sqcup X_2} \cong H_{X_1} \otimes H_{X_2}$ for any compact oriented $(d-1)$-dimensional manifolds X_1 and X_2. Moreover, for any compact oriented d-dimensional manifolds W_1 and W_2 whose boundaries are $\partial W_1 = X_1 \sqcup X$ and $W_2 = X^* \sqcup X_2$, let $W_1 \cup W_2$ denote the compact oriented manifolds obtained by gluing W_1 and W_2 along X. Then the natural pairing $\mathrm{Tr}_X : H_X \otimes H_X^* \to \mathbb{C}$ carries $Z_{W_1 \sqcup W_2} \in H_{\partial(W_1 \sqcup W_2)}$ to $Z_{W_1 \cup W_2} \in H_{\partial(W_1 \cup W_2)}$.

- (Non-trivial) $H_\emptyset = \mathbb{C}$ and $Z_{X\times[0,1]} = \mathrm{id}$.

In this note, we construct a $2n$-dimensional TQFT $\hat{\mathcal{Z}}_n^{2n}$ based on ordinary cohomology groups of degree n with coefficients in any finite field F. A similar construction gives a $2n$-dimesional TQFT $\check{\mathcal{Z}}_n^{2n}$. The constructions of these TQFT's use the cup product, the fundamental classes of manifolds, and the Poincare-Lefschetz duality. We also construct d-dimensional TQFT's $\mathcal{Z}_p^d$ and $\mathcal{Z}_{\le p}^d$ from certain generalized cohomology theory, and explore relations among these TQFT's.

The drawback of these TQFT's is that they only capture information of Betti numbers. But, the simpleness of the construction may be of the advantage.

As a convention of this note, for a finite group A, we will write $|A|$ for the number of elements in A.

2 Preliminary

2.1 Some facts about (skew-)symmetric form

Let V be a finite-dimensional vector space over a field R. A bilinear form $I : V \times V \to R$ is said to be *symmetric* if $I(x,y) = I(y,x)$ for all $x, y \in V$. Also, I is said to be *skew-symmetric* if $I(x,y) = -I(y,x)$ for all $x, y \in V$ instead. A (skew-)symmetric form I is called *non-degenerate* if any $x \in V\backslash\{0\}$ admits $y \in V$ such that $I(x,y) \neq 0$. The non-degeneracy of I is equivalent to that the homomorphism $I^\sharp : V \to \mathrm{Hom}(V,R)$ given by $I^\sharp(x)(y) = I(x,y)$ is injective. If this is the case, then the finite-dimensionality of V implies that $I^\sharp$ is an isomorphism.

Lemma 2.1. *Let V be a finite-dimensional vector space over a field R, and $I : V \times V \to R$ a non-degnerate (skew-)symmetric bilinear form. For any subspace $W \subset V$, let $W^\perp$ denote the complement of W in V with respect to I:*

$$W^\perp = \{x \in V |\ I(x,y) = 0 \text{ for all } y \in W\}.$$

Then the following holds:

(a) There is a natural isomorphism $W^\perp \cong \mathrm{Hom}(V/W,R)$.

(b) $(W^\perp)^\perp = W$.

Proof. Since R is a field, the inclusion $W \subset V$ induces the exact sequence:

$$0 \to \mathrm{Hom}(V/W,R) \to \mathrm{Hom}(V,R) \xrightarrow{j_W} \mathrm{Hom}(W,R) \to 0.$$

Since $I^\sharp$ is an isomorphism, we see

$$W^\perp = \mathrm{Ker}(j_W I^\sharp) \cong \mathrm{Ker}(j_W) \cong \mathrm{Hom}(V/W,R),$$

which shows (a). For (b), we use (a) to get the formulae:

$$\dim W + \dim W^{\perp} = \dim V,$$
$$\dim W^{\perp} + \dim (W^{\perp})^{\perp} = \dim V.$$

Hence we have $\dim W = \dim (W^{\perp})^{\perp}$. But, by the (skew-)symmetry of I, we have $W \subset (W^{\perp})^{\perp}$. Thus, by the dimensional reason, we see $W = (W^{\perp})^{\perp}$. $\square$

A non-degenerate skew-symmetric bilinear form $I : V \times V \to R$ is called *symplectic* if $I(x, x) = 0$ for all $x \in V$.

Lemma 2.2. *If I is a symplectic, then* $\mathrm{rank}_R V = 0 \mod 2$.

Proof. This is standard: a proof constructs a symplectic basis inductively. $\square$

2.2 The intersection pairing

Definition 2.3. Let R be a principal ideal domain, and W a compact d-dimensional manifold with boundary ∂W which is oriented over R (i.e. it has fundamental class in the cohomology with its coefficients in R). We define an R-module $'H^q = {}'H^q(W; R)$ to be the kernel of the restriction $r : H^q(W; R) \to H^q(\partial W; R)$:

$$'H^q(W; R) = \mathrm{Ker}\{r : H^q(W; R) \to H^q(\partial W; R)\}.$$

Lemma 2.4. *For p, q such that $p + q = d$, there exists a bilinear form*

$$I : \ 'H^p(W; R) \times {}'H^q(W; R) \longrightarrow R.$$

Proof. Recall the exact sequence for the pair $(W, \partial W)$:

$$H^{q-1}(\partial W; R) \xrightarrow{\delta} H^q(W, \partial W; R) \xrightarrow{j} H^q(W; R) \xrightarrow{r} H^q(\partial W; R).$$

Now, suppose that $x \in {}'H^p$ and $y \in {}'H^q$ are given. Since $r(y) = 0$ by definition, there is $\tilde{y} \in H^q(W, \partial W; R)$ such that $j(\tilde{y}) = y$. We then define

$$I(x, y) = \langle x \cup \tilde{y}, [W] \rangle,$$

where $x \cup \tilde{y} \in H^d(W, \partial W; R)$ is the cup product, and $[W] \in H_d(W, \partial W; R)$ is the fundamental class of W. If $\tilde{y}' \in H^q(W, \partial W; R)$ is another choice such that $j(\tilde{y}') = y$, then there is $z \in H^{q-1}(\partial W; R)$ such that $\delta(z) = \tilde{y}' - \tilde{y}$. Now, we get

$$\langle x \cup (\tilde{y}' - \tilde{y}), [W] \rangle = \langle x \cup \delta(z), [W] \rangle = \langle r(x) \cup z, [\partial W] \rangle = 0,$$

so that $I(x, y)$ is well-defined. $\square$

Lemma 2.5. *Let p, q be such that $p + q = d$. Then we have*

$$I(x, y) = (-1)^{pq} I(y, x)$$

for any $x \in {}'H^p$ and $y \in {}'H^q$.

Proof. We choose $\tilde{x} \in H^p(W, \partial W; R)$ and $\tilde{y} \in H^q(W, \partial W; R)$ such that $j(\tilde{x}) = x$ and $j(\tilde{y}) = y$. Then the following holds in $H^d(W, \partial W; R)$:

$$x \cup \tilde{y} = \tilde{x} \cup \tilde{y} = (-1)^{pq} \tilde{y} \cup \tilde{x} = (-1)^{pq} y \cup \tilde{x},$$

which leads to the lemma. □

Proposition 2.6. *Let p, q be such that $p + q = d$. We define a homomorphism*

$$I^\sharp : \, 'H^p(W; R) \longrightarrow \mathrm{Hom}_R('H^q(W; R), R)$$

by $I^\sharp(x)(y) = I(x, y)$. The the following holds:

(a) The kernel of $I^\sharp$ is the torsion submodule of $'H^p(W; R)$.

(b) $I^\sharp$ is surjective if and only if:

$$'H^p(W; R) = r^{-1}(T(H^p(\partial W; R))),$$

where we write $r : H^p(W; R) \to H^p(\partial W; R)$ for the restriction, and $T(H^p(\partial W; R)) \subset H^p(\partial W; R)$ for the torsion submodule.

Proof. We have the following commutative diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & T('H^p) & \longrightarrow & 'H^p(W; R) & \xrightarrow{I^\sharp} & \mathrm{Hom}('H^q(W; R), R) & & \\
 & & \downarrow & & \downarrow & & \downarrow{\scriptstyle j^*} & & \\
0 & \longrightarrow & T(H^p) & \longrightarrow & H^p(W; R) & \xrightarrow{\tilde{I}^\sharp} & \mathrm{Hom}(H^q(W, \partial W; R), R) & \longrightarrow & 0 \\
 & & \| & & \| & & \| & & \\
0 & \longrightarrow & T(H^p) & \longrightarrow & H^p(W; R) & \longrightarrow & \mathrm{Hom}(H_p(W; R), R) & \longrightarrow & 0,
\end{array}$$

where $T(H^p) \cong \mathrm{Ext}(H_{q-1}(W, \partial W; R), R)$ is the torsion part in $H^p(W; R)$, and $T('H^p) = T(H^p) \cap {}'H^p(W; R)$ that in $'H^p(W; R)$. The homomorphism $\tilde{I}^\sharp$ is defined by $\tilde{I}^\sharp(x)(y) = \langle x \cup y, [W] \rangle$ for $x \in H^q(W; R)$ and $y \in H^p(W, \partial W; R)$. The isomorphism $H^q(W, \partial W; R) \cong H_p(W; R)$ is the Lefschetz duality, and the universal coefficient theorem implies that sequence in the lowest row is exact. Since $j : H^q(W, \partial W; R) \to {}'H^q(W; R)$ is surjective by definition, the induced homomorphism j^* is injective. Hence we see the kernel of $I^\sharp$ is exactly $T('H^p)$ and (a) is shown. For (b), let $h_x : H^q(W, \partial W; R) \to R$ denote the homomorphism determined by an element $x \in H^p(W; R)$, that is, $h_x(y) = \langle x \cup y, [W] \rangle$ for all $y \in H^q(W, \partial W; R)$. To get the necessary and sufficient condition for h_x belongs to the image of j^*, notice the identification:

$$'H^q(W; R) \cong H^q(W, \partial W; R)/\mathrm{Ker}(j) = H^q(W, \partial W; R)/\mathrm{Im}(\delta),$$

where $\delta : H^{q-1}(\partial W; R) \to H^q(W, \partial W; R)$ is the connecting homomorphism. For any $z \in H^{q-1}(\partial W; R)$, we have the formula

$$h_x(\delta(z)) = \langle x \cup \delta(z), [W] \rangle = \langle r(x) \cup z, [\partial W] \rangle = f_{r(x)}(z \cap [\partial W]),$$

where f is the homomorphism in the universal coefficient theorem:

$$0 \to \mathrm{Ext}(H_{p-1}(\partial W, R)) \to H^p(\partial W; R) \xrightarrow{f} \mathrm{Hom}(H_p(\partial W), R) \to 0,$$

and $z \cap [\partial W] \in H_{p-1}(\partial W)$ is the image of the cap product:

$$\cap : \ H^{q-1}(\partial W; R) \times H_{d-1}(\partial W; R) \longrightarrow H_p(\partial W; R).$$

By the Poincaré duality, $\cap[\partial W] : H^{q-1}(\partial W; R) \to H_p(\partial W; R)$ is an isomorphism. Therefore the condition for h_x to be in $\mathrm{Im}(j^*)$ is that $r(x) \in H^p(\partial W; R)$ is a torsion, which implies (b). $\square$

Corollary 2.7. *If R is a field, then $I^\sharp$ is an isomorphism.*

Corollary 2.8. *Suppose that $d = 4k+2$ and $p = q = 2k+1$ for some k. Suppose also that: R is a field in which 2 is invertible; or $R = \mathbb{Z}$. Then we have*

$$\mathrm{rank}_R {}'H^{2k+1}(W; R) = 0 \mod 2.$$

Proof. In the case that R is a field in which 2 is invertible, we have $I(x, x) = 0$ for all $x \in {}'H^{2k+1}(W; R)$. Thus, with the fact that $I^\sharp$ is an isomorphism, the bilinear form I is a symplectic form. This implies that the rank of ${}'H^{2k+1}(W; R)$ is divisible by 2. Then the case of $R = \mathbb{Z}$ follows from the fact that the rank of $H^{2k+1}(W; \mathbb{Z})$ agrees with that of $H^{2k+1}(W; \mathbb{Z}) \otimes \mathbb{R} = H^{2k+1}(W; \mathbb{R})$. $\square$

We slightly generalize the construction above: For a pair of integers p and q, and a pair of manifolds (X, Y) such that $Y \subset X$, we will write

$$H^{(p,q)}(X, Y; R) = H^p(X, Y; R) \oplus H^q(X, Y; R)$$

for the direct sum of the cohomology groups of degree p and q. We will also write $T^{(p,q)}(X, Y)$ for the torsion submodule in $H^{(p,q)}(X, Y; R)$. It is then easy to derive the following as a corollary to Proposition 2.6:

Corollary 2.9. *For p and q such that $p + q = d$, we put*

$${}'H^{(p,q)}(W; R) = \mathrm{Ker}\{r : H^{(p,q)}(W; R) \to H^{(p,q)}(\partial W; R)\},$$

where r is the restriction. On ${}'H^{(p,q)}(W; R)$, we define a bilinear form

$$I : \ {}'H^{(p,q)}(W; R) \times {}'H^{(p,q)}(W; R) \longrightarrow R$$

by $I((a, b), (a', b')) = \langle a \cup \tilde{b}' + b \cup \tilde{a}', [W] \rangle$, where $(\tilde{a}', \tilde{b}') \in H^{(p,q)}(W, \partial W; R)$ is such that $j(\tilde{a}', \tilde{b}') = (a', b')$. Then the following holds:

1. *I is well-defined.*
2. *$I(x,y) = (-1)^{pq} I(y,x)$ for all $x, y \in {}'H^{(p,q)}(W;R)$.*
3. *If p and q are odd, then $I(x,x) = 0$ for all $x \in {}'H^{(p,q)}(W;R)$.*
4. *Let $I^\sharp : {}'H^{(p,q)}(W;R) \to \mathrm{Hom}({}'H^{(p,q)}(W;R), R)$ be the homomorphism defined by $I^\sharp(x)(y) = I(x,y)$. Then we have*

$$\mathrm{Ker} I^\sharp = {}'H^{(p,q)}(W;R) \cap T^{(p,q)}(W).$$

5. *$I^\sharp$ is surjective if and only if ${}'H^{(p,q)}(W;R) = r^{-1}(T^{(p-1,q-1)}(\partial W))$.*

Corollary 2.10. *Suppose that $d = 2n$ for some n and that p and q are odd numbers such that $p + q = d$. Suppose also that R is a field or $\mathbb{Z}$. Then,*

$$\mathrm{rank}_R {}'H^{(p,q)}(W;R) = 0 \mod 2.$$

Remark 1. In the case that R is a field in which 2 is not invertible, a compact oriented $(4k+1)$-dimensional manifold W may admits a non-trivial element $x \in {}'H^{2k+1}(W;R)$ such that $x^2 \neq 0$. In particular, in the case of $R = \mathbb{Z}/2$, there exist such elements if and only if the $(2k+1)$th Wu class $\nu_{2k+1}(W) \in H^{2k+1}(W;\mathbb{Z}/2)$ is non-trivial.

3 TQFT constructed from cohomology

3.1 Construction

Definition 3.1. Let F be a finite field, and n a positive integer.

(a) We assign to a compact oriented $(2n-1)$-dimensional manifold X the vector space $\hat{H}^{2n}_n(X)$ over $\mathbb{C}$ generated by elements in $H^n(X;F)$:

$$\hat{H}^{2n}_n(X) = \bigoplus_{c \in H^n(X;F)} \mathbb{C}|c\rangle.$$

We also define a Hermitian metric $\langle\ |\ \rangle : H_X \times H_X \to \mathbb{C}$ by $\langle \alpha c_1 | \beta c_2 \rangle = \bar{\alpha}\beta\delta_{c_1,c_2}$ for $\alpha, \beta \in \mathbb{C}$ and $c_1, c_2 \in H^n(X;F)$, where $\delta_{c_1,c_2} = 1$ if $c_1 = c_2$ while $\delta_{c_1,c_2} = 0$ otherwise.

(b) Let W be a compact oriented $2n$-dimensional manifold W. In the case of $\partial W \neq \emptyset$, we assign to W the vector $\hat{Z}^{2n}_n(W) \in \hat{H}^{2n}_n(\partial W)$ defined by

$$\hat{Z}^{2n}_n(W) = \sqrt{|\mathrm{Ker}(r)|} \sum_{c \in \mathrm{Im}(r)} |c\rangle,$$

where $r : H^n(W;F) \to H^n(\partial W;F)$ is induced by the restriction. In the case of $\partial W = \emptyset$, we assign to W the number $\hat{Z}^{2n}_n(W) \in \mathbb{C}$ defined by

$$\hat{Z}^{2n}_n(W) = \sqrt{|H^n(W;F)|}.$$

Theorem 3.2. *The assignments $X \mapsto \hat{H}_n^{2n}(X)$ and $W \mapsto \hat{Z}_n^{2n}(W)$ in Definition 3.1 give rise to a $2n$-dimensional topological quantum field theory $\hat{\mathcal{Z}}_n^{2n}$.*

The remainder of this subsection is devoted to the proof of this theorem. To suppress notations, we write $\hat{H}_n^{2n}(X) = H_X$ and $\hat{Z}_n^{2n}(W) = Z_W$ simply.

In the axioms of topological quantum field theory, the functoriality axiom is clear. For the involutority (orientation) axiom, we use the Hermitian metric $\langle \mid \rangle$ to construct an isomorphism $H_{X^*} \cong H_X^*$. (Since $Z_{W^*} = Z_W$ in $H_{X^*} = H_X$), the isomorphism carries $Z_W \in H_{\partial W}$ to $Z_{W^*} \in H_{\partial W^*}$. Thus, the TQFT is in particular "unitary".) For the non-triviality axiom, we adapt the convention $H_\emptyset = \mathbb{C}$. It is easy to check that $Z_{X\times[0,1]}$ gives rise to the identity on H_X.

Finally, we prove the multiplicativity axiom. It is clear that $H_{X\sqcup X'} \cong H_X \otimes H_{X'}$. Now, let W_1 and W_2 be compact oriented $2n$-dimensional manifolds whose boundaries are $\partial W_1 = X_1 \sqcup X$ and $W_2 = X \sqcup X_2$. We assume that the induced orientations on $X \subset \partial W_1$ and $X \subset \partial W_2$ are opposite to each other, so that we glue W_1 and W_2 along X to get a compact oriented $2n$-dimensional manifold $W_1 \cup W_2$ whose boundary is $\partial(W_1 \cup W_2) = X_1 \sqcup X_2$. We write R_1 and R_2 for the homomorphisms induced by restriction:

$$R_1: \ H^n(W_1) \to H^n(X_1) \oplus H^n(X), \quad R_2: \ H^nW_2) \to H^n(X) \oplus H^n(X_2).$$

In the above, we omit the coefficients from the notations of cohomology groups. We also write R and R' for the following restrictions:

$$\begin{aligned} R: \ & H^n(W_1 \cup W_2) \to H^n(X_1) \oplus H^n(X) \oplus H^n(X_2), \\ R': \ & H^n(W_1 \cup W_2) \to H^n(X_1) \oplus H^n(X_2), \end{aligned}$$

By definition, the vectors assigned to W_1 and W_2 are expressed as:

$$Z_{W_1} = \sqrt{|\mathrm{Ker} R_1|} \sum_{(a_1,b_1)\in \mathrm{Im} R_1} a_1 \otimes b_1, \quad Z_{W_2} = \sqrt{|\mathrm{Ker} R_2|} \sum_{(b_2,a_2)\in \mathrm{Im} R_1} b_2 \otimes a_2,$$

where $a_i \in H^n(X_i)$ and $b_i \in H^n(X)$ for $i = 1, 2$. Now, by the contraction

$$\mathrm{Tr}_X: \ H_{X_1} \otimes H_X \otimes H_X \otimes H_{X_2} \longrightarrow H_{X_1} \otimes H_{X_2},$$

we evaluate $Z_{W_1 \sqcup W_2} = Z_{W_1} \otimes Z_{W_2}$ to get:

$$\begin{aligned} \mathrm{Tr}_X(Z_{W_1\sqcup W_2}) &= \sqrt{|\mathrm{Ker}(R_1 \oplus R_2)|} \sum_{\substack{(a_1,b_1)\in \mathrm{Im} R_1 \\ (b_2,a_2)\in \mathrm{Im} R_2}} \delta_{b_1,b_2} a_1 \otimes a_2 \\ &= \sqrt{|\mathrm{Ker}(R_1 \oplus R_2)|} \sum_{\substack{(a_1,b_1,b_2,a_2)\in \mathrm{Im}(R_1\oplus R_2) \\ b_1=b_2}} a_1 \otimes a_2. \end{aligned}$$

To analyze the summation in the above, we consider the following commutative diagram involving a part of the Mayer-Vietoris exact sequence:

$$\begin{tikzcd}
H^n(W_1 \cup W_2) \arrow[r, "R"] \arrow[d, "f"'] & H^n(X_1) \oplus H^n(X) \oplus H^n(X_2) \arrow[r, "\pi"] \arrow[d, "\Delta"] & H^n(X_1) \oplus H^n(X_2) \arrow[d, equal] \\
H^n(W_1) \oplus H^n(W_2) \arrow[r, "R_1 \oplus R_2"] \arrow[d, "\rho_1 - \rho_2"'] & H^n(X_1) \oplus H^n(X) \oplus H^n(X) \oplus H^n(X_2) \arrow[r, "\tilde{\pi}"] \arrow[d] & H^n(X_1) \oplus H^n(X_2) \\
H^n(X) \arrow[r, equal] & \mathrm{Coker}\Delta, &
\end{tikzcd}$$

where f, ρ_1, ρ_2 are the restrictions, Δ the diagonal map, and π and $\tilde{\pi}$ the projections. The exactness of the first column shows

$$\{(a_1, b_1, b_2, a_2) \in \mathrm{Im}(R_1 \oplus R_2) |\ b_1 = b_2\} = \mathrm{Im}(\Delta R).$$

For any $(a_1, a_2) \in \mathrm{Im}(\pi \Delta R)$ given, we have

$$\begin{aligned}
&|\{(a_1', b', b', a_2') \in \mathrm{Im}(\Delta R) |\ a_1' = a_1, a_2' = a_2\}| \\
&\quad = |\{(a_1'', b'', b'', a_2'') \in \mathrm{Im}(\Delta R) |\ a_1'' = 0, a_2'' = 0\}| = |\mathrm{Im}(\Delta R) \cap \mathrm{Ker}(\tilde{\pi})|.
\end{aligned}$$

Since $\tilde{\pi}\Delta = \pi$, the injection Δ induces the isomorphism:

$$\mathrm{Im}(R) \cap \mathrm{Ker}(\pi) \cong \mathrm{Im}(\Delta R) \cap \mathrm{Ker}(\tilde{\pi}).$$

Further, the homomorphism R induces the isomorphism

$$\mathrm{Ker}(R')/\mathrm{Ker}(R) = \mathrm{Ker}(\pi R)/\mathrm{Ker}(R) \cong \mathrm{Im}(R) \cap \mathrm{Ker}(\pi).$$

Thus, in view of $\tilde{\pi}\Delta R = \pi \Delta = R'$, we arrive at:

$$\mathrm{Tr}_X(Z_{W_1 \sqcup W_2}) = \sqrt{|\mathrm{Ker}(R_1 \oplus R_2)|} |\mathrm{Ker}(R')/\mathrm{Ker}(R)| \sum_{(a_1, a_2) \in \mathrm{Im}(R')} a_1 \otimes a_2.$$

To rewrite the formula above, we use

Lemma 3.3. *There is the following exact sequence:*

$$0 \longrightarrow \mathrm{Ker} f \longrightarrow \mathrm{Ker} R \xrightarrow{\ f\ } \mathrm{Ker}(R_1 \oplus R_2) \longrightarrow 0.$$

Proof. The exact sequence follows from the following commutative diagram:

$$\begin{tikzcd}
 & H^{n-1}(X) \arrow[d] & & 0 \arrow[d] \\
H^n(W_1 \cup W_2, X_1 \sqcup X \sqcup X_2) \arrow[r] \arrow[d, equal] & H^n(W_1 \cup W_2) \arrow[r, "R"] \arrow[d, "f"] & & H^n(X_1) \oplus H^n(X) \oplus H^n(X_2) \arrow[d] \\
H^n(W_1, X_1 \sqcup X) \oplus H^n(W_2, \sqcup X \sqcup X_2) \arrow[r] & H^n(W_1) \oplus H^n(W_2) \arrow[r, "R_1 \oplus R_2"] \arrow[d, "\rho_1 - \rho_2"] & & H^n(X_1) \oplus H^n(X) \oplus H^n(X) \oplus H^n(X_2) \arrow[d] \\
 & H^n(X) \arrow[rr, equal] & & H^n(X) \arrow[d] \\
 & & & 0.
\end{tikzcd}$$

In this diagram, the first and second columns are the Mayer-Vietoris exact sequences. Also, the second row is the exact sequence for the pair $(W_1 \cup W_2, X_1 \sqcup X \sqcup X_2)$, and the third row the direct sum of the exact sequences for the pairs $(W_1, X_1 \sqcup X)$ and $(W_2, X \sqcup X_2)$. $\square$

As a consequence of the lemma, we get

$$\sqrt{|\mathrm{Ker}(R_1 \oplus R_2)|\mathrm{Ker}R'/\mathrm{Ker}R|} = \sqrt{|\mathrm{Ker}R'|}\sqrt{\frac{|\mathrm{Ker}R'/\mathrm{Ker}R|}{|\mathrm{Ker}f|}}.$$

Lemma 3.4. $|\mathrm{Ker}R'/\mathrm{Ker}R| = |\mathrm{Ker}f|$.

Proof. Since W is oriented and F is a field, W is also oriented over F. Thus, by Proposition 2.6, $\mathrm{Ker}R' = {}'H^n(W_1 \cup W_2)$ is endowed with the non-degenerate (skew-)symmetric form I. Recall $\mathrm{Ker}f \subset \mathrm{Ker}R'$. Then, by Lemma 2.1, we get

$$\mathrm{rank}\mathrm{Ker}f = \mathrm{rank}\mathrm{Hom}(\mathrm{Ker}R'/\mathrm{Ker}f^{\perp}, F) = \mathrm{rank}(\mathrm{Ker}R'/\mathrm{Ker}f^{\perp}).$$

We now claim $\mathrm{Ker}f^{\perp} = \mathrm{Ker}R$. To see this, let $\delta : H^{n-1}(X) \to H^n(W_1 \cup W_2)$ be the connecting homomorphism in the Mayer-Vietoris sequence, so that $\mathrm{Ker}f = \mathrm{Im}\delta$. For any $x \in H^{n-1}(X)$ and $y \in \mathrm{Ker}R'$, we have

$$I(\delta x, y) = \langle \delta x \cup \tilde{y}, [W_1 \cup W_2]\rangle = \langle x \cup \tilde{y}|_X, [X]\rangle = \langle x \cup \rho(y), [X]\rangle,$$

where $\tilde{y} \in H^n(W_1 \cup W_2, X_1 \cup X_2)$ is such that $j(\tilde{y}) = y$, and $\rho : H^n(W_1 \cup W_2) \to H^n(X)$ is the restriction. Thus, $y \in \mathrm{Ker}R'$ belongs to $\mathrm{Ker}f^{\perp} = \mathrm{Im}\delta^{\perp}$ if and only if $x \in \mathrm{Ker}R = \mathrm{Ker}R' \cap \mathrm{Ker}\rho$. $\square$

Lemma 3.4 above completes the proof of the multiplicativity axiom:

$$\mathrm{Tr}_X(Z_{W_1 \sqcup W_2}) = \sqrt{|\mathrm{Ker}R'|}\sum_{(a_1,a_2)\in \mathrm{Im}(R')} a_1 \otimes a_2 = Z_{W_1 \cup W_2}.$$

Remark 2. In the case of $n = 1$, we can relax the condition on the coefficients of the cohomology: Instead of a fintie field F, we can allow a principal ideal domain R with $|R| < +\infty$. This is because the ordinary cohomology with coefficients in R of compact oriented manifolds of dimesion less than or equal to 2 are free as R-modules. Thus, for a compact oriented 2-dimensional manifold X, the intersection pairing I on ${}'H^1(X; R)$ is non-degenerate, and hence we can apply the argument in the proof of Theorem 3.2 to this case.

3.2 Application

Proposition 3.5. *Let W_1 and W_2 be a compact oriented $(4k+2)$-dimensional manifolds, where k is a non-negative integer. Assume that a compact oriented $(4k+1)$-dimensional manifold X is a component of the boundary of each W_1 and W_2 with opposite induced orientations, so that we glue W_1 and W_2 together along X to get a compact oriented $(4k+2)$-dimensional manifold $W_1 \cup W_2$. If $\nu_{2k+1}(W_1)$ and $\nu_{2k+1}(W_2)$ are trivial, then so is $\nu_{2k+1}(W_1 \cup W_2)$.*

Proof. We can prove this claim directly by using the Mayer-Vietoris sequence. But, we appeal to our TQFT with $F = \mathbb{Z}/2$: In general, if $\nu_{2k+1}(W)$ is trivial, then the intersection pairing on ${}'H^{2k+1}(W; \mathbb{Z}/2)$ is symplectic, so that the coefficients $\sqrt{|\mathrm{Ker}(r)|} = |F|^{\mathrm{rank}' H^{2k+1}(W;F)/2}$ of $Z_W \in H_{\partial W}$ are integers. Thus, by the assumption, the coefficients of Z_{W_1} and Z_{W_2} are integer. Clearly, the coefficients of $Z_{W_1} \otimes Z_{W_2} = Z_{W_1 \sqcup W_2}$ are integers. Since Tr_X preserves the lattice of vectors with integer coefficients, the coefficients of $\mathrm{Tr}_X(Z_{W_1 \sqcup W_2}) = Z_{W_1 \cup W_2}$ are also integers, so that $\nu_{2k+1}(W_1 \cup W_2)$ is trivial. □

3.3 'Dual' or 'complementary' construction

We here construct a TQFT $\check{\mathcal{Z}}^{2n}_n$ which is dual to $\hat{\mathcal{Z}}^{2n}_n$ in a sense.

Definition 3.6. Let F be a finite field and n a positive integer.

(a) We assign to a compact oriented $(2n-1)$-dimensional manifold X the vector space $\check{H}^{2n}_n(X)$ over $\mathbb{C}$ generated by elements in $H^{n-1}(X; F)$:

$$\check{H}^{2n}_n(X) = \bigoplus_{x \in H^{n-1}(X;F)} \mathbb{C}|x\rangle.$$

We also define a Hermitian metric $\langle\ |\ \rangle$ on $\check{H}^{2n}_n(X)$ by $\langle x|x'\rangle = \delta_{x,x'}$.

(b) Let W be a compact oriented $2n$-dimensional manifold W. In the case of $\partial W \neq \emptyset$, we assign to W the vector $\check{Z}^{2n}_n(W) \in \check{H}^{2n}_n(\partial W)$ defined by

$$\check{Z}^{2n}_n(W) = \frac{\sqrt{|H^{n-1}(\partial W; F)||\mathrm{Ker}(r^n)|}}{|\mathrm{Im}(r^{n-1})|} \sum_{x \in \mathrm{Im}(r^{n-1})} |x\rangle,$$

where $r^i : H^i(W; F) \to H^i(\partial W; F)$, $(i = n-1, n)$ are the restrictions. In the case of $\partial W = \emptyset$, we assign to W the number $\check{Z}^{2n}_n(W) \in \mathbb{C}$ defined by

$$\check{Z}^{2n}_n(W) = \sqrt{|H^n(W; F)|}.$$

The coefficient of $\check{Z}^{2n}_n(W)$ has the following equivalent expressions:

$$\frac{\sqrt{|H^{n-1}(\partial W; F)||\mathrm{Ker}(r^n)|}}{|\mathrm{Im}(r^{n-1})|} = \sqrt{\frac{|H^n(W, \partial W; F)|}{|\mathrm{Im}(r^{n-1})|}} = \sqrt{\frac{|H^n(W; F)|}{|\mathrm{Im}(r^{n-1})|}}.$$

Theorem 3.7. *The assignments $X \mapsto \check{H}^{2n}_n(X)$ and $W \mapsto \check{Z}^{2n}_n(W)$ in Definition 3.6 give rise to a $2n$-dimensional topological quantum field theory $\check{\mathcal{Z}}^{2n}_n$.*

Proof. The proof is essentially the same as that of Theorem 3.2: We use the relations among some cohomology groups derived from the properties of the cup product. □

Notice that in the case of $n = 1$ the TQFT $\check{\mathcal{Z}}^{2n}_n$ is also defined by using a finite PID R instead of a finite field F.

Remark 3. As will be seen later, $\hat{\mathcal{Z}}^2_1$ and $\check{\mathcal{Z}}^2_1$ are (non-canonically) equivalent. A conjecture is that $\hat{\mathcal{Z}}^{2n}_n$ and $\check{\mathcal{Z}}^{2n}_n$ is equivalent generally.

3.4 Generalization

Definition 3.8. Let F be a finite field, n a positive integer, and p and q are numbers such that $2n = p + q$.

(a) We assign to a compact oriented $(2n-1)$-dimensional manifold X the vector space $\hat{H}^{2n}_{p,q}(X)$ over $\mathbb{C}$ generated by elements in $H^{(p,q)}(X;F)$:

$$\hat{H}^{2n}_{p,q}(X) = \bigoplus_{c \in H^{(p,q)}(X;F)} \mathbb{C}|c\rangle.$$

We also define a Hermitian metric $\langle\ |\ \rangle : \hat{H}^{2n}_{p,q}(X) \times \hat{H}^{2n}_{p,q}(X) \to \mathbb{C}$ by $\langle \alpha c_1 | \beta c_2 \rangle = \bar{\alpha}\beta\delta_{c_1,c_2}$ for $\alpha, \beta \in \mathbb{C}$ and $c_1, c_2 \in H^{(p,q)}(X;F)$.

(b) Let W be a compact oriented $2n$-dimensional manifold W. In the case of $\partial W \neq \emptyset$, we assign to W the vector $\hat{Z}^{2n}_{p,q}(W) \in \hat{H}^{2n}_{p,q}(\partial W)$ defined by

$$\hat{Z}^{2n}_{p,q}(W) = \sqrt{|\mathrm{Ker}(r)|} \sum_{c \in \mathrm{Im}(r)} |c\rangle,$$

where $r : H^{(p,q)}(W;F) \to H^{(p,q)}(\partial W;F)$ is induced by the restriction. In the case of $\partial W = \emptyset$, we assign to W the number $\check{Z}^{2n}_{p,q}(W) \in \mathbb{C}$ defined by

$$\hat{Z}^{2n}_{p,q}(W) = \sqrt{|H^{(p,q)}(W;F)|}.$$

Theorem 3.9. *The assignments $X \mapsto \hat{H}^{2n}_{p,q}(X)$ and $W \mapsto \hat{Z}^{2n}_{p,q}(W)$ in Definition 3.8 give rise to a $2n$-dimensional topological quantum field theory $\hat{\mathcal{Z}}^{2n}_{p,q}$.*

Proof. The argument proving Theorem 3.2 is adapted without any change. □

In a similar way, we incorporate the other combinations of degree to define various TQFT's. For example, we use all even degree and all odd degree to construct $\hat{\mathcal{Z}}^{2n}_{\mathrm{even}}$ and $\hat{\mathcal{Z}}^{2n}_{\mathrm{odd}}$. With a slight generalization, we can construct a TQFT involving all degree $\hat{\mathcal{Z}}^{2n}_{\mathrm{all}}$ as well. There also exist the dual versions such as $\check{\mathcal{Z}}^{2n}_{p,q}$, $\check{\mathcal{Z}}^{2n}_{\mathrm{even}}$, $\check{\mathcal{Z}}^{2n}_{\mathrm{odd}}$, $\check{\mathcal{Z}}^{2n}_{\mathrm{all}}$, etc. For instance, $\check{\mathcal{Z}}^{2n}_{p,q}$ with $p + q = 2n$ is defined by

$$\check{H}^{2n}_{p,q}(X) = \bigoplus_{z \in H^{(p-1,q-1)}(X;F)} \mathbb{C}|z\rangle,$$

$$\check{Z}^{2n}_{p,q}(W) = \frac{\sqrt{|H^{(p-1,q-1)}(\partial W;F)||\mathrm{Ker}(r^p \oplus r^q)|}}{|\mathrm{Im}(r^{p-1} \oplus r^{q-1})|} \sum_{x \in \mathrm{Im}(r^{p-1} \oplus r^{q-1})} |x\rangle.$$

3.5 Possibility of other generalizations

- The construction of our TQFT's only uses basic facts about ordinary cohomology theory with its coefficients in a finite field F. In particular, the essential fact is that the cup product and the fundamental class combine to give a non-degenerate, (skew-)symmetric bilinear form on cohomology groups with coefficients in F. Thus, it would be possible to apply our construction to other generalized cohomology theories h with some extra structures. (For this direction, we will shortly axiomatize the properties of a cohomology theory we used in the construction of our TQFT's.)

- In the construction of our TQFT's, we restrict ourselves to consider a finite field as the coefficient of cohomology. But, there might be a generalization based on cohomology groups with integer coefficients. We expect the generalization in 2-dimension recover the TQFT constructed from the $U(1)$ Wess-Zumino-Witten model at level ℓ.
- Related to the two generalizations above, we may generalize the construction of TQFT by introducing "local coefficients" to the underlying cohomology theory.
- The notion of extended TQFT, including manifolds of higher codimensions and higher categories, is of recent interest. It may be possible to construct an extended version of our TQFT in a simple manner also.

A related issue is to explain our construction from viewpoint of physics, namely, to find out field theories whose quantizations yield the TQFT's.

4 Axiomatization

4.1 Axioms

Let $\mathscr{F}$ denote the category of pairs of finite CW complexes (X, Y) such that $Y \subset X$. The morphisms $f : (X, Y) \to (X', Y')$ in $\mathscr{F}$ are continuous maps $f : X \to X'$ such that $f(Y) \subset Y'$. A finite CW complex X may be regarded as an object $(X, \emptyset)$ in $\mathscr{F}$. There is the subcategory $\mathscr{M}$ in $\mathscr{F}$: an object in $\mathscr{F}$ is a pair $(X, \partial X)$ consisting of a compact manifold X and its boundary ∂X. A morphism $f : (X, \partial X) \to (X', \partial X')$ in $\mathscr{M}$ is a smooth map $f : X \to X'$ such that $f(\partial X) \subset \partial X'$. A subcategory in $\mathscr{M}$ is said to be *closed under gluing* if, for manifolds $X_1, X_2 \in \mathscr{M}$ which share a boundary component, the manifold $X_1 \cup X_2$ obtained by gluing X_1 and X_2 along the boundary component belongs to $\mathscr{M}$.

Axiom 4.1. Let h be a generalized cohomology theory defined on $\mathscr{F}$.

(a) (finite field) There is a finite field F, and $h^p(X, Y)$ is an F-module for any $(X, Y) \in \mathscr{F}$ and $p \in \mathbb{Z}$.

(b) (multiplication) There exists an biadditive map

$$\cup : \ h^p(X, A) \times h^q(X, B) \to h^{p+q}(X, A \cup B)$$

for any $(X, A), (X, B) \in \mathscr{F}$ and $p, q \in \mathbb{Z}$ satisfying the following:

- If h satisfies the finite-field axiom, then $\cup$ is F-bilinear.
- $\cup$ is natural: for any $p,q \in \mathbb{Z}$, $(X,A),(X,B),(X',A'),(X',B') \in \mathscr{F}$ and a continuous map $f : X \to X'$ such that $f(A) \subset A'$ and $f(B) \subset B'$, the following diagram is commutative:

$$\begin{tikzcd}
h^p(X,A) \times h^q(X,B) \arrow[r, "\cup"] & h^{p+q}(X, A \cup B) \\
h^p(X',A') \times h^q(X',B') \arrow[r, "\cup"] \arrow[u, "f^* \times f^*"] & h^{p+q}(X', A' \cup B'). \arrow[u, "f^*"']
\end{tikzcd}$$

- $\cup$ is compatible with the exactness axiom: for any $p,q \in \mathbb{Z}$ and $(X,Y) \in \mathscr{F}$, the following diagram is commutative:

$$\begin{tikzcd}
h^{p-1}(Y) \times h^q(X) \arrow[r, "1\times r"] \arrow[d, "\delta\times 1"'] & h^{p-1}(Y) \times h^q(Y) \arrow[r, "\cup"] & h^{p+q-1}(Y) \arrow[d, "\delta"] \\
h^p(X,Y) \times h^q(X) \arrow[rr, "\cup"] & & h^{p+q}(X,Y),
\end{tikzcd}$$

where δ and r are the maps in the exactness axiom for (X,Y):

$$\cdots \to h^{*-1}(Y) \xrightarrow{\delta} h^*(X,Y) \to h^*(X) \xrightarrow{r} h^*(Y) \xrightarrow{\delta} \cdots .$$

- $\cup$ is graded-commutative.

(c) (integration) Under the finite-field axiom, there are a subcategory $\mathscr{M}^+$ in $\mathscr{M}$ closed under gluing, and, for any compact d-dimensional manifold W possibly with boundary such that $(W, \partial W) \in \mathscr{M}^+$, a F-linear map

$$\int_W : \ h^d(W, \partial W) \longrightarrow F,$$

satisfying the following exists:

- $\int$ is compatible with boundary: for any $(W, \partial W) \in \mathscr{M}^+$ such that $\dim W = d$, the following diagram is commutative:

$$\begin{tikzcd}
h^{d-1}(\partial W) \arrow[rr, "\delta"] \arrow[dr, "\int_{\partial W}"'] & & h^d(W, \partial W) \arrow[dl, "\int_W"] \\
& F, &
\end{tikzcd}$$

where δ is the connecting map in the exact sequence for $(W, \partial W)$.

- $\int$ is compatible with excision: Suppose $(W_1, X_1 \sqcup X), (W_2, X \sqcup X_2) \in \mathscr{M}^+$ such that $\dim W_1 = \dim W_2 = d$ are given. We denote by $(W_1 \cup W_2, X_1 \sqcup X_2) \in \mathscr{M}^+$ the manifold obtained by gluing. Then the natural inclusions induce the homomorphisms making the following diagram commutative:

$$\begin{array}{ccc} h^d(W_1 \cup W_2, X_1 \sqcup W_2) & \longrightarrow & h^d(W_1 \cup W_2, X_1 \sqcup X_2) \\ \cong \Big\downarrow & & \Big\downarrow \int_{W_1 \cup W_2} \\ h^d(W_1, X_1 \sqcup X) & \xrightarrow{\int_{W_1}} & F. \end{array}$$

(d) (duality) Under the above three axioms, let $(W, \partial W) \in \mathscr{M}^+$ be a d-dimensional compact manifold possibly with boundary. For any $p \in \mathbb{Z}$, we define an F-module ${}'h^p(W)$ by

$$'h^p(W) = \mathrm{Ker}\{r : h^p(W) \to h^p(\partial W)\},$$

which agrees with $h^p(W)$ in the case of $\partial W = \emptyset$. For any $p, q \in \mathbb{Z}$, we also define an F-bilinear form

$$I : \ 'h^p(W) \times {}'h^q(W) \longrightarrow F$$

by $\int_X x \cup \tilde{y}$, where $\tilde{y} \in h^q(W, \partial W)$ maps to $y \in h^q(W)$ under the natural map. Then the F-linear map

$$I^\sharp : \ 'h^p(X) \longrightarrow \mathrm{Hom}_F('h^q(X), F),$$

given by $I^\sharp(x)(y) = I(x, y)$, is an isomorphism.

An example of a cohomology theory satisfying the above axioms is of course the ordinary cohomology theory with coefficients in a finite field F: $\cup$ is the usual cup product, $\mathscr{M}^+$ is the subcategory of oriented manifolds, and $\int_W$ is defined by the evaluation of the fundamental class of $(W, \partial W) \in \mathscr{M}^+$.

4.2 Consequence of axiom

Lemma 4.2. *Suppose that a cohomology theory h satisfies the multiplication axiom. Then, for any $(X, A), (X, B) \in \mathscr{F}$ and $p, q \in \mathbb{Z}$, the following diagram is commutative:*

$$\begin{array}{ccccc} h^{p-1}(A \cup B) \times h^q(X, B) & \xrightarrow{1 \times r} & h^{p-1}(A \cup B) \times h^q(A \cup B, B) & \xrightarrow{\cup} & h^{p+q-1}(A \cup B, B) \\ {\scriptstyle \delta \times 1}\Big\downarrow & & & & \Big\downarrow{\scriptstyle \delta} \\ h^p(X, A \cup B) \times h^q(X, B) & & \xrightarrow{\qquad\qquad\cup\qquad\qquad} & & h^{p+q}(X, A \cup B), \end{array}$$

where δ and r are the maps in the exactness sequence for the triple $(X, A \cup B, B)$:

$$\cdots \to h^{*-1}(A \cup B, B) \xrightarrow{\delta} h^*(X, A \cup B) \to h^*(X, B) \xrightarrow{r} h^*(A \cup B, B) \xrightarrow{\delta} \cdots .$$

Proof. The connecting homomorphism δ is the composition of

$$h^{*-1}(A \cup B, B) \to h^{*-1}(A \cup B) \to h^*(X, A \cup B).$$

With this decomposition, the present lemma follows from a diagram chasing by using the naturality and the compatibility with the exactness in the multiplication axiom together with the excision axiom of h. $\square$

Lemma 4.3. *Suppose that a cohomology theory h satisfies the multiplication axiom. Then, for any $(W_1, X_1 \sqcup X), (W_2, X \sqcup X_2) \in \mathscr{M}^+$ such that $\mathrm{dim} W_1 = \mathrm{dim} W_2 = d$, and for any $p, q \in \mathbb{Z}$ such that $p + q = d$, the following diagram is commutative:*

$$\begin{tikzcd}
h^{p-1}(X) \times h^q(W_1 \cup W_2, X_1 \sqcup X_2) \arrow[r, "1\times r"] \arrow[d, "\delta\times 1"'] & h^{p-1}(X) \times h^q(X) \arrow[r, "\cup"] & h^{p+q-1}(X) \arrow[d, "\delta'"] \\
h^p(W_1 \cup W_2) \times h^q(W_1 \cup W_2, X_1 \sqcup X_2) \arrow[rr, "\cup"] & & h^{p+q}(W_1 \cup W_2, X_1 \sqcup X_2),
\end{tikzcd}$$

where $W_1 \cup W_2$ is the manifold obtained by gluing W_1 and W_2 along X, δ and δ' are the connecting homomorphisms in the Mayer-Vietoris exact sequences for $\{W_1, W_2\}$ and $\{(W_1, X_1), (W_2, X_2)\}$, respectively, and r is the restriction.

Proof. The lemma also follows from the naturality and the compatibility with the exactness in the multiplication axiom together with the excision axiom of h. Notice that the equivalent form of the compatibility with the exactness shown in the previous lemma is useful. $\square$

Lemma 4.4. *Suppose that a cohomology theory h satisfies the integration axiom. Then, for any $(W_1, X_1 \sqcup X), (W_2, X \sqcup X_2) \in \mathscr{M}^+$ such that $\mathrm{dim} W_1 = \mathrm{dim} W_2 = d$, the following diagram is commutative:*

$$\begin{tikzcd}
h^{d-1}(X) \arrow[rr, "\delta'"] \arrow[dr, "\int_X"'] & & h^d(W_1 \cup W_2, X_1 \sqcup X_2) \arrow[dl, "\int_{W_1 \cup W_2}"] \\
& F, &
\end{tikzcd}$$

where $W_1 \cup W_2$ is the manifold obtained by gluing W_1 and W_2 along X, and δ' is the connecting map in the Mayer-Vietoris sequence for $\{(W_1, X_1), (W_2, X_2)\}$.

Proof. The connecting map δ' is the composition of:

$$h^{d-1}(X) \cong h^{d-1}(X \sqcup X_1, X_1) \to h^{d-1}(X \sqcup X_1),$$
$$h^{d-1}(X \sqcup X_1) \to h^d(W_1, X \sqcup X_1),$$
$$h^d(W_1, X \sqcup X_1) \cong h^d(W_1 \cup W_2, X_1 \sqcup W_2) \to h^d(W_1 \cup W_2, X_1 \sqcup X_2).$$

Thus the integration axiom leads to the present lemma. $\square$

Theorem 4.5. *Let h be a cohomology theory satisfying the finite-field, multiplication, integration and duality axioms, and n a positive integer. To a $(2n-1)$-dimensional oriented manifold $X \in \mathscr{M}^+$, we assign a $\mathbb{C}$-vector space:*

$$X \mapsto H_X = \bigoplus_{x \in h^n(X)} \mathbb{C}x.$$

To a $2n$-dimensional oriented manifold $W \in \mathscr{M}^+$, we assign a vector:

$$W \mapsto Z_W = \sqrt{|\mathrm{Ker}(r)|} \sum_{x \in \mathrm{Im}(r)} x \quad \in H_{\partial W},$$

where $r : h^n(W) \to h^n(\partial W)$ is the restriction. Then the assignments above gives rise to a $2n$-dimensional topological quantum field theory in which compact oriented manifolds in $\mathscr{M}^+$ are only considered.

Proof. Since the underlying manifolds M are assumed to be compact, the cohomology groups $h^*(M)$ are finitely generated abelian groups for each degree. Hence the rank of the F-module $h^{2k+1}(X)$ is finite, and so is H_X. Now, the argument in the proof of Theorem 3.2 can be directly adapted. □

There also exist generalizations such as in Definition 3.8.

5 Other constructions of TQFT

5.1 Construction, I

We let h be a generalized cohomology theory such that the abelian group $h^p(X, Y)$ is finite for all $p \in \mathbb{Z}$ and $(X, Y) \in \mathscr{F}$.

Definition 5.1. Let d be a non-negative integer, and p an integer.

(a) We assign to a compact oriented $(d-1)$-dimensional manifold X the vector space $H^d_p(X)$ over $\mathbb{C}$ generated by elements in $h^{p-1}(X) \oplus h^p(X)$:

$$H^d_p(X) = \bigoplus_{x \in h^{p-1}(X), y \in h^p(X)} \mathbb{C}|x, y\rangle.$$

We also define a Hermitian metric $\langle\ |\ \rangle : H^d_p(X) \times H^d_p(X) \to \mathbb{C}$ by extending $\langle x, y | x', y' \rangle = \delta_{x,x'}\delta_{y,y'}$ for $x, x' \in h^{p-1}(X)$ and $y, y' \in h^p(X)$.

(b) Let W be a compact oriented d-dimensional manifold W. In the case of $\partial W \neq \emptyset$, we assign to W the vector $Z^d_p(W) \in H^d_p(\partial W)$ defined by

$$Z^d_p(W) = \sqrt{|h^{p-1}(\partial W)|} \frac{|\mathrm{Ker}(r^p)|}{|\mathrm{Im}(r^{p-1})|} \sum_{\substack{x^{p-1} \in \mathrm{Im}(r^{p-1}) \\ x^p \in \mathrm{Im}(r^p)}} |x^{p-1}, x^p\rangle,$$

where $r^i : h^i(W) \to h^i(\partial W)$, $(i = p-1, p)$ are the restrictions. In the case of $\partial W = \emptyset$, we assign to W the number $Z_p^d(W) \in \mathbb{C}$ defined by

$$Z_p^d(W) = |h^p(W)|.$$

Theorem 5.2. *The assignments $X \mapsto H_p^d(X)$ and $W \mapsto Z_p^d(W)$ in Definition 5.1 give rise to a d-dimensional topological quantum field theory $\mathcal{Z}_p^d$.*

We prove this theorem in the remainder of this subsection. To suppress notatins, we write $H_X = H_p^d(X)$ and $Z_W = Z_p^d(W)$ simply. The functoriality, involutority, non-triviality axioms and the former part of the multiplicativity axiom are straightforward. To prove the latter part of the multiplicativity axiom, let W_1 and W_2 be compact oriented d-dimensionla manifolds whose boundaries are $\partial W_1 = X_1 \sqcup Y^*$ and $\partial W_2 = Y \sqcup X_2$. For $i = p-1, p$ and $j = 1, 2$, we write R_j^i and δ_j^i for the following homomorphisms in the exact sequence for the pair $(W_j, \partial W_j)$:

$$h^{p-1}(W_1) \overset{R_1^{p-1}}{\to} h^{p-1}(X_1 \sqcup Y) \to h^p(W_1, X_1 \sqcup Y) \to h^p(W_1) \overset{R_1^p}{\to} h^{p-1}(X_1 \sqcup Y),$$
$$h^{p-1}(W_2) \overset{R_2^{p-1}}{\to} h^{p-1}(Y \sqcup X_2) \to h^p(W_2, Y \sqcup X_2) \to h^p(W_2) \overset{R_2^p}{\to} h^{p-1}(X_2 \sqcup Y),$$

Then the vectors assigned to W_j are:

$$\begin{aligned} Z_{W_1} &= \sqrt{|h^{p-1}(X_1 \sqcup Y)|}\frac{|\mathrm{Ker}(R_1^p)|}{|\mathrm{Im}(R_1^{p-1})|} \\ &\times \sum_{\substack{(x_1^{p-1}, y_1^{p-1}) \in \mathrm{Im}(R_1^{p-1}) \subset h^{p-1}(X_1 \sqcup Y) \\ (x_1^p, y_1^p) \in \mathrm{Im}(R_2^p) \subset h^p(X_1 \sqcup Y)}} |x_1^{p-1}, x_1^p\rangle \otimes \langle y_1^{p-1}, y_1^p|, \\ Z_{W_2} &= \sqrt{|h^{p-1}(Y \sqcup X_2)|}\frac{|\mathrm{Ker}(R_2^p)|}{|\mathrm{Im}(R_2^{p-1})|} \\ &\times \sum_{\substack{(y_2^{p-1}, x_2^{p-1}) \in \mathrm{Im}(R_2^{p-1}) \subset h^{p-1}(Y \sqcup X_2) \\ (y_2^p, x_2^p) \in \mathrm{Im}(R_2^p) \subset h^p(Y \sqcup X_2)}} |y_2^{p-1}, y_2^p\rangle \otimes |x_2^{p-1}, x_2^p\rangle. \end{aligned}$$

The same argument as in the proof of Theorem 3.2 leads to:

$$\begin{aligned} &\mathrm{Tr}_Y(Z_{W_1} \otimes Z_{W_2}) \\ &\qquad = C \sum_{\substack{(x_1^{p-1}, x_2^{p-1}) \in \mathrm{Im}(R'^{p-1}) \subset h^{p-1}(X_1 \sqcup X_2) \\ (x_1^p, x_2^p) \in \mathrm{Im}(R'^p) \subset h^p(X_1 \sqcup X_2)}} |x_1^{p-1}, x_1^p\rangle \otimes |x_2^{p-1}, x_2^p\rangle. \end{aligned}$$

The coefficient $C \in \mathbb{C}$ is

$$C = \sqrt{|h^{p-1}(X_1 \sqcup X_2)|}|h^{p-1}(Y)|\frac{|\mathrm{Ker}(R_1^p \oplus R_2^p)|}{|\mathrm{Im}(R_1^{p-1} \oplus R_2^{p-1})|}\frac{|\mathrm{Ker}R'^{p-1}|}{|\mathrm{Ker}R^{p-1}|}\frac{|\mathrm{Ker}R'^p|}{|\mathrm{Ker}R^p|},$$

where R^i and R'^i are the restriction homomorphisms:

$$\begin{aligned} R^i:\ & h^i(W_1 \cup W_2) \to h^i(X_1) \oplus h^i(Y) \oplus h^i(X_2), \\ R'^i:\ & h^i(W_1 \cup W_2) \to h^i(X_1) \oplus h^i(X_2). \end{aligned}$$

Lemma 5.3. *The following holds:*

$$\begin{aligned} |\mathrm{Ker} R^p| &= |\mathrm{Im}\nabla^{p-1}||\mathrm{Ker}(R_1^p \oplus R_2^p)|, \\ |\mathrm{Ker} R'^{p-1}|/|\mathrm{Ker} R^{p-1}| &= |\mathrm{Im}\tilde{\rho}_1^{p-1} \cap \mathrm{Im}\tilde{\rho}_2^{p-1}|, \end{aligned}$$

where ∇^{p-1} is the connecting homomorphism in the Mayer-Vietoris exact sequence for $\{W_1, W_2\}$, and $\tilde{\rho}_j^{p-1}$ the restriction homomorphisms:

$$\begin{aligned} &\nabla^{p-1} : h^{p-1}(Y) \to h^p(W_1 \cup W_2), \\ &\tilde{\rho}_j^{p-1} : h^{p-1}(W_j, X_j) \to h^{p-1}(Y) = h^{p-1}(Y \sqcup Y_j, Y_j). \end{aligned}$$

Proof. The first formula follows from Lemma 3.3 and the exact sequence:

$$h^{p-1}(Y) \xrightarrow{\nabla^{p-1}} h^p(W_1 \cup W_2) \xrightarrow{f^p} h^p(W_1) \oplus h^p(W_2),$$

For the second formula, we use the map of short exact sequences:

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & h^{p-1}(W_1 \cup W_2) \arrow[r, equal] \arrow[d, "R^{p-1}"'] & h^{p-1}(W_1 \cup W_2) \arrow[d, "R'^{p-1}"'] \\ h^{p-1}(Y) \arrow[r] & h^{p-1}(X_1 \sqcup Y \sqcup X_2) \arrow[r, "\text{proj}"] & h^{p-1}(X_1 \sqcup X_2). \end{tikzcd}$$

This induces the exact sequence

$$0 \to \mathrm{Ker} R^{p-1} \to \mathrm{Ker} R'^{p-1} \to h^{p-1}(Y) \to \mathrm{Coker} R^{p-1} \to \mathrm{Coker} R'^{p-1} \to 0.$$

A close look at the map to $h^{p-1}(Y)$ gives the exact sequence:

$$0 \longrightarrow \mathrm{Ker} R^{p-1} \longrightarrow \mathrm{Ker} R'^{p-1} \longrightarrow \rho^{p-1}(\mathrm{Ker} R'^{p-1}) \longrightarrow 0,$$

where $\rho^{p-1} : h^{p-1}(W_1 \cup W_2) \to h^{p-1}(Y)$ is the restriction. To complete the proof, consider the commutative diagram:

$$\begin{tikzcd} h^{p-1}(W_1 \cup W_2, X_1 \sqcup X_2) \arrow[r] \arrow[d] & h^{p-1}(W_1 \cup W_2) \arrow[r, "R'^{p-1}"] \arrow[d, "f^{p-1}"] & h^{p-1}(X_1 \sqcup X_2) \arrow[d, equal] \\ h^{p-1}(W_1, X_1) \oplus h^{p-1}(W_2, X_2) \arrow[r] \arrow[d, "\tilde{\rho}_1^{p-1} - \tilde{\rho}_2^{p-1}"] & h^{p-1}(W_1) \oplus h^{p-1}(W_2) \arrow[r] \arrow[d, "\rho_1^{p-1} - \rho_2^{p-1}"] & h^{p-1}(X_1 \sqcup X_2) \\ h^{p-1}(Y) \arrow[r, equal] & h^{p-1}(Y), & \end{tikzcd}$$

In view of this diagram, we understand $\rho^{p-1}(\mathrm{Ker} R'^{p-1}) = \mathrm{Im}\tilde{\rho}_1^{p-1} \cap \mathrm{Im}\tilde{\rho}_2^{p-1}$. $\square$

Lemma 5.4. *We have*

$$|\mathrm{Im}\tilde{\nabla}^{p-1}| = |\mathrm{Im}\nabla^{p-1}||\mathrm{Im}(r_1^{p-1} \oplus r_2^{p-1})/\mathrm{Im}R'^{p-1}|,$$

where $r_j : h^{p-1}(W_j) \to h^{p-1}(X_j)$ are the restrictions.

Proof. Let $\rho_j^{p-1} : h^{p-1}(W_j) \to h^{p-1}(Y)$ be the restrictions. The diagram:

$$\begin{array}{ccc}
h^{p-1}(W_1, X_1) \oplus h^{p-1}(W_2, X_2) & \longrightarrow & h^{p-1}(W_1) \oplus h^{p-1}(W_2) \\
\downarrow{\scriptstyle \tilde{\rho}_1^{p-1} - \tilde{\rho}_2^{p-1}} & & \downarrow{\scriptstyle \rho_1^{p-1} - \rho_2^{p-1}} \\
h^{p-1}(Y) & ===== & h^{p-1}(Y) \\
\downarrow{\scriptstyle \tilde{\nabla}^{p-1}} & & \downarrow{\scriptstyle \nabla^{p-1}} \\
h^{p-1}(W_1 \cup W_2, X_1 \sqcup X_2) & \longrightarrow & h^{p-1}(W_1 \cup W_2)
\end{array}$$

implies $\mathrm{Ker}\tilde{\nabla}^{p-1} \subset \mathrm{Ker}\nabla^{p-1}$. The obvious exact sequence

$$0 \to \mathrm{Ker}\nabla^{p-1}/\mathrm{Ker}\tilde{\nabla}^{p-1} \to h^{p-1}(Y)/\mathrm{Ker}\tilde{\nabla}^{p-1} \to h^{p-1}(Y)/\mathrm{Ker}\nabla^{p-1} \to 0$$

and isomorphisms

$$\begin{aligned}
&\mathrm{Ker}\nabla^{p-1}/\mathrm{Ker}\tilde{\nabla}^{p-1} = \mathrm{Im}(\rho_1^{p-1} - \rho_2^{p-1})/\mathrm{Im}(\tilde{\rho}_1^{p-1} - \tilde{\rho}_2^{p-1}), \\
&h^{p-1}(Y)/\mathrm{Ker}\tilde{\nabla}^{p-1} \cong \mathrm{Im}\tilde{\nabla}^{p-1}, \\
&h^{p-1}(Y)/\mathrm{Ker}\nabla^{p-1} \cong \mathrm{Im}\nabla^{p-1},
\end{aligned}$$

lead to the formula:

$$|\mathrm{Im}\tilde{\nabla}^{p-1}| = |\mathrm{Im}\nabla^{p-1}||\mathrm{Im}(\rho_1^{p-1} - \rho_2^{p-1})/\mathrm{Im}(\tilde{\rho}_1^{p-1} - \tilde{\rho}_2^{p-1})|.$$

Noting $\mathrm{Im}R'^{p-1} \subset \mathrm{Im}(r_1^{p-1} \oplus r_2^{p-1}) \subset h^{p-1}(X_1 \sqcup X_2)$, consider the diagram:

$$\begin{array}{ccccc}
h^{p-1}(W_1 \cup W_2, X_1 \cup X_2) & \longrightarrow & h^{p-1}(W_1 \cup W_2) & \xrightarrow[R'^{p-1}]{} & \mathrm{Im}R'^{p-1} \\
\downarrow{\scriptstyle \tilde{f}^{p-1}} & & \downarrow{\scriptstyle f^{p-1}} & & \downarrow \\
h^{p-1}(W_1, X_1) \oplus h^{p-1}(W_2, X_2) & \longrightarrow & h^{p-1}(W_1) \oplus h^{p-1}(W_2) & \xrightarrow{r_1^{p-1} \oplus r_2^{p-1}} & \mathrm{Im}(r_1^{p-1} \oplus r_2^{p-1}) \\
\downarrow{\scriptstyle \tilde{\rho}_1^{p-1} - \tilde{\rho}_2^{p-1}} & & \downarrow{\scriptstyle \rho_1^{p-1} - \rho_2^{p-1}} & & \\
h^{p-1}(Y) & & h^{p-1}(Y). & &
\end{array}$$

The upper part of this diagram leads to the exact sequence:

$$0 \to \mathrm{Coker}\tilde{f}^{p-1} \to \mathrm{Coker}f^{p-1} \to \mathrm{Im}(r_1^{p-1} \oplus r_2^{p-1})/\mathrm{Im}R'^{p-1} \to 0.$$

Hence we get

$$\begin{aligned}
\mathrm{Im}(\rho_1^{p-1} - \rho_2^{p-1})/\mathrm{Im}(\tilde{\rho}_1^{p-1} - \tilde{\rho}_2^{p-1}) &\cong \mathrm{Coker}f^{p-1}/\mathrm{Coker}\tilde{f}^{p-1} \\
&\cong \mathrm{Im}(r_1^{p-1} \oplus r_2^{p-1})/\mathrm{Im}R'^{p-1},
\end{aligned}$$

and the lemma is proved. $\square$

Lemma 5.5. *For $j = 1, 2$, it holds that:*

$$|\mathrm{Im}R_j^{p-1}| = |\mathrm{Im}\tilde{\rho}_j^{p-1}||\mathrm{Im}(r_j^{p-1})|.$$

Proof. We have the following diagram:

$$\begin{tikzcd}
h^{p-1}(W_j, X_j) \arrow[r] \arrow[d, "\tilde{\rho}_j^{p-1}"] & h^{p-1}(W_j) \arrow[r, "r_j^{p-1}"] \arrow[d, "R_j^{p-1}"] & \mathrm{Im}(r_j^{p-1}) \arrow[d] \\
h^{p-1}(Y) \arrow[r] & h^{p-1}(Y \sqcup X_j) \arrow[r] & h^{p-1}(X_j),
\end{tikzcd}$$

in which the lower row is split. Noting that $\mathrm{Im}(r_j^{p-1}) \subset h^{p-1}(X_j)$, we get:

$$0 \to \mathrm{Coker}\tilde{\rho}_j^{p-1} \to \mathrm{Coker}R_j^{p-1} \to \mathrm{Coker}(r_j^{p-1}) \to 0.$$

This exact sequence also splits, and leads to $\mathrm{Im}R_j^{p-1} \cong \mathrm{Im}\tilde{\rho}_j^{p-1} \oplus \mathrm{Im}(r_j^{p-1})$. $\square$

Lemma 5.6. *It holds that:*

$$|\mathrm{Im}(\tilde{\nabla}^{p-1})| = \frac{|h^{p-1}(Y)||\mathrm{Im}\tilde{\rho}_1^{p-1} \cap \mathrm{Im}\tilde{\rho}_2^{p-1}|}{|\mathrm{Im}\tilde{\rho}_1^{p-1}||\mathrm{Im}\tilde{\rho}_2^{p-1}|}.$$

Proof. Let γ_1 be the connecting homomorphism in the exact sequence for the triple $(W_1, X_1 \sqcup X, X_1)$:

$$h^{p-1}(W_1, X_1) \xrightarrow{\tilde{\rho}_1^{p-1}} h^{p-1}(X_1 \sqcup Y, X_1) = h^{p-1}(Y) \xrightarrow{\gamma_1} h^p(W_1, X_1 \sqcup Y).$$

Then the connecting homomorphism $\tilde{\nabla}^{p-1} : h^{p-1}(Y) \to h^p(W_1 \cup W_2, X_1 \sqcup X_2)$ in the Mayer-Vietoris sequence for $\{(W_1, X_1), (W_2, X_2)\}$ is realized as $\tilde{\nabla}^{p-1} = j\epsilon^{-1}\gamma_1$, where j is induced from the inclusion and ϵ is the excision isomorphism:

$$\begin{tikzcd}
h^{p-1}(W_2, X_2) \arrow[r] \arrow[d, "\tilde{\rho}_1^{p-1}"] & h^p(W_1 \cup W_2, X_1 \sqcup W_2) \arrow[r, "j"] \arrow[d, "\epsilon"] & h^p(W_1 \cup W_2, X_1 \sqcup X_2) \\
h^{p-1}(Y) \arrow[r, "\gamma_1"] & h^p(W_1, X_1 \sqcup Y). &
\end{tikzcd}$$

Thus, $\mathrm{Ker}\gamma_1 \subset \mathrm{Ker}\tilde{\nabla}^{p-1}$ clearly, and we get an exact sequence:

$$0 \to \mathrm{Ker}\tilde{\nabla}^{p-1}/\mathrm{Ker}\gamma_1 \to h^{p-1}(Y)/\mathrm{Ker}\gamma_1 \to h^{p-1}(Y)/\mathrm{Ker}\tilde{\nabla}^{p-1} \to 0.$$

We know $\mathrm{Im}\tilde{\rho}_1^{p-1} = \mathrm{Ker}\gamma_1$ and $\mathrm{Im}(\tilde{\rho}_1^{p-1} - \tilde{\rho}_2^{p-1}) = \mathrm{Ker}\tilde{\nabla}^{p-1}$, so that:

$$\mathrm{Ker}\tilde{\nabla}^{p-1}/\mathrm{Ker}\gamma_1 = \mathrm{Im}(\tilde{\rho}_1^{p-1} - \tilde{\rho}_2^{p-1})/\mathrm{Im}\tilde{\rho}_1^{p-1} \cong \mathrm{Im}\tilde{\rho}_2^{p-1}/(\mathrm{Im}\tilde{\rho}_1^{p-1} \cap \mathrm{Im}\tilde{\rho}_2^{p-1}).$$

Hence we have the exact sequence:

$$0 \to \mathrm{Im}\tilde{\rho}_2^{p-1}/(\mathrm{Im}\tilde{\rho}_1^{p-1} \cap \mathrm{Im}\tilde{\rho}_2^{p-1}) \to h^{p-1}(Y)/\mathrm{Im}\tilde{\rho}_1^{p-1} \to \mathrm{Im}\tilde{\nabla}^{p-1} \to 0,$$

which establishes the formula in this lemma. $\square$

Now, the formulae in the last three lemmas show:

$$C = \sqrt{|h^{p-1}(X_1 \sqcup X_2)|} \frac{|\mathrm{Ker} R'^p|}{|\mathrm{Im} R'^{p-1}|}.$$

This completes the proof of the multiplicativity $\mathrm{Tr}_Y(Z_{W_1} \otimes Z_{W_2}) = Z_{W_1 \cup W_2}$.

Remark 4. Assume that $d = 2n-1$ and $p = n$ for an integer $n \geq 2$, and that $h(\ \ ) = H(\ \ ; F)$ is ordinary cohomology with coefficients in a fintie field F. (In the particular case of $n = 2$, we allow F to be any principal ideal domain such that $|R| < \infty$.) In this case, the coefficients of the vector $Z_n^{2n-1}(W)$ assigned to a compact oriented $(2n-1)$-dimensional manifold W with boundary get simple:

$$Z_n^{2n-1}(W) = |\mathrm{Ker}(r^n)| \sum_{\substack{x^{n-1} \in \mathrm{Im}(r^{n-1}) \\ x^n \in \mathrm{Im}(r^n)}} |x^{n-1}, x^n\rangle.$$

The reason is as follows: Under the assumption, there is the intersection pairing I on $H^{n-1}(\partial W; F)$. By some properties of the cup product, we can prove that the complement of image $\mathrm{Im}(r^{n-1}) \subset H^{n-1}(\partial W; F)$ with respect to I satisfies:

$$\mathrm{Im}(r^{n-1})^\perp = \mathrm{Im}(r^{n-1}).$$

Then, by Lemma 2.1, we get $|\mathrm{Im}(r^{n-1})|^2 = |H^{n-1}(\partial W; F)|$. (This also implies that: for any compact oriented $(2n-1)$-dimensional manifold W with boundary, the rank of $H^{n-1}(\partial W; F)$ is divisible by 2.) We notice that $\mathcal{Z}_n^{2n-1}$ is equivalent to the compactification of $\hat{\mathcal{Z}}_n^{2n}$ along S^1, as will be shown in Proposition 6.4.

Remark 5. We can readily generalized the construction of $\mathcal{Z}_p^d$ invoving other degree. But, the essense of the construction can be transparetly accounted for in view of Proposition 6.2 given later.

5.2 Construction, II

Let h be a generalized cohomology theory such that:

- The abelina group $h^p(X, Y)$ is finite for all $p \in \mathbb{Z}$ and $X, Y \in \mathscr{F}$,
- $h^p(X, Y)$ is bounded below for all $X, Y \in \mathscr{F}$.

Definition 5.7. Let d be a non-negative integer, and p an integer.

(a) We assign to a compact oriented $(d-1)$-dimensional manifold X the vector space $H^d_{\leq p}(X)$ over $\mathbb{C}$ generated by elements in $h^p(X)$:

$$H^d_{\leq p}(X) = \bigoplus_{x \in h^p(X)} \mathbb{C}|x\rangle.$$

We define a Hermitian metric $\langle\ |\ \rangle : H^d_{\leq p}(X) \times H^d_{\leq p}(X) \to \mathbb{C}$ by extending $\langle x|x'\rangle = \delta_{x,x'}$ for $x, x' \in h^p(X)$.

(b) Let W be a compact oriented d-dimensional manifold W. In the case of $\partial W \neq \emptyset$, we assign to W the vector $Z^d_{\leq p}(W) \in H^d_{\leq p}(\partial W)$ defined by

$$Z^d_{\leq p}(W) = |\mathrm{Ker}(r^p)| \prod_{i \geq 1} \left(\frac{|h^{p-i}(W)|}{\sqrt{|h^{p-i}(\partial W)|}} \right)^{(-1)^i} \sum_{x \in \mathrm{Im}(r^p)} |x\rangle,$$

where $r^p : h^p(W) \to h^p(\partial W)$ is the restriction. In the case of $\partial W = \emptyset$, we assign to W the number $Z^d_{\leq p}(W) \in \mathbb{C}$ defined by

$$Z^d_{\leq p}(W) = \prod_{i \geq 0} |h^{p-i}(W)|^{(-1)^i}.$$

Theorem 5.8. *The assignments $X \mapsto H^d_{\leq p}(X)$ and $W \mapsto Z^d_{\leq p}(W)$ in Definition 5.7 give rise to a d-dimensional topological quantum field theory $\mathcal{Z}^d_{\leq p}$.*

Proof. We just indicate how to prove the multiplicative axiom in the case where compact oriented d-dimensional manifolds W_1 and W_2 such that $\partial W_1 = X$ and $\partial W_2 = X^*$ are given. In this case, the Mayer-Vietoris sequence and the homomorphism theorem yield the formula

$$|\mathrm{Im}(f^j)| = \frac{|h^j(W_1 \cup W_2)||h^{j-1}(W_1)||h^{j-1}(W_2)|}{|h^{j-1}(X)||\mathrm{Im}(f^{j-1})|},$$

where $j = p, p-1, p-2, \ldots$, and f^j appears in the exact sequence

$$\cdots \to h^j(W_1 \cup W_2) \xrightarrow{f^j} h^j(W_1) \oplus h^j(W_2) \xrightarrow{r_1^j - r_2^j} h^j(X) \to \cdots.$$

Now, we use the formula recursively to compute $\mathrm{Tr}_X(Z^d_p(W_1 \sqcup W_2))$. Since h^* is bounded below, the computation terminates and we get $Z^d_p(W_1 \cup W_2)$. $\square$

Remark 6. In the case where $p = 1$ and $h^*(\ ) = H^*(\ ; A)$ with A a finite abelian group, $\mathcal{Z}^d_{\leq 1}$ is equivalent to the untwisted Dijkgraaf-Witten theory. The TQFT $\mathcal{Z}^d_{\leq p}$ with general p may be an example of the TQFT's considered in [1].

6 Relations

We here observe relations among the TQFT's introduced in this note.

6.1 Tensor product

In general, d-dimensional TQFT's $\mathcal{Z}$ and $\mathcal{Z}'$ yield via tensor products a d-dimensional TQFT $\mathcal{Z} \otimes \mathcal{Z}'$. We study relations among TQFT's introduced so far from the viewpoint of tensor product operations.

The relations among $\hat{\mathcal{Z}}^{2n}$ such as

$$\hat{\mathcal{Z}}^{2n}_{n,n} \cong \hat{\mathcal{Z}}^{2n}_n \otimes \hat{\mathcal{Z}}^{2n}_n, \qquad \hat{\mathcal{Z}}^{2n}_{\mathrm{all}} \cong \hat{\mathcal{Z}}^{2n}_{\mathrm{even}} \otimes \hat{\mathcal{Z}}^{2n}_{\mathrm{odd}}$$

are quie easy to see.

The following relation accounts for that $\check{\mathcal{Z}}_n$ is termd 'dual' or complementary.

Proposition 6.1. *For any n and p, q such that $p + q = n$, we have:*

$$\hat{\mathcal{Z}}_n^{2n} \otimes \check{\mathcal{Z}}_n^{2n} \cong \mathcal{Z}_n^{2n}, \qquad \hat{\mathcal{Z}}_{p,q}^{2n} \otimes \check{\mathcal{Z}}_{p,q}^{2n} \cong \mathcal{Z}_p^{2n} \otimes \mathcal{Z}_q^{2n}.$$

Proof. The TQFT $\mathcal{Z}_n^{2n}$ is defined by using the finite field F that is used in $\hat{\mathcal{Z}}_n^{2n}$ and $\check{\mathcal{Z}}_n^{2n}$. Then the proposition follows directly from the definitions. In fact, $\check{\mathcal{Z}}_n^{2n}$ is defined so that the relation holds true. Notice that the Poincare-Lefschetz duality is used implicitly. The case involving p, q is similar. □

A relation among $\mathcal{Z}_p^d$ and $\mathcal{Z}_{\leq p}^d$ is as follows:

Proposition 6.2. *For any d and p, we have a natural equivalence:*

$$\hat{\mathcal{Z}}_p^d \cong \mathcal{Z}_{\leq p-1}^d \otimes \mathcal{Z}_{\leq p}^d.$$

Proof. The TQFT's $\hat{\mathcal{Z}}_p^d$, $\mathcal{Z}_{\leq p-1}^d$ and $\mathcal{Z}_{\leq p}^d$ are defined by using the same generalized cohomology theory h^*. Then proposition can be verified by just comparing the definitions the TQFT's. □

By this proposition, we can see that the construction of $\mathcal{Z}_p^d$ is readily generalized so that, for any positive integer k given, the resulting TQFT assings to a compact oriented d-dimensional manifold W without boundary the invariant

$$\prod_{i=0}^{k} |h^{p-i}(W)|^{(-1)^i}.$$

The combination of propositions above provide us:

Corollary 6.3. *For any n and p, q such that $p + q = n$, we have*

$$\hat{\mathcal{Z}}_n^{2n} \otimes \check{\mathcal{Z}}_n^{2n} \cong \mathcal{Z}_{\leq n-1}^{2n} \otimes \mathcal{Z}_{\leq n}^{2n}, \quad \hat{\mathcal{Z}}_{p,q}^{2n} \otimes \check{\mathcal{Z}}_{p,q}^{2n} \cong \mathcal{Z}_{p-1}^{2n} \otimes \mathcal{Z}_p^{2n} \otimes \mathcal{Z}_{q-1}^{2n} \otimes \mathcal{Z}_q^{2n}.$$

6.2 Compactification or dimensional reduction

Besides the tensor product construction of TQFT's, there is another general construction of TQFT's called compactifications or dimensional reductions: Let $\mathcal{Z}$ be a d-dimensional TQFT, and K a compact oriented k-dimensional manifold without boundary. Then we have a $(d-k)$-dimensional TQFT $\mathcal{Z}/K$ by 'compactifying' d- and $(d-1)$-dimensional manifolds along K. This construction produces some relationship among our TQFT's.

Proposition 6.4. *For any n, we have:*

$$\hat{\mathcal{Z}}_n^{2n}/S^1 \cong \mathcal{Z}_n^{2n-1} \cong \mathcal{Z}_{\leq n-1}^{2n-1} \otimes \mathcal{Z}_{\leq n}^{2n-1}.$$

Proof. The TQFT $\mathcal{Z}_n^{2n-1}$ is defined by using $H^*(\ ) = H^*(\ ; F)$ with F a finite field. By the Kunneth formula, we have a canonical identification $\hat{H}_n^{2n}/S^1(Y) = \hat{H}_n^{2n-1}(Y) \otimes \hat{H}_n^{2n-1}(Y)$ for any compact oriented $(2n-2)$-dimensional manifold

Y without boundary. Now, for any compact oriented $(2n-1)$-dimensional manifold, the Kunneth formula gives

$$\check{Z}_n^{2n}/S^1(X) = \sqrt{|\mathrm{Ker}(r^{n-1})||\mathrm{Ker}(r^n)|} \sum_{z \in \mathrm{Im}(r^{n-1} \oplus r^n)} |z\rangle,$$

$$\check{Z}_n^{2n-1}(X) = \sqrt{|H^{n-1}(\partial W)|}\frac{|\mathrm{Ker}(r^n)|}{|\mathrm{Im}(r^{n-1})|} \sum_{z \in \mathrm{Im}(r^{n-1} \oplus r^n)} |z\rangle.$$

The Mayer-Vietoris sequence and the homomorphism theorem shows

$$|\mathrm{Ker}(r^n)| = \frac{|H^n(X, \partial X)|}{|H^{n-1}(\partial X)|}|\mathrm{Im}(r^{n-1})|,$$

which leads to

$$\sqrt{|H^{n-1}(\partial W)|}\frac{|\mathrm{Ker}(r^n)|}{|\mathrm{Im}(r^{n-1})|} = \frac{|H^n(X, \partial X)|}{\sqrt{|H^{n-1}(\partial X)|}},$$

$$\sqrt{|\mathrm{Ker}(r^{n-1})||\mathrm{Ker}(r^n)|} = \sqrt{\frac{|H^n(X, \partial X)||H^{n-1}(X)|}{|H^{n-1}(\partial X)|}}.$$

Comparing these formulae, we get

$$\sqrt{|H^{n-1}(\partial W)|}\frac{|\mathrm{Ker}(r^n)|}{|\mathrm{Im}(r^{n-1})|} = \sqrt{|\mathrm{Ker}(r^{n-1})||\mathrm{Ker}(r^n)|} \times \sqrt{\frac{|H^n(X, \partial X)|}{|H^{n-1}(X)|}}.$$

Now, the Poincare-Lefshcetz duality completes the proof. □

A few examples of compactifications of $\hat{\mathcal{Z}}_n^{2n}$ along other manifolds are:

$$\hat{\mathcal{Z}}_{1,3}^4/S^2 \cong \hat{\mathcal{Z}}_1^2 \otimes \hat{\mathcal{Z}}_1^2, \qquad \hat{\mathcal{Z}}_3^6/S^2 \cong \hat{\mathcal{Z}}_{1,5}^6/S^2 \cong \hat{\mathcal{Z}}_{1,3}^4,$$

$$\hat{\mathcal{Z}}_3^6/(S^2 \times S^2) \cong \hat{\mathcal{Z}}_1^2 \otimes \hat{\mathcal{Z}}_1^2, \qquad \hat{\mathcal{Z}}_3^6/\mathbb{C}P^2 \cong \hat{\mathcal{Z}}_1^2.$$

It happens that a dimensional reduction gives a trivial TQFT, e.g. $\hat{\mathcal{Z}}_3^6/S^4$.

Proposition 6.5. *For any d, p and k, we have:*

$$\mathcal{Z}_p^d/S^k \cong \mathcal{Z}_p^{d-k} \otimes \mathcal{Z}_{p-k}^{d-k}.$$

Proof. This directly follows from the Kunneth formula. □

Remark 7. In general, the compactification a d-dimensional TQFT $\mathcal{Z}$ along S^1 assigns to a compact oriented $(d-1)$-dimensional manifold X without boundary the integer $Z/S^1(X) = Z(X \times S^1) = \dim H_X$. Thus, the TQFT $\hat{\mathcal{Z}}_{2k+1}^{4k+2}$ with the coefficients $F = \mathbb{Z}/2$ cannot be the compactification of any $(4k+3)$-dimensional TQFT along S^1, provided that there exits a $(4k+2)$-dimensional manifold W without boundary such that $\nu_{2k+1}(W) \neq 0$.

7 Examples in low dimension

7.1 $\hat{\mathcal{Z}}_n^{2n}$ and $\check{\mathcal{Z}}_n^{2n}$

The information of a 2-dimensional TQFT is encoded in the so-called commutative Frobenius algebra. We here describe the Frobenius algebras $\hat{\mathcal{A}}_1^2$ and $\check{\mathcal{A}}_1^2$ corresponding to $\hat{\mathcal{Z}}_1^2$ and $\check{\mathcal{Z}}_1^2$ associated to a principal ideal domain R:

	space	unit	multiplication
$\hat{\mathcal{A}}_1^2$	$\bigoplus_{x\in R}\mathbb{C}\lvert x\rangle$	$\lvert 0\rangle$	$\lvert x\rangle * \lvert x'\rangle = \lvert x+x'\rangle$
$\check{\mathcal{A}}_1^2$	$\bigoplus_{x\in R}\mathbb{C}\lvert x\rangle$	$\frac{1}{\sqrt{\lvert R\rvert}}\sum_{x\in R}\lvert x\rangle$	$\lvert x\rangle * \lvert x'\rangle = \sqrt{\lvert R\rvert}\delta_{x,x'}\lvert x'\rangle$

Thus, $\hat{\mathcal{A}}_1^2$ is the group algebra $\mathbb{C}[R]$ of the additive group underlies R. In other words, $\hat{\mathcal{A}}_1^2$ is the space $C(R,\mathbb{C})$ of $\mathbb{C}$-valued functions on R equipped with the convlution product. On the other hand, $\check{\mathcal{A}}_1^2$ is $C(R,\mathbb{C})$ equipped with the usual pointwise product of functions. They are of course different on the nose. However, we can construct an algebra isomorphism $\hat{\mathcal{A}}_1^2 \to \check{\mathcal{A}}_1^2$ by specifying the primitive idempotents in the group algebra $\mathbb{C}[R]$. This proves that $\hat{\mathcal{Z}}_1^2$ and $\check{\mathcal{Z}}_1^2$ are (non-canonically) equivalent.

7.2 $\mathcal{Z}_p^d$

We here take $h^*(\quad) = H^p(\quad; A)$ to be the ordinary cohomology with coefficients in a finite abelian group A. In 2-dimensions, the Frobenius algebras $\mathcal{A}_p^2$ corresponding to the 2-dimensional TQFT's $\mathcal{Z}_p^2$ are:

p	$\mathcal{A}_p^2$	unit	multiplication
0	$\bigoplus_{x\in A}\mathbb{C}\lvert x\rangle$	$\sum_{x\in A}\lvert x\rangle$	$\lvert x\rangle * \lvert x'\rangle = \delta_{x,x'}\lvert x'\rangle$
1	$\bigoplus_{x,y\in A}\mathbb{C}\lvert x,y\rangle$	$\frac{1}{\sqrt{\lvert A\rvert}}\sum_{x\in A}\lvert x,0\rangle$	$\lvert x,y\rangle * \lvert x',y'\rangle = \sqrt{\lvert A\rvert}\delta_{x,x'}\lvert x',y+y'\rangle$
2	$\bigoplus_{y\in A}\mathbb{C}\lvert y\rangle$	$\sqrt{\lvert A\rvert}\lvert 0\rangle$	$\lvert y\rangle * \lvert y'\rangle = \frac{1}{\sqrt{\lvert A\rvert}}\lvert y+y'\rangle$

Also, the 2-dimensionla TQFT $\mathcal{Z}3_p/S^1$ given by the dimensional reduction of the 3-dimensional TQFT $\mathcal{Z}_p^3$, $(p=0,1,2,3)$ along S^1 produces the following Frobenius algebras:

p	$\mathcal{A}^p/S^1$	unit	multiplication
0	$\bigoplus_{x\in A}\mathbb{C}\vert x\rangle$	$\sum_{x\in A}\vert x\rangle$	$\vert x\rangle * \vert x'\rangle = \delta_{x,x'}\vert x'\rangle$
1	$\bigoplus_{x,y,z\in A}\mathbb{C}\vert x,y,z\rangle$	$\frac{1}{\sqrt{\vert A\vert}}\sum_{x,z\in A}\vert x,0,z\rangle$	$\vert x,y,z\rangle * \vert x',y',z'\rangle$ $= \sqrt{\vert A\vert}\delta_{x,x'}\delta_{z,z'}\vert x',y+y',z'\rangle$
2	$\bigoplus_{x,y,z\in A}\mathbb{C}\vert x,y,z\rangle$	$\sum_{y\in A}\vert 0,y,0\rangle$	$\vert x,y,z\rangle * \vert x',y',z'\rangle$ $= \delta_{y,y'}\vert x+x',y',z+z'\rangle$
3	$\bigoplus_{x\in A}\mathbb{C}\vert x\rangle$	$\sqrt{\vert A\vert}\vert 0\rangle$	$\vert x\rangle * \vert x'\rangle = \frac{1}{\sqrt{\vert A\vert}}\vert x+x'\rangle$

As is seen, we have the equivalence

$$\mathcal{Z}_p^d/S^k \cong \mathcal{Z}_p^{d-k} \otimes \mathcal{Z}_{p-k}^{d-k}.$$

The examples above illustrate this phnomena explicitly.

References

[1] Martins and Porter, *On Yetter's invariant and extension of the Dijkgraaf-Witten invariant to categorical groups*, QA/0608484.