# Twisted K-theory and finite-dimensional approximation

Kiyonori Gomi

**Problem in twisted $K$-theory**

Realize twisted $K$-theory generally by means of finite dimensional geometric objects.

**Main theorem**

We can define a group by means of "*twisted $\mathbb{Z}_2$-graded Hermitian general vector bundles*", into which there exists a monomorphism from twisted $K$-theory.

## Plan

§1 Twisted $K$-theory

§2 Hermitian general vector bundle

## §1 Twisted $K$-theory

Origin

P. Donovan and M. Karoubi (1970)

J. Rosenberg (1989)

Application

$D$-brane charges
[Witten, Kapustin, ...]

The Verlinde algebras
[Freed-Hopkins-Teleman]

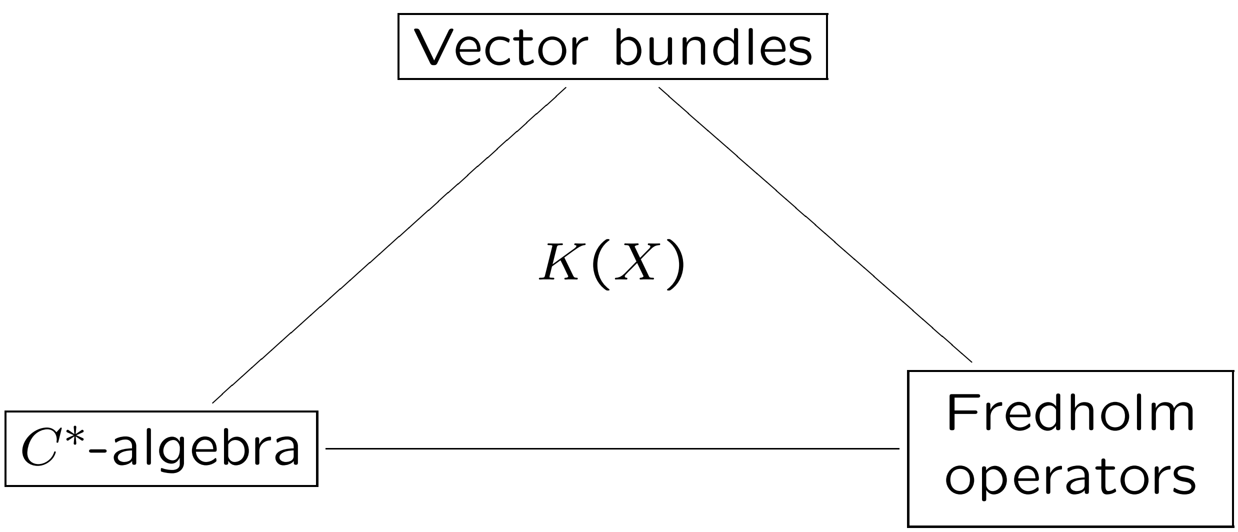
The quantum Hall effect
[Carey-Hannabuss-Mathai-McCann]

## $K$-theory  $X$ : compact

Vect($X$) = the isomorphism classes of finite dimensional vector bundles over $X$

**Definition**

$$K(X) = K(\text{Vect}(X))$$
$$= \text{Vect}(X) \times \text{Vect}(X)/\Delta(\text{Vect}(X))$$



# Fredholm operators

$\mathcal{H}$ : separable Hilbert space ($\dim\mathcal{H} = \infty$)

A Fredholm operator $f : \mathcal{H} \to \mathcal{H}$

$$\overset{\text{def}}{\Longleftrightarrow} \begin{cases} \text{bounded linear,} \\ \text{Image}(f) \subset \mathcal{H} : \text{closed,} \\ \dim\text{Ker}(f), \dim\text{Coker}(f) < \infty. \end{cases}$$

$\mathcal{F}(\mathcal{H}) = \{\text{Fredholm operators } f : \mathcal{H} \to \mathcal{H}\}$

**Fact [Atiyah, Jänich]**

$X$ : compact

$$C(X, \mathcal{F}(\mathcal{H}))/\text{htpy} \xrightarrow{\text{iso}} K(X)$$

## Twisted $K$-theory

$$PU(\mathcal{H}) = U(\mathcal{H})/U(1) \overset{\text{Ad}}{\curvearrowright} \mathcal{F}(\mathcal{H})$$

**Definition**

$P \to X$ : principal $PU(\mathcal{H})$-bundle

$$K(X;P) = \Gamma(X, P \times_{Ad} \mathcal{F}(\mathcal{H}))/\text{htpy}$$

- $P \cong X \times PU(\mathcal{H}) \Rightarrow K(X;P) \cong K(X)$.

- $\begin{cases} U(\mathcal{H}) \simeq \text{pt}, \\ PU(\mathcal{H}) \simeq K(\mathbb{Z}, 2), \\ BPU(\mathcal{H}) \simeq K(\mathbb{Z}, 3). \end{cases}$

  Principal $PU(\mathcal{H})$-bundles $P$ are classified by their Dixmier-Douady classes:

$$\delta(P) \in H^3(X;\mathbb{Z}).$$

Examples

$$\begin{cases} H^3(X;\mathbb{Z}) & \cong & \mathbb{Z}, \\ \delta(P) & = & k \neq 0. \end{cases}$$

$X = S^3$

$$K(S^3;k) \cong 0$$

$X = S^1 \times S^2$

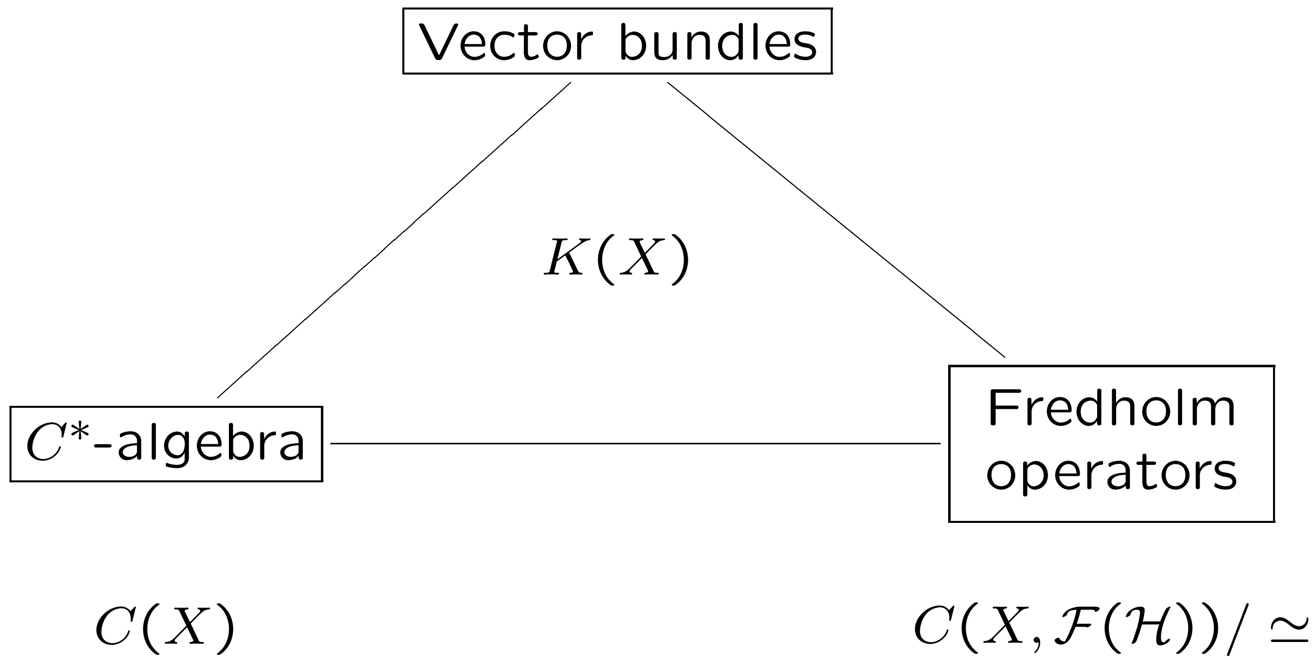
$$K(S^1 \times S^2;k) \cong \mathbb{Z}$$

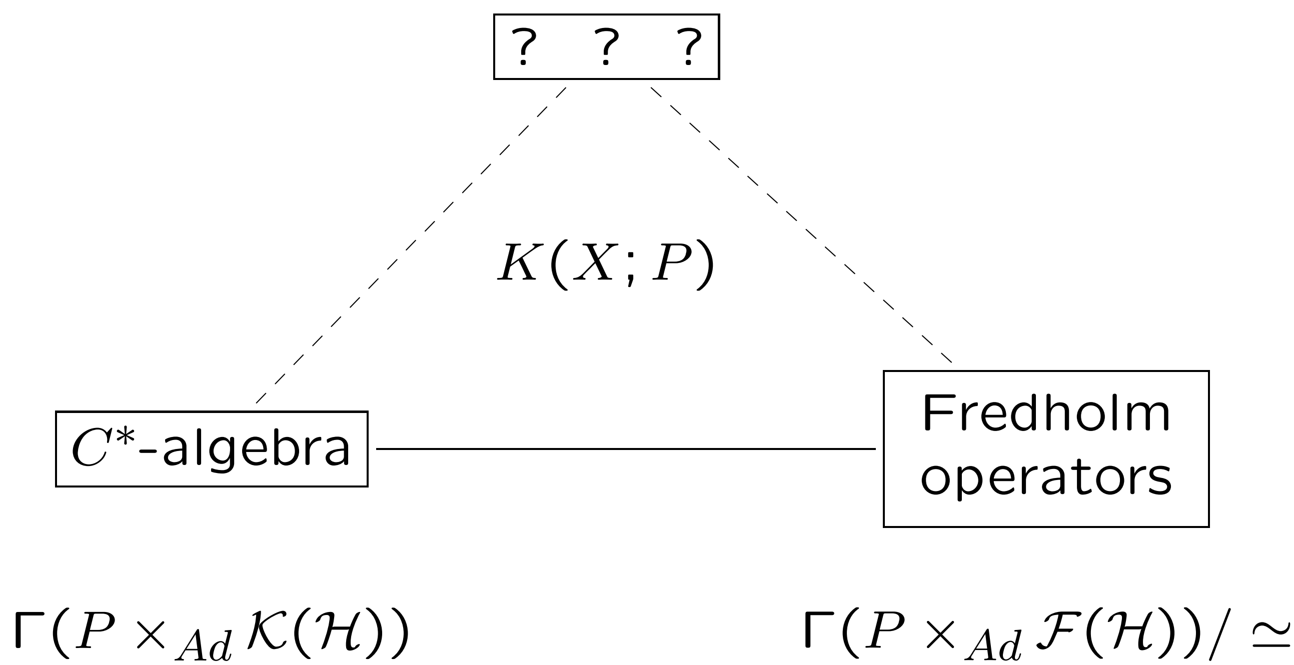
$X = S^3/\mathbb{Z}_p$, ($p$: prime)

$$K(S^3/\mathbb{Z}_p;k) \cong \mathbb{Z}_p$$

$X = SU(3)$

$$K(SU(3);k) \cong \begin{cases} \mathbb{Z}_k & k \text{ odd} \\ \mathbb{Z}_{k/2} & k \text{ even} \end{cases}$$


Vector bundles
$K(X)$
$C^*$-algebra
Fredholm operators
$C(X)$
$C(X, \mathcal{F}(\mathcal{H}))/\simeq$


? ? ?
$K(X; P)$
$C^*$-algebra
Fredholm operators
$\Gamma(P \times_{Ad} \mathcal{K}(\mathcal{H}))$
$\Gamma(P \times_{Ad} \mathcal{F}(\mathcal{H}))/\simeq$

**Problem**

Realize twisted $K$-theory generally by means of finite dimensional geometric objects.

$\delta(P)$ : finite order $\Rightarrow$ $^{\exists}$answer

**Fact**

$$\begin{cases} X & : \text{ compact} \\ P & : \delta(P) \text{ is finite order} \end{cases}$$

We can define a group by means of "*twisted vector bundles*", to which there exists an isomorphism from $K(X; P)$.

<u>Remark</u> There are a number of works on twisted vector bundles.

# Twisted vector bundle

- $\mathcal{U} = \{U_\alpha\}$ : open cover of $X$

- $(z_{\alpha\beta\gamma}) \in \check{Z}^2(\mathcal{U}, \underline{U(1)})$ : 2-cocycle representing $\delta(P) \in H^3(X, \mathbb{Z}) \cong H^2(X, \underline{U(1)})$.

twisted vector bundle $(E_\alpha, \phi_{\alpha\beta})$

$$\Leftrightarrow \begin{cases} E_\alpha \to U_\alpha & \text{finite rank vector bundle} \\ \phi_{\alpha\beta} : E_\alpha|_{U_{\alpha\beta}} \to E_\beta|_{U_{\alpha\beta}} & \text{isomorphism} \end{cases}$$

$$\phi_{\alpha\beta}\phi_{\beta\gamma} = z_{\alpha\beta\gamma}\phi_{\alpha\gamma}$$

<u>Remark</u> $(E_\alpha, \phi_{\alpha\beta})$ : rank $r \Rightarrow r \cdot \delta(P) = 0$.

$$(\det\phi_{\alpha\beta})(\det\phi_{\beta\gamma}) = (z_{\alpha\beta\gamma})^r(\det\phi_{\alpha\gamma})$$

## §2 Hermitian general vector bundle

M. Furuta, "*Index theorem, II*". (Japanese) Iwanami Series in Modern Mathematics. Iwanami Shoten, Publishers, Tokyo, 2002.

- to approximate Dirac-type operators;

  linear version of the finite dimensional approximation of the Seiberg-Witten equations

- to define $K(X)$.

**Theorem[Furuta]**

$X$ : compact

We can define a group by means of $\mathbb{Z}_2$-graded Hermitian general vector bundles, which is isomorphic to $K(X)$.

## Hermitian general vector bundle on $X$

$$(\mathcal{U}, (E_\alpha, h_\alpha), \phi_{\alpha\beta})$$

$$\begin{cases} \mathcal{U} = \{U_\alpha\} & \text{open cover of } X; \\ E_\alpha \to U_\alpha & \mathbb{Z}_2\text{-gr. Hermitian vector bundle;} \\ h_\alpha : E_\alpha \to E_\alpha & \text{Hermitian map of degree 1;} \\ \phi_{\alpha\beta} : E_\alpha|_{U_{\alpha\beta}} \to E_\beta|_{U_{\alpha\beta}} & \text{map of degree 0;} \end{cases}$$

1. "$h_\alpha \phi_{\alpha\beta} = \phi_{\alpha\beta} h_\beta$",

$$\left( \begin{array}{c} {}^\forall x \in U_{\alpha\beta};\ \begin{cases} x \in {}^\exists V \subset U_{\alpha\beta}, \\ {}^\exists \mu > 0, \end{cases} \text{such that :} \\ \begin{cases} {}^\forall y \in V, \\ {}^\forall v \in \bigoplus_{\lambda < \mu} \{v \in (E_\alpha)_y |\ h_\alpha^2 v = \lambda v\}, \end{cases} \\ \\ h_\alpha \phi_{\alpha\beta}(v) = \phi_{\alpha\beta} h_\beta(v). \end{array} \right)$$

2. "$\phi_{\alpha\beta}\phi_{\beta\alpha} = 1$",

3. "$\phi_{\alpha\beta}\phi_{\beta\gamma} = \phi_{\alpha\gamma}$".

Fredholm operator $f : \mathcal{H} \to \mathcal{H}$

$$\rightsquigarrow \text{approximate}$$

$(E, h)$

$$\begin{cases} E = E^0 \oplus E^1 & \mathbb{Z}_2\text{-gr. Herm. vector space} \\ h : E \to E & \text{Hermitian map of degree 1} \end{cases}$$

<u>Step 1</u>
$$\begin{cases} \widehat{\mathcal{H}} = \mathcal{H} \oplus \mathcal{H} & \mathbb{Z}_2\text{-graded} \\ \widehat{f} = \begin{pmatrix} 0 & f^* \\ f & 0 \end{pmatrix} & \text{self-adjoint, degree 1} \end{cases}$$

<u>Step 2</u> $\sigma(\widehat{f}^2) \ni 0$ : discrete $\Rightarrow {}^{\exists}\mu > 0$ s.t.

- $\mu \notin \sigma(\widehat{f}^2)$;
- $\sigma(\widehat{f}^2) \cap [0, \mu)$ consists of a finite number of eigenvalues: $0 = \lambda_1 < \lambda_2 < \cdots < \lambda_n < \mu$;
- $(\mathcal{H}, \widehat{f})_{\lambda_i} = \{v \in \widehat{\mathcal{H}} |\ \widehat{f}^2 v = \lambda_i v\}$ : finite dim.

( $(\mathcal{H}, \widehat{f})_0 = \mathrm{Ker}\widehat{f}^2 \cong \mathrm{Ker} f \oplus \mathrm{Coker} f$ )

$$
\begin{array}{ccc}
\widehat{\mathcal{H}} & \xrightarrow{\widehat{f}} & \widehat{\mathcal{H}} \\
\| & & \| \\
(\widehat{\mathcal{H}}, \widehat{f})_0 & \xrightarrow{0} & (\widehat{\mathcal{H}}, \widehat{f})_0 \\
\oplus & & \oplus \\
(\widehat{\mathcal{H}}, \widehat{f})_{\lambda_2} & \cong & (\widehat{\mathcal{H}}, \widehat{f})_{\lambda_2} \\
\oplus & & \oplus \\
(\widehat{\mathcal{H}}, \widehat{f})_{\lambda_3} & \cong & (\widehat{\mathcal{H}}, \widehat{f})_{\lambda_3} \\
\oplus & & \oplus \\
(\widehat{\mathcal{H}}, \widehat{f})_{\lambda_4} & \cong & (\widehat{\mathcal{H}}, \widehat{f})_{\lambda_4} \\
\oplus & & \oplus \\
\vdots & & \vdots \\
\oplus & & \oplus \\
(\widehat{\mathcal{H}}, \widehat{f})_{\lambda_n} & \cong & (\widehat{\mathcal{H}}, \widehat{f})_{\lambda_n} \\
\oplus & & \oplus \\
\text{complement} & \cong & \text{complement}
\end{array}
$$

Step 3 Put $\begin{cases} E = \oplus_{\lambda<\mu}(\widehat{\mathcal{H}}, \widehat{f})_\lambda, \\ h = \widehat{f}|_E. \end{cases}$

Remark $\{\widehat{f}_x : \widehat{\mathcal{H}} \to \widehat{\mathcal{H}}\}_{x\in U}$ : family

- dimKer$\widehat{f}_x^2$ may jump.

- $\mu \notin \sigma(\widehat{f}_{x_0}^2)$
  $\Rightarrow$ dim $\bigoplus_{\lambda<\mu} (\widehat{\mathcal{H}}, \widehat{f}_x)_\lambda$ is constant near $x_0$.

family $\{f_x : \mathcal{H} \to \mathcal{H}\}_{x\in X}$

$\rightsquigarrow$ approximate

$(\mathcal{U}, (E_\alpha, h_\alpha), \phi_{\alpha\beta})$

$\mathbb{Z}_2$-gr. Herm. general vector bundle on $X$

## Main theorem

$$\begin{cases} X: & \text{compact} \\ P: & PU(\mathcal{H})\text{-bundle} \end{cases}$$

We can define a group by means of *twisted* $\mathbb{Z}_2$-graded Hermitian general vector bundles, into which there exists a *monomorphism* from $K(X;P) = \Gamma(P \times_{Ad} \mathcal{F}(\mathcal{H}))/\simeq$.

- twisting $\Leftarrow$ "$\phi_{\alpha\beta}\phi_{\beta\gamma} = z_{\alpha\beta\gamma}\phi_{\alpha\gamma}$"
- monomorphism $\Leftarrow$ finite dimensional approximation