Topics in K-theory of operator algebras: Dadarlat-Pennig's generalization of Dixmier-Douady theory and group actions on Kirchberg algebras I, II

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C*-algebras

H: Hilbert space.
$\mathbb{B}(H)$: the set of bounded operators on H.

- $\mathbb{B}(H)$ is an algebra over $\mathbb{C}$.
- $\mathbb{B}(H)$ is a Banach space w.r.t. operator norm

$$\|T\| = \sup_{\xi \in H \setminus \{0\}} \frac{\|T\xi\|}{\|\xi\|}.$$

- $\mathbb{B}(H)$ has a $*$-operation $T \mapsto T^*$, where $\langle T\xi, \eta\rangle = \langle \xi, T^*\eta\rangle$, $\forall \xi, \eta \in H$.

Definition

A C*-algebra is a subalgebra of $\mathbb{B}(H)$ closed under the norm topology and the $*$-operation.

Theorem (Gelfand-Naimark)

A Banach $*$-algebra A satisfying the C-condition $\|T^*T\| = \|T\|^2$ for any $T \in A$ is isomorphic to a C*-algebra.*

Examples of C*-algebras

Example

- $\mathbb{B}(H)$, $\mathbb{M}_n = \mathbb{B}(\mathbb{C}^n)$.
- $C(X)$ with $\|f\| = \max_{x \in X} |f(x)|$ and $f^*(x) = \overline{f(x)}$.
  Here $C(X)$ is the set of continuous functions on a compact Hausdorff space X.

In what follows,
H is a separable infinite dimensional Hilbert space, and
X is a compact metric space (often a finite CW-complex).

Example

A: C*-algebra,
The set of continuous A-valued functions on X, denoted by $C(X, A)$, is a C*-algebra with pointwise operations and $\|f\| = \max_{x \in X} \|f(x)\|$.

$C(X, A) \cong C(X) \otimes A$.

(Locally trivial) Continuous fields of C*-algebras

Let A be a C*-algebra.

For a fiber bundle $\pi : E \to X$ with a fiber A and structure group $\mathrm{Aut}(A)$, the set of continuous sections $\Gamma(E)$ is a C*-algebra with fiberwise operations and norm $\|s\| = \max_{x \in X} \|s(x)\|$.

We call it a locally trivial continuous field of A over X.

Example

$C(X, A) = \Gamma(E)$ for a trivial bundle E.

Whenever we discuss isomorphisms between (locally trivial) continuous fields over X, we assume that they are $C(X)$-module maps.

We denote by $\mathfrak{Bun}_X(A)$ the set of isomorphism classes of locally trivial fields of A over X.

The C*-algebra $\mathbb{K}$

The set of compact operators $\mathbb{K}$ on H is a C*-algebra.

Definition

A C*-algebra A is said to be stable if A is isomorphic to $A \otimes \mathbb{K}$.
C*-algebras A and B are stably isomorphic if $A \otimes \mathbb{K} \cong B \otimes \mathbb{K}$.

Since $H \otimes H \cong H$, we have $\mathbb{K} \otimes \mathbb{K} \cong \mathbb{K}$, and $A \otimes \mathbb{K}$ is always stable for any C*-algebra A.

We have a short exact sequence

$$0 \to \mathbb{T} \to \mathcal{U}(H) \to \mathrm{Aut}(\mathbb{K}) \to 0, \tag{1}$$

with $\mathcal{U}(H) \ni U \mapsto \mathrm{Ad}\, U \in \mathrm{Aut}(\mathbb{K})$, where $\mathcal{U}(H)$ is the set of unitary operators on H equipped with the strong operator topology, which is contractible.

Dixmier-Douady theorem

Let $A = \Gamma(E)$, $B = \Gamma(E')$ be locally trivial continuous fields of $\mathbb{K}$ over X.

Then $A \otimes_{C(X)} B = \Gamma(E \otimes E')$, which is also a locally trivial continuous field of $\mathbb{K}$ over X as $\mathbb{K} \otimes \mathbb{K} \cong \mathbb{K}$.

Theorem (Dixmier-Douady 1963)

The set $\mathfrak{Bun}_X(\mathbb{K})$ of isomorphism classes of locally trivial fields of $\mathbb{K}$ over X forms an abelian group isomorphic to $H^3(X;\mathbb{Z})$ under operation of tensor product over $C(X)$.

The Dixmier-Douady class $\delta(A) \in H^3(X;\mathbb{Z})$ for $A = \Gamma(E)$ is a characteristic class of the fiber bundle $p : E \to X$ with fiber $\mathbb{K}$.

Homotopy theory approach

Recall that $\mathbb{T}$ is an Eilenberg-MacLane space of type $K(\mathbb{Z}, 1)$, i.e.

$$\pi_i(\mathbb{T}) = \begin{cases} \{0\}, & i \neq 1 \\ \mathbb{Z}, & i = 1 \end{cases} .$$

Since $\mathcal{U}(H)$ is contractible, $\mathcal{U}(H) \to \mathrm{Aut}(\mathbb{K}) \cong \mathcal{U}(H)/\mathbb{T}$ is a universal principal $\mathbb{T}$-bundle and $\mathrm{Aut}(\mathbb{K})$ is a classifying space of $\mathbb{T}$, i.e. $\mathcal{U}(H) = E\mathbb{T}$, $\mathrm{Aut}(\mathbb{K}) = B\mathbb{T}$, and in particular $\mathrm{Aut}(\mathbb{K})$ is a $K(\mathbb{Z}, 2)$ space.

The classifying space $B\,\mathrm{Aut}(\mathbb{K})$ is a $K(\mathbb{Z}, 3)$ space, and

$$\mathfrak{Bun}_X(\mathbb{K}) \cong [X, B\,\mathrm{Aut}(\mathbb{K})] = [X, K(\mathbb{Z}, 3)] \cong H^3(X; \mathbb{Z}).$$

Goal of the talk

For any strongly self-absorbing C*-algebra D, the homotopy set $[X, B\,\mathrm{Aut}(D \otimes \mathbb{K})]$ gives the first group $E^1_D(X)$ of a generalized cohomology theory $E^*_D(X)$ (Dadarlat-Pennig 2016).

K_0-groups

For a C*-algebra A, set $\mathcal{P}(A) := \{p \in A;\ p = p^2 = p^*\}$.

Recall that $p \in \mathcal{P}(C(X, \mathbb{K})) = \mathcal{P}(C(X) \otimes \mathbb{K})$ gives a vector bundle:

$$\bigsqcup_{x \in X} p(x)H \to X.$$

Definition (Murray-von Neumann equivalence)

For $p, q \in A$, we set $p \sim q$ if $\exists v \in A$ s.t. $v^*v = p$ and $vv^* = q$.

Since $\mathbb{K} \otimes \mathbb{M}_2 = \mathbb{K}(H \otimes \mathbb{C}^2) \cong \mathbb{K}$, the set $\mathcal{P}(A \otimes \mathbb{K})/\sim$ is a semigroup with

$$p \oplus q = \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix} \in \mathcal{P}(A \otimes \mathbb{K} \otimes \mathbb{M}_2) \cong \mathcal{P}(A \otimes \mathbb{K}).$$

Definition (Unital case)

$K_0(A)$ is the Grothendieck group of $\mathcal{P}(A \otimes \mathbb{K})/\sim$.
We denote by $K_0(A)_+$ the image of $\mathcal{P}(A \otimes \mathbb{K})/\sim$ in $K_0(A)$.

K_1-groups

For a unital C*-algebra A, set $\mathcal{U}(A) := \{u \in A;\ u^*u = uu^* = 1\}$, and $\mathcal{U}_n(A) = \mathcal{U}(A \otimes \mathbb{M}_n)$.

Recall $K^1(X) = [X, U(n)]$ for sufficiently large n.
Since $\mathrm{Map}(X, U(n)) = \mathcal{U}(C(X, \mathbb{M}_n)) = \mathcal{U}(C(X) \otimes \mathbb{M}_n) = \mathcal{U}_n(C(X))$, we have $[X, U(n)] = \mathcal{U}_n(C(X))/\mathcal{U}_n(C(X))_0$.

Definition

$K_1(A)$ is the inductive limit $\varinjlim \mathcal{U}_n(A)/\mathcal{U}_n(A)_0$ with a connection map $\mathcal{U}_n(A) \ni u \mapsto u \oplus 1 \in U_{n+1}(A)$.

$K_*(C(X)) = K^*(X)$.
$K_*(A) = K_*(A \otimes \mathbb{M}_n) = K_*(A \otimes \mathbb{K})$.

Example

For $A = \mathbb{M}_n$, $(K_0(A), K_0(A)_+, [1]_0, K_1(A)) = (\mathbb{Z}, \mathbb{Z}_+, n, \{0\})$.

Non-unital case

For a based space (X, x_0), denote $C(X, x_0) = \{f \in C(X);\ f(x_0) = 0\}$.

Recall $\tilde{K}^i(X) = \ker(\iota^* : K^i(X) \to K^i(\{x_0\}))$.
We need a definition of $K_i(A)$ satisfying $K_i(C(X, x_0)) = \tilde{K}^i(X)$.

Definition (Non-unital case)

For a non-unital C*-algebra A, set $K_i(A) := \ker(\pi_* : K_i(A + \mathbb{C}1) \to K_i(\mathbb{C}))$.

For a locally compact Hausdorff space Y, set $C_0(Y) = C(Y \cup \{\infty\}, \infty)$.
The suspension of A is $SA := A \otimes C_0(0,1) \cong A \otimes C_0(\mathbb{R})$.

$SC(X, x_0) = C(\Sigma X, \Sigma x_0)$ where $\Sigma X = (X \times I)/(\{x_0\} \times I \cup X \times \partial I)$.

Theorem (Bott periodicity)

$K_0(SA) \cong K_1(A)$ *and* $K_1(SA) \cong K_0(A)$.

Twisted K-theory

Recall $K_*(C(X,\mathbb{K})) = K^*(X)$.

What happens if the trivial continuous field $C(X,\mathbb{K})$ of $\mathbb{K}$ over X is replaced with a locally trivial one?

Definition (J. Rosenberg 1988)

The twisted K-theory $K^{\tau+*}(X)$ with $\tau \in H^3(X,\mathbb{Z})$ is defined by $K_*(A)$, where A is a locally trivial continuous field of $\mathbb{K}$ over X whose Dixmier-Douady class $\delta(A)$ is τ (cf. Donovan-Karoubi 1970).

Kasparov KK-theory

$KK(A, B)$ is a bivariant functor from the category of (separable) C*-algebras to the category of abelian groups satisfying

- $\exists$ associative product $KK(A, B) \times KK(B, C) \to KK(A, C)$.
- $KK(\mathbb{C}, B) = K_0(B)$.
- $KK(A, \mathbb{C}) = K^0(A)$, where $K^0(C(X)) = K_0(X)$.
- If A is in a bootstrap category $\mathcal{N}$ of Rosenberg-Schochet, the UCT sequence

$$0 \to \mathrm{Ext}(K(A), K(B)) \to KK(A, B) \to \mathrm{Hom}(K(A), K(B)) \to 0$$

  is exact.

If A and B are Kirchberg algebras,

$$KK(A, B) \cong \mathrm{Hom}(A \otimes \mathbb{K}, B \otimes \mathbb{K}) / \sim$$

where $\rho \sim \sigma$ iff $\exists \{u_t\}_{t \in [0,\infty)}$ in $\mathcal{U}(B \otimes \mathbb{K} + \mathbb{C}1)$ s.t. $\lim_{t \to \infty} u_t \rho(a) u_t^* = \sigma(a)$.
ρ and σ as above is said to be asymptotically unitarily equivalent.

KK-equivalence

Definition

A and B are KK-equivalent if $\exists x \in KK(A, B)$, $\exists y \in KK(B, A)$ s.t. $x \hat{\otimes} y = KK(\mathrm{id}_A) \in KK(A, A)$ and $y \hat{\otimes} x = KK(\mathrm{id}_B) \in KK(B, B)$.

A, $A \otimes \mathbb{M}_n$, and $A \otimes \mathbb{K}$ are mutually KK-equivalent.

$\mathbb{C}$ and $S^2\mathbb{C} = C_0(\mathbb{R}^2) = C_0(\mathbb{C})$ are KK-equivalent.
Define $p_B \in \mathcal{P}((C_0(\mathbb{C}) + \mathbb{C}1) \otimes \mathbb{M}_2) = \mathcal{P}(C(\mathbb{C}P^1) \otimes \mathbb{M}_2)$ by

$$p_B(z) = \frac{1}{1 + |z^2|} \begin{pmatrix} |z|^2 & z \\ \overline{z} & 1 \end{pmatrix}.$$

Then $x = [p_B]_0 - [1]_0 \in K_0(C_0(\mathbb{R}^2)) = KK(\mathbb{C}, C_0(\mathbb{R}^2))$ is an invertible element.

The UHF algebras

For a sequence $\{n_k\}_{k=1}^{\infty}$ of integers greater than 1, we set

$$A_m = \mathbb{M}_{n_1} \otimes \mathbb{M}_{n_2} \otimes \cdots \otimes \mathbb{M}_{n_m} \cong \mathbb{M}_n, \quad n = \prod_{k=1}^{m} n_k.$$

Embedding A_m into A_{m+1} by $a \mapsto a \otimes 1_{\mathbb{M}_{n_{m+1}}}$, we get an inductive system of C*-algebras $\{A_m\}_{m=1}^{\infty}$.

The norm completion of its inductive limit is a C*-algebra, called the UHF algebra, and denoted by $\bigotimes_{k=1}^{\infty} \mathbb{M}_{n_k}$.

A UHF algebra A is said to be of infinite type if $A \otimes A \cong A$, e.g.
$\mathbb{M}_{n^\infty} = \bigotimes_{k=1}^{\infty} \mathbb{M}_n$,
$\mathbb{M}_{\mathbb{Q}} = \bigotimes_{k=1}^{\infty} \mathbb{M}_{k!}$. The universal UHF algebra.

UHF algebras of infinite type are strongly self-absorbing.

The classification of UHF algebras

UHF algebras are completely classified by the supernatural number

$$\prod_k^\infty n_k = \prod_{p:\text{prime}} p^{a_p}, \quad a_p = 0, 1, 2, \ldots, \infty,$$

or more precisely, the set $\{a_p\}_p$ is a complete invariant (Glimm 1960).

Definition

For a C*-algebra A, a linear functional $\tau : A \to \mathbb{C}$ is a trace if $\|\tau\| = \tau(1) = 1$ and $\tau(xy) = \tau(yx)$ for $\forall x, y \in A$.

The normalized trace $\tau_m = \frac{1}{n_1 n_2 \cdots n_m} \mathrm{Tr}$ on $\bigotimes_{k=1}^m \mathbb{M}_{n_k}$ extends to a unique trace τ on $\bigotimes_{k=1}^\infty \mathbb{M}_{n_k}$, and

$$\{\tau(p);\ p \in \mathcal{P}(\bigotimes_{k=1}^\infty \mathbb{M}_{n_k})\} = (\bigcup_{m=1}^\infty \frac{1}{n_1 n_2 \cdots n_m} \mathbb{Z}) \cap [0,1].$$

Elliott Theorem

An inductive limit of finite dimensional C*-algebras is said to be AF algebra.

Theorem (Elliott 1976)

The isomorphism class of an AF algebras A is completely determined by $(K_0(A), K_0(A)_+, [1]_0)$.

Since $K_1(\mathbb{M}_n) = \{0\}$, the K_1-group is trivial for any AF algebra.

For $A = \bigotimes_{k=1}^{\infty} \mathbb{M}_{n_k}$,

$$(K_0(A), K_0(A)^+, [1]_0) = (\bigcup_{m=1}^{\infty} \frac{1}{n_1 n_2 \cdots n_m}\mathbb{Z}, K_0(A) \cap \mathbb{R}_+, 1).$$

$K_0(\mathbb{M}_{n^\infty}) = \mathbb{Z}[\frac{1}{n}]$.
$K_0(\mathbb{M}_{\mathbb{Q}}) = \mathbb{Q}$.
They have ring structure.

The Cuntz algebras $\mathcal{O}_2$ and $\mathcal{O}_\infty$

The Cuntz algebra $\mathcal{O}_n$ for $n \geq 2$ is the universal C*-algebra with generators $\{S_i\}_{i=1}^n$ and relations

$$S_i^* S_j = \delta_{i,j} 1, \quad \sum_{i=1}^{n} S_i S_i^* = 1.$$

For $n = \infty$, we define the Cuntz algebra $\mathcal{O}_\infty$ by imposing only the first relation.

Their K-groups are

$$K_0(\mathcal{O}_n) = \begin{cases} \mathbb{Z}/(n-1)\mathbb{Z}, & n < \infty \\ \mathbb{Z}, & n = \infty \end{cases}, \quad K_1(\mathcal{O}_n) = \{0\}.$$

$K^*(X; \mathbb{Z}_p) \cong K_*(C(X) \otimes \mathcal{O}_{p+1})$.

The Cuntz algebras $\mathcal{O}_2$ and $\mathcal{O}_\infty$ are strongly self-absorbing.

Kirchberg-Phillips Theorem

The Cuntz algebras are typical examples of Kirchberg algebras, separable nuclear simple purely infinite C*-algebras.

Theorem (Phillips 2000)

Any two Kirchberg algebras A and B are stably isomorphic iff they are KK-equivalent.
If moreover they are in the bootstrap class $\mathcal{N}$ of Rosenberg-Schochet, they are isomorphic iff

$$(K_0(A), [1_A]_0, K_1(A)) \cong (K_0(B), [1_B]_0, K_1(B)).$$

$\mathcal{O}_2$ is KK-equivalent to $\{0\}$, and it plays the role of 0, i.e. $\mathcal{O}_2 \otimes A \cong \mathcal{O}_2$ for any unital simple separable nuclear C^*-algebra A.

$\mathcal{O}_\infty$ is KK-equivalent to $\mathbb{C}$, and it plays the role of 1, i.e. $\mathcal{O}_\infty \otimes B \cong B$ for any Kirchberg algebra B.

Jiang-Su algebra $\mathcal{Z}$

The Jiang-Su algebra $\mathcal{Z}$ is a mysterious simple stably finite C*-algebra without having projections other than 0 and 1.

$\mathcal{Z}$ is KK-equivalent to $\mathbb{C}$, having $K_0(\mathcal{Z}) = \mathbb{Z}$ and $K_1(\mathcal{Z}) = \{0\}$.

$\mathcal{Z}$ is absorbed by every strongly self-absorbing C*-algebra by tensor product.

In fact, whether a given C*-algebra absorbs $\mathcal{Z}$ or not is the most important criterion of classifiability of it.

$\mathcal{Z}$ is an inductive limit of building blocks $I_{p,q}$ where p and q are mutually prime natural numbers and

$$I_{p,q} = \{f \in C([0,1], \mathbb{M}_p \otimes \mathbb{M}_q);\ f(0) \in \mathbb{M}_p \otimes \mathbb{C}1,\ f(0) \in \mathbb{C}1 \otimes \mathbb{M}_q\}.$$

Strongly self-absorbing C*-algebras

The notion of strongly self-absorbing C*-algebras was introduced by Toms-Winter in 2007 to single out the class of C*-algebras playing distinguished roles in the classification of nuclear C*-algebras.

Definition (Toms-Winter, 2007)

A C*-algebra D with unit is said to be strongly self-absorbing if there exist an isomorphism $\psi : D \to D \otimes D$ and a sequence of unitaries $\{U_n\}_{n=1}^{\infty} \subset \mathcal{U}(D \otimes D)$ such that for any $T \in D$, and we have

$$\lim_{n\to\infty} U_n(T \otimes 1)U_n^* = \psi(T).$$

The sequence $\{U_n\}_{n=1}^{\infty}$ can be replaced by a continuous family $\{U(t)\}_{t\in[0,\infty)}$ with $U(0) = 1$, and ψ is arbitrary once it exists.

Basic properties

Theorem

Let D be a strongly self-absorbing C-algebra.*

(1) *D is simple nuclear either stably finite or purely infinite; if it is stably finite, then it admits a unique trace.*

(2) *$K_0(D)$ has a ring structure with unit $[1]$ given by $[p][q] = [\psi^{-1}(p \otimes q)] \in K_0(A)$ for any projections $p, q \in D$.*

(3) *$K_1(D) = \{0\}$ if D is in the bootstrap class $\mathcal{N}$ of Rosenberg-Schochet.*

(4) *$\mathrm{Aut}(D)$ is contractible.*

(5) *$\mathrm{Aut}_0(D \otimes \mathbb{K})$ has homotopy type of a CW-complex.*

All the known strongly self-absorbing C*-algebras are in the following list: $\mathbb{C}$, the Cuntz algebras $\mathcal{O}_2$ and $\mathcal{O}_\infty$, the UHF algebras of infinite type, the Jiang-Su algebra $\mathcal{Z}$, and the tensor product of the Cuntz algebra $\mathcal{O}_\infty$ and the UHF algebras of infinite type.

Proof of (4) of $\mathrm{Aut}(D)$

Lemma

If there exists a continuous path $\{\psi\}_{t\in[0,1]}$ *in* $\mathrm{Hom}(D, D \otimes D)$ *satisfying* $\psi_0(x) = x \otimes 1$, $\psi_1(x) = 1 \otimes x$, *and* ψ_t *is an isomorphism for any* $0 < t < 1$, *then* $\mathrm{Aut}(D)$ *is contractible.*

Proof.

For $\alpha \in \mathrm{Aut}(D)$, set

$$H(\alpha, t) = \begin{cases} \alpha, & t = 0, \\ \psi_t^{-1} \circ (\alpha \otimes \mathrm{id}) \circ \psi_t, & 0 < t < 1, \\ \mathrm{id}, & t = 1 \end{cases} .$$

□

Infinite loop space

For a pointed topological space E, we denote by ΩE its loop space i.e.

$$\Omega E = \{f \in \mathrm{Map}([0,1], E);\ f(0) = f(1) = *\}.$$

Definition

A pointed topological space $E = E_0$ is said to be an infinite loop space if there exists a sequence of pointed topological spaces $\{E_n\}_{n=1}^{\infty}$ such that ΩE_n is homotopy equivalent to E_{n-1}.
Such a sequence is called an Ω-spectrum.

A Ω-spectrum $\{E_n\}_{n=0}^{\infty}$ gives rise to a generalized cohomology via the homotopy sets $[X, E_n]$.

Example

$E_n = K(A, n)$, $[X, E_n] = H^n(X; A)$.

Dadarlat-Pennig theorem

Theorem (Dadarlat-Pennig, 2016)

Let X be a compact metrizable space, and let D be a strongly self-absorbing C^*-algebra.
The set $\mathfrak{Bun}_X(D \otimes \mathbb{K})$ of isomorphism classes of locally trivial fields of $D \otimes \mathbb{K}$ over X is an abelian group under operation of tensor product over $C(X)$.
Moreover, the group is isomorphic to $E^1_D(X)$, the first group of a generalized connective cohomology theory $E^*_D(X)$ defined by the infinite loop space $B\,\mathrm{Aut}(D \otimes \mathbb{K})$.

There exists an Ω-spectrum $\{E_n\}$ with $E_0 = \mathrm{Aut}(D \otimes \mathbb{K})$, $E_1 = B\,\mathrm{Aut}(D \otimes \mathbb{K})$, and $E^n_D(X) = [X, E_n]$.

Symmetric monoidal category $\mathcal{B}_\otimes$

Let $\mathcal{B}_\otimes$ be a topological category whose objects are $\mathbb{Z}_+$ (n is identified with $(D \otimes \mathbb{K})^{\otimes n}$) and morphsims are

$$\mathrm{Hom}(0, n) = \{p \in \mathcal{P}((D \otimes \mathbb{K})^{\otimes n});\ [p]_0 \in K_0(D^{\otimes n})_+^\times\},$$

$$\mathrm{Hom}(m, n) = \{\rho \in \mathrm{Hom}((D \otimes \mathbb{K})^{\otimes m}), (D \otimes \mathbb{K})^{\otimes n});\ KK(\rho) \text{ is invertible}\},$$

for $m \geq 1$.

$\mathcal{B}_\otimes$ is a symmetric monoidal category with $m \otimes n = m + n$.

Note $\mathrm{Aut}(D \otimes \mathbb{K}) \subset \mathrm{Hom}(1, 1) = \{\rho \in \mathrm{End}(D \otimes \mathbb{K}); KK(\rho) \text{ is invertible}\}$.

$B\,\mathrm{Aut}(D \otimes \mathbb{K}) \cong B\mathcal{B}_\otimes$.

Coefficients

The E_2-page of the Atiyah-Hirzebruch spectral sequence for $E_D^*(X)$ is given by

$$E_2^{p,q} = H^p(X, E_D^q(pt)) = H^p(X, \pi_{-q}(\mathrm{Aut}(D \otimes \mathbb{K}))).$$

Theorem (Dadarlat-Pennig, 2016)

Let D be a strongly self-absorbing C-algebra not isomorphic to $\mathbb{C}$ satisfying the UCT. Then*

$$\pi_{2i}(\mathrm{Aut}(D \otimes \mathbb{K})) \cong \begin{cases} K_0(D)_+^\times, & i = 0 \\ K_0(D), & i \geq 1 \end{cases},$$

$$\pi_{2i-1}(\mathrm{Aut}(D \otimes \mathbb{K})) \cong \{0\}.$$

Recall that $K_0(D)$ has a ring structure, which also has a positive cone $K_0(D)_+$ generated by the classes represented by projections.

	$\mathcal{O}_2$	$\mathcal{O}_\infty$	$\mathcal{O}_\infty \otimes M_{n^\infty}$	$\mathcal{O}_\infty \otimes M_{\mathbb{Q}}$	M_{n^∞}	$M_{\mathbb{Q}}$	$\mathcal{Z}$
$K_0(D)$	$\{0\}$	$\mathbb{Z}$	$\mathbb{Z}[\frac{1}{n}]$	$\mathbb{Q}$	$\mathbb{Z}[\frac{1}{n}]$	$\mathbb{Q}$	$\mathbb{Z}$
$K_0(D)_+^\times$	$\{0\}$	$\{1, -1\}$	$\mathbb{Z}[\frac{1}{n}]^\times$	$\mathbb{Q}^\times$	$\mathbb{Z}[\frac{1}{n}]_+^\times$	$\mathbb{Q}_+^\times$	$\{1\}$

Examples

Since the differentials of the Atiyah-Hirzebruch spectral sequence are known to be torsion operators, we get

Corollary

(1) *For a connected compact metrizable space* X*,*

$$\mathfrak{Bun}_X(M_{\mathbb{Q}} \otimes \mathbb{K}) \cong H^1(X, \mathbb{Q}_+^\times) \oplus \bigoplus_{k \geq 1} H^{2k+1}(X, \mathbb{Q}),$$

$$\mathfrak{Bun}_X(\mathcal{O}_\infty \otimes M_{\mathbb{Q}} \otimes \mathbb{K}) \cong H^1(X, \mathbb{Q}^\times) \oplus \bigoplus_{k \geq 1} H^{2k+1}(X, \mathbb{Q}).$$

(2) *For a connected finite CW-complex* X *with* $H^*(X, \mathbb{Z})$ *torsion free,*

$$\mathfrak{Bun}_X(\mathcal{Z} \otimes \mathbb{K}) \cong \bigoplus_{k \geq 1} H^{2k+1}(X, \mathbb{Z}),$$

$$\mathfrak{Bun}_X(\mathcal{O}_\infty \otimes \mathbb{K}) \cong H^1(X, \mathbb{Z}/2\mathbb{Z}) \oplus \bigoplus_{k \geq 1} H^{2k+1}(X, \mathbb{Z}).$$

$[X, \mathrm{Aut}(D \otimes \mathbb{K})]$

Theorem

Assume X is connected and D is a strongly self-absorbing. Then

$$[X, \mathrm{Aut}(D \otimes \mathbb{K})] \cong K_0(C(X) \otimes D)_+^\times.$$

$$[X, \mathrm{Aut}(\mathcal{Z} \otimes \mathbb{K})] \cong (1 + \widetilde{K}^0(X))^\times,$$
$$[X, \mathrm{Aut}(\mathcal{O}_\infty \otimes \mathbb{K})] \cong (\pm 1 + \widetilde{K}^0(X))^\times.$$

$[X, \mathrm{Aut}(A)]$ for Kirchberg algebras

Theorem (Dadarlat 2007)

For a uital Kirchberg algebra A, there are bijection

$$\chi : [X, \mathrm{Aut}(A)_0] \to KK(C_\nu A, SC(X, x_0) \otimes A),$$

$$\overline{\chi} : [X, \mathrm{Aut}(A \otimes \mathbb{K})] \to KK(A, C(X, x_0) \otimes A),$$

where $C_\nu A = \{f \in C([0,1], A);\ f(0) \in \mathbb{C}1_A,\ f(1) = 0\}$ is the mapping cone of the inclusion map $\nu : \mathbb{C} \to A$.
If moreover (X, x_0) is a H'-space, they are group isomorphisms.

The exact sequence $0 \to SA \to C_\nu A \to \mathbb{C} \to 0$ implies the exact sequence

$$KK(A, SC(X, x_0) \otimes A) \xrightarrow{\nu^*} K_1(C(X, x_0) \otimes A) \to KK(C_\nu A, SC(X, x_0) \otimes A)$$

$$\to KK(A, C(X, x_0) \otimes A) \xrightarrow{\nu^*} K_0(C(X, x_0) \otimes A) \to \cdots$$

$[X, \mathrm{Aut}(\mathcal{O}_{n+1})]$

For the Cuntz algebra $\mathcal{O}_{n+1}$, we have $[X, \mathrm{Aut}(\mathcal{O}_{n+1})] \cong \widetilde{K}^1(X; \mathbb{Z}_n)$ as sets.

Recall the universal coefficient exact sequence:

$$0 \to \widetilde{K}^1(X) \otimes \mathbb{Z}_n \to \widetilde{K}(X; \mathbb{Z}_n) \xrightarrow{\delta} \mathrm{Tor}(\widetilde{K}^0(X), \mathbb{Z}_n) \to 0.$$

Theorem (I, 2018)

$[X, \mathrm{Aut}(\mathcal{O}_{n+1})]$ *is isomorphic to* $\widetilde{K}^1(X; \mathbb{Z}_n)$ *equipped with group structure*

$$x \circ y = x + y - x\delta(y).$$

In particular, $[X, \mathrm{Aut}(\mathcal{O}_{n+1})]$ *is a group extension*

$$0 \to \widetilde{K}^1(X) \otimes \mathbb{Z}_n \to [X, \mathrm{Aut}(\mathcal{O}_{n+1})] \to (1 + \mathrm{Tor}(\widetilde{K}^0(X), \mathbb{Z}_n))^{\times} \to 0.$$

Since $y^{-1}xy = x(1 - \delta(y))$ for $x \in \widetilde{K}^1(X; \mathbb{Z}_n)$, $y \in \widetilde{K}^1(X) \otimes \mathbb{Z}_n$, if $(\widetilde{K}^1(X) \otimes \mathbb{Z}_n)\mathrm{Tor}(\widetilde{K}^0(X), \mathbb{Z}_n)) \neq \{0\}$, the group $[X, \mathrm{Aut}(\mathcal{O}_{n+1})]$ is non-commutative.

$[X, B\,\mathrm{Aut}(\mathcal{O}_{n+1})]$

Theorem (Dadarlat 2012+Sogabe-I. 2018)

If $H^*(X;\mathbb{Z})$ has no n-torsion, $\#[X, B\,\mathrm{Aut}(\mathcal{O}_{n+1})] = \#\widetilde{K}^0(X) \otimes \mathbb{Z}_n$.

The canonical map $U(n+1) \to \mathrm{Aut}(\mathcal{O}_{n+1})$ induces a surjetion $[X, BU(n+1)] \to [X, B\,\mathrm{Aut}(\mathcal{O}_{n+1})]$ if n is sufficiently large compared to $\dim X$.

Group actions

Recall for two discrete groups Γ, Λ, the classifying map

$$\mathrm{Hom}(\Gamma, \Lambda)/\mathrm{conjugacy} \ni [\varphi] \mapsto [B\varphi] \in [B\Gamma, B\Lambda]$$

is a bijection.

We try to classify group actions on C*-algebra follwing the same spirit.

Let G be a discrete group and let A be a C*-algebra.
A G-action on A is a homomorphism from G to $\mathrm{Aut}(A)$.

- Two G-actions α, β are coycle conjugate if $\exists \theta \in \mathrm{Aut}(A)$ and $\exists \{u_g\}_{g \in G} \subset \mathcal{U}(A)$ satisfying $u_{gh} = u_g \alpha_g(u_h)$ and $\mathrm{Ad}\, u_g \circ \alpha_g = \theta \circ \beta_g \circ \theta^{-1}$.
- A G-action α is outer if α_g is outer for any $g \in G \setminus \{e\}$.

Denote by $\mathcal{OA}(G, A)$ the set of outer actions of G on A.

Conjecture

Conjecture

Let G be a discrete torsion-free amenable group, and let A be a stable Kirchberg algebra. Then the map

$$\mathcal{OA}(G,A)/\mathit{cocycle\ conjugacy} \ni [\alpha] \mapsto [B\alpha^s] \in [BG, B\,\mathrm{Aut}(A\otimes\mathbb{K})]$$

is a bijection, where $\alpha^s_g = \alpha_g \otimes \mathrm{Ad}\,\rho_g$ acts on $A\otimes\mathbb{K}(\ell^2(G))$ and ρ is the right regular representation.

Theorem (Matui-I. 2018)

The conjecture is true for any poly-$\mathbb{Z}$ group G.

Poly-$\mathbb{Z}$ group actions

A group G is poly-$\mathbb{Z}$ if there exists a subnormal series

$$\{e\} = G_0 \leq G_1 \leq \cdots \leq G_n = G,$$

satisfying $G_k/G_{k-1} \cong \mathbb{Z}$ for any $k = 1, 2, \ldots, n$.

The number n is called the Hirsch length of G, denoted by $h(G)$, which coincides with the cohomological dimension of G.

In fact, there exists a free cocompact polynomial action of G on $\mathbb{R}^n$, and we can choose $EG = \mathbb{R}^n$ and $BG = \mathbb{R}^n/G$.

Example

Cocycle conjugacy classes of outer $\mathbb{Z}^3$-actions on $\mathcal{O}_\infty \otimes \mathbb{K}$ are in one-to-one correspondence with

$$\mathfrak{Bun}_{\mathbb{T}^3}(\mathcal{O}_\infty \otimes \mathbb{K}) \cong H^1(\mathbb{T}^3; \mathbb{Z}/2\mathbb{Z}) \oplus H^3(\mathbb{T}^3; \mathbb{Z}) \cong (\mathbb{Z}/2\mathbb{Z})^3 \oplus \mathbb{Z}.$$

Unital case

For a G-action α on A, we define a principal $\mathrm{Aut}(A)$-bundle $\mathcal{P}_\alpha$ over BG by

$$\mathcal{P}_\alpha := (EG \times \mathrm{Aut}(A))/G \to BG,$$

where $g \cdot (x, \gamma) = (g \cdot x, \alpha_g \circ \gamma)$.

Theorem (Matui-I. 2018)

Let G be a poly-$\mathbb{Z}$ group and let A be a unital Kirchberg algebra. Then $\alpha, \beta \in \mathcal{OA}(G, A)$ are cocycle conjugate iff $\mathcal{P}_{\alpha^s} \cong \mathcal{P}_{\beta^s}$.

Example

Cocycle conjugacy classes of outer $\mathbb{Z}^3$-actions on $\mathcal{O}_n$ are in one-to-one correspondence with

$$H^2(\mathbb{T}^3; \pi_1(\mathrm{Aut}(\mathcal{O}_n \otimes \mathbb{K}))) = H^2(\mathbb{T}^3; \mathbb{Z}_{n-1}) \cong (\mathbb{Z}/2(n-1)\mathbb{Z})^3.$$