Is magnitude related to physics of patterned resonators?

Kiyonori Gomi

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- **One of my recent works concerns with the magnitude (or better the magnitude homology) of metric spaces.**

Question

Is magnitude or magnitude homology related to physics?

- **Magnitude has at least superficial relationship to patterned resonators of Prodan and Shmalo.**
- **But, meaningful relationship is still missing.**
- **I talk about basics of magnitude (homology), which hopefully helps discovery of meaningful relationship.**

Plan of my talk

1. **Resonators on point patterns**
2. **Magnitude**
3. **Magnitude homology**

Resonators on point patterns

- **A notion of resonators is introduced in a work of Prodan and Shmalo [JGP2019] in their development of the bulk-boundary correspondence:**

Definition

A resonator is a 0-dimensional physical system, i.e. a system confined to a small region of the physical space, whose physical observables and dynamics can be described by linear operators over a finite dimensional Hilbert space.

- **A typical example is a quantum system with a finite number of quantum states.**

Resonators on point patterns

- **We are interested in physics of identical resonators placed on a point pattern.**

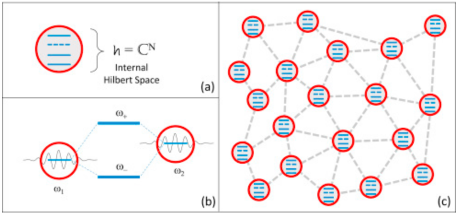

Figure: Fig 2.1 in the paper of Prodan and Shmalo (Journal of Geometry and Physics, Volume 135, January 2019, Pages 135–171)

- **Prodan and Shmalo focused on dynamically generated patterns, and established a bulk-boundary correspondence for resonators placed on such patterns.**

Hamiltonian of resonators on a point pattern

- **Prodan and Shmalo mentioned an example of a Hamiltonian of single-state (N = 1) resonators on a general point pattern $\mathscr{P}$**

$$H(\mathscr{P}) = \sum_{p,p' \in \mathscr{P}} e^{-\beta|p-p'|} |p\rangle \langle p'|$$

where β is a constant.

- **This essentially coincides with the zeta matrix of the point pattern $\mathscr{P}$ regarded as a metric space.**
- **The zeta matrix is used in the definition of the magnitude of a metric space.**
- **Noting this coincidence, I wondered whether there is relation between magnitude and physics of resonators, but I could not yet find meaningful relationship.**

Plan of my talk

1. Resonators on point patterns
2. Magnitude
3. Magnitude homology

Magnitude

- Magnitude is a real number defined for a certain metric space. (More generally, magnitude is defined for certain categories [Leinster-Shulman(2017)].)
- It provides an effective number of points, and has an origin in a work to formulate a measurement of diversity of species [Solow-Polasky(1994)].
- Recall that a metric space (X, d) is a set X equipped with a distance function $d : X \times X \to \mathbb{R}_{\geq 0}$,
  1. $d(x, y) = 0$ if and only if $x = y$,
  2. $d(x, y) = d(y, x)$,
  3. $d(x, y) + d(y, z) \geq d(x, z)$. (triangle inequality)
- Example: $X = \mathbb{R}^d$, $d(x, y) = \|x - y\|$.

Definition of magnitude

- **For simplicity, consider a finite metric space consisting of n points $X = \{1, 2, \cdots, n\}$.**
- **The zeta matrix of this finite metric space is the $n \times n$ matrix whose (i, j)-component is $\exp(-d(i, j))$.**

$$H_X = (e^{-d(i,j)}) = \sum_{i,j} e^{-d(i,j)} |i\rangle\langle j|$$

- **A weight is a real vector $\phi = (\phi_i)$ such that $H_X \phi = \mathbb{1}$, where $\mathbb{1} = (1)$ is the vector whose components are 1.**
- **If H_X is invertible, then there is the unique weight $\phi = H_X^{-1}\mathbb{1}$.**

Definition of magnitude

$$H_X = (e^{-d(i,j)}), \qquad H_X\phi = \mathbb{1}.$$

Definition (magnitude)

Let (X, d) be a finite metric space. When its zeta matrix H_X admits a weight ϕ, the magnitude of (X, d) is defined as the sum of all the components of $\phi = (\phi_i)$.

$$\mathrm{Mag}(X) := \sum_i \phi_i = \langle \mathbb{1}, \phi \rangle$$

- **The definition is independent of the choice of ϕ.**
- **If H_X is invertible, then $\mathrm{Mag}(X)$ is the sum of all the components of $H_X^{-1} = (H_X^{-1}(i,j))$.**

$$\mathrm{Mag}(X) = \sum_{i,j} H_X^{-1}(i,j) = \langle \mathbb{1}, H_X^{-1}\mathbb{1} \rangle = \langle \mathbb{1}, \phi \rangle$$

Example: two-point set

$$H_X = (e^{-d(i,j)}), \quad H_X \phi = \mathbb{1}, \quad \mathrm{Mag}(X) = \sum_i \phi_i.$$

- **Let $X = \{1, 2\}$ be the two-point set with $d(1, 2) = \delta$.**
- **The zeta matrix, its weight, and the magnitude:**

$$H_X = \begin{pmatrix} 1 & q^\delta \\ q^\delta & 1, \end{pmatrix}, \quad \phi = \frac{1}{1+q^\delta}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \mathrm{Mag} = \frac{2}{1+q^\delta},$$

where $q = e^{-1}$.

- **$\lim_{\delta\to 0} \mathrm{Mag} = 1$ and $\lim_{\delta\to\infty} \mathrm{Mag} = 2$.**

A generalization of entropy

- The magnitude has a relationship to a generalization of entropy (in information theory).
- A probability distribution on a finite set is a vector $p = (p_i)$ such that $p_i \geq 0$ and $\sum_i p_i = 1$.
- The entropy of p is given by $S(p) = -\sum_i p_i \log p_i$.
- In the presence of the zeta matrix H_X, a generalization of the entropy is

$$S^X(p) = -\sum_i p_i \log(H_X p)_i,$$

where $(H_X p)_i = \sum_j e^{-d(i,j)} p_j$ is the ith component of the vector $H_X p$.

Magnitude and generalization of quadratic entropy

$$S^X(p) = -\sum_i p_i \log(Hp)_i$$

Proposition [Leinster]

Let (X, d) be a finite metric space such that its zeta matrix $H = (e^{-d(i,j)})$ admits a weight $\phi = (\phi_i)$ such that $\phi_i \geq 0$. Then it holds that

$$\max_p S^X(p) = \log \mathrm{Mag}(X),$$

and the maximum is attained by $p = \phi/\mathrm{Mag}(X)$.

- **Generally, a weight is not an eigenvector of H.**
- **A statistical approach may be useful.**

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Magnitude homology

- The magnitude homology is a "categorification" of magnitude [Hepworth-Willerton, Leinster-Shulman].
- "Categorification" can be thought of as a way to generalize notions in mathematics:

number	$\Longrightarrow$	set
(an element of a set)		(an object of a category)

- A basic example is the homology of polyhedra: Given a finite polyhedron X, its homology group $\{H_n(X)\}$ categorifies the Euler characteristic:

$$\chi(X) = \sum_{n \geq 0} (-1)^n \mathrm{rank} H_n(X)$$

- A more sophisticated example is Khovanov homology, which categorifies the Jones polynomial of links.

Magnitude homology categorifies magnitude

- **Similarly, magnitude homology categorifies magnitude.**
- **Recall that the magnitude of a finite metric space is computed from $H_X = (e^{-d(i,j)}) = (q^{d(i,j)})$.**
- **In a certain ring $\mathbb{Q}((q^{\mathbb{R}}))$ of functions in q, H_X is invertible, and the magnitude always makes sense.**

Theorem [Leinster-Shulman(2017)]

Let (X, d) be a finite metric space. Then, in $\mathbb{Q}((q^{\mathbb{R}}))$,

$$\mathrm{Mag}(X) = \sum_{\ell \geq 0} \left(\sum_{n \geq 0} (-1)^n \mathrm{rank} M_\ell H_n(X) \right) q^\ell,$$

with $M_\ell H_n(X)$ the nth magnitude homology of length ℓ.

Definition of magnitude homology

- Given a metric space (X, d), an n-chain $\langle x_0, \cdots, x_n \rangle$ is a sequence of points on X such that

$$x_0 \neq x_1 \neq \cdots \neq x_{n-1} \neq x_n.$$

- The length of $\gamma = \langle x_0, \cdots, x_n \rangle$ is defined by

$$\ell(\gamma) = d(x_0, x_1) + d(x_1, x_2) + \cdots + d(x_{n-1}, x_n).$$

- For $i = 0, \cdots, n$, we remove x_i to define

$$\partial_i \gamma = \langle x_0, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n \rangle.$$

Then $\ell(\partial_i \gamma) \leq \ell(\gamma)$ by triangle inequality.

Definition of magnitude homology

$$\partial_i\langle x_0,\cdots,x_n\rangle = \langle x_0,\cdots,x_{i-1},x_{i+1},\cdots,x_n\rangle$$

- Let $M_\ell C_n(X)$ denote the free abelian group generated by n-chains γ of length ℓ.
- We define $\partial : M_\ell C_n(X) \to M_\ell C_{n-1}(X)$ by

$$\partial\gamma = \sum_i (-1)^i \partial_i\gamma,$$

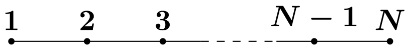
  where the sum is over i such that $\ell(\partial_i\gamma) = \ell(\gamma)$.
- The composition $M_\ell C_{n+1}(X) \xrightarrow{\partial} M_\ell C_n(X) \xrightarrow{\partial} M_\ell C_{n-1}(X)$ is trivial.

Definition (magnitude homology)

The homology of the chain complex $(M_\ell C_*(X), \partial)$ is the magnitude homology: $M_\ell H_n(X) = \mathrm{Ker}\partial/\mathrm{Im}\partial$

Example

- **Let $P_N = \{1, \cdots, N\} \subset \mathbb{R}$ be the N-point set.**



$$M_\ell H_n(P_N) = \begin{cases} \mathbb{Z}^{N}, & (n = \ell = 0) \\ \mathbb{Z}^{2N-2}, & (n = \ell = 1, 2, 3, \ldots) \\ 0. & (\text{otherwise}) \end{cases}$$

$$\mathrm{Mag}(P_N) = \frac{N - (N-2)q}{1+q} = N - (2N-2)q + (2N-2)q^2 - \cdots$$

- **Generators of $M_n H_n(P_N)$:**

$$\langle \overbrace{i, i+1, i, i+1, \cdots}^{n+1} \rangle, \quad \langle \overbrace{i, i-1, i, i-1, \cdots}^{n+1} \rangle.$$

A refinement of the magnitude homology

- The "end points" x_0, x_n of an n-chain $\langle x_0, \cdots, x_n \rangle$ are preserved by $\partial : M_\ell C_n(X) \to M_\ell C_{n-1}(X)$.
- This fact leads to the direct sum decomposition

$$M_\ell H_n(X) = \bigoplus_{x,y \in X} M_\ell H_n(x,y).$$

- We have the following equality in $\mathbb{Q}((q^{\mathbb{R}}))$

$$H_X^{-1}(x,y) = \sum_{\ell \geq 0} \left(\sum_{n \geq 0} (-1)^n \mathrm{rank} M_\ell H_n(x,y) \right) q^\ell.$$

- This can be seen as a series expansion of the Feynman propagator (two point function, Green function) in the theory of free scalar fields $\phi(x)$ on X with Lagrangian

$$\mathscr{L}(\phi) = \frac{1}{2} \langle \phi, H_X \phi \rangle.$$