Lecture 1:
A quick review of Bivariant Theory

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Lecture 1 is **a quick review or recall** of

"Introduction to Bivariant Theory, I, II, III"

which I gave for

"The 9th (Non-)Commutative Algebra and Topology"

February 18 - 20, 2020, Faculty of Science, Shinshu University.

"Bivariant Theory **入門**、I, II, III"

「第 9 回（非）可換代数とトポロジー」

2020 **年** 2 **月** 18 **日**〜2 **月** 20 **日、信州大学理学部**

Bivariant Theory is one introduced by W. Fulton and R. MacPherson in

[FM] "Categorical frameworks for the study of singular spaces"

Mem. Amer. Math. Soc. **243** (1981)

Part I: Bivariant Theories (pp.1-117)
Part II: Products in Riemann-Roch (pp.119-161)

Menu

§1 Hirzebruch–Riemann–Roch (HRR)

§2. Grothendieck–Riemann–Roch (GRR)

§3. Fulton–MacPherson's bivariant theory

- §3.1. Ingredients of Fulton–MacPherson's bivariant theory
- §3.2. Bivariant operations on $\mathbb{B}$
- §3.3. Seven axioms required on these 3 operations
- §3.4. Grothendieck transformation

§4. Associated covariant & contravariant functors $\mathbb{B}_*$,$\mathbb{B}^*$

§5. Canonical orientation

§6. Gysin maps induced by bivariant elements

§7. Gysin maps induced by canonical orientations

§8. Riemann–Roch formula by Fulton-MacPherson

§9. A remark on RR-formulas

§1 Hirzebruch–Riemann–Roch (HRR)

E, a holomorphic vector bundle on compact manifold X over $\mathbb{C}$

$$\chi(X,E) := \sum_{i=0}^{\dim X} (-1)^i \dim_{\mathbb{C}} H^i(X,E)\ , \qquad \text{Euler characteristic of } E.$$

Serre's conjecture (1953, 9/29, a letter to Kodaira-Spencer, IAS)
$\exists$ a polynomial $P(X,E)$ of Chern classes of the tangent bundle TX and E such that

$$\chi(X,E) = \int_X P(X,E) \cap [X]$$

Hirzebruch–Riemann–Roch (HRR) (1953, 12/9, at IAS of Princeton):

$$\chi(X,E) = \int_X (td(TX) \cup ch(E)) \cap [X].$$

$td(TX) := \prod_{j=1}^{\dim X} \frac{\beta_j}{1-e^{-\beta_j}}$ Todd class of TX, $ch(E) = \sum_{i=1}^{\operatorname{rank} E} e^{\alpha_i}$ Chern character. β_j and α_i are the Chern roots of TX and E respectively.

"private memo" : 9/29 $\xrightarrow{\text{in 36 days}}$ 11/4 $\xrightarrow{\text{in 35 days}}$ 12/9. (In the very middle of the birth of HRR!)

§2. Grothendieck–Riemann–Roch (GRR)

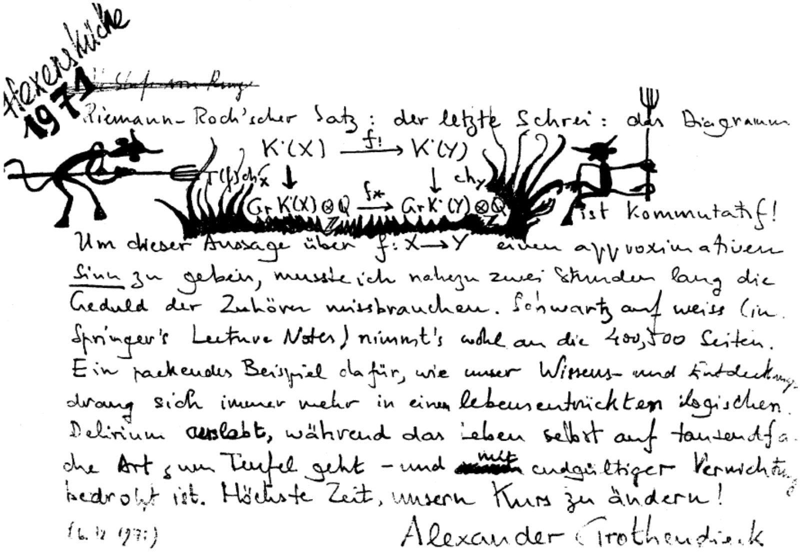
Grothendieck said, "No, the Riemann-Roch theorem is not a theorem about varieties, it's a theorem about morphisms between varieties."
He extended **HRR** to the **natural transformation**:

$$ch(--) \cup td(-) : K^0(-) \to H^*(-) \otimes \mathbb{Q}.$$

$K^0(Z)$ is K-theory of vector bundles, $H^*(Z)$ is cohomology.
Namely, for a holomorphic map $f : X \to Y$ of algebraic manifolds (=non-singular complex projective varieties) X and Y, the following diagram is commutative:

$$\begin{array}{ccc} K^0(X) & \xrightarrow{ch(-)\cup td(TX)} & H^*(X) \otimes \mathbb{Q} \\ {\scriptstyle f_!}\downarrow & & \downarrow{\scriptstyle f_!} \\ K^0(Y) & \xrightarrow[ch(-)\cup td(TY)]{} & H^*(Y) \otimes \mathbb{Q}. \end{array}$$

Note $K^0(-)$ and $H^*(-)$ are contravariant! So $f_!$ are Gysin (wrong-way) maps.
Grothendieck gave **4 lectures (12 hours for 4 days)** of his proof "Classes de faisceaux et théorème de Riemann–Roch" (1957) at **1st Arbeitstagung at Bonn in 1957** (founded by Friedrich Hirzebruch), published in SGA 6(1971), 20-71. His proof was also **published by Borel-Serre in Bull.Soc.Math. France** (1958), p. 97-136.)
Borel said, "Grothendieck's version of Riemann–Roch is a fantastic theorem. This is really a masterpiece of mathematics."



Hexenküche 1971

Riemann-Roch'scher Satz: der letzte Schrei: das Diagramm

$$\begin{array}{ccc} K^{\cdot}(X) & \xrightarrow{f_!} & K^{\cdot}(Y) \\ \downarrow{\scriptstyle T(f)\,ch_X} & & \downarrow{\scriptstyle ch_Y} \\ Gr_{\mathbb{Z}}K^{\cdot}(X)\otimes\mathbb{Q} & \xrightarrow{f_*} & Gr_{\mathbb{Z}}K^{\cdot}(Y)\otimes\mathbb{Q} \end{array}$$

ist kommutativ!

Um dieser Aussage über $f: X \to Y$ einen approximativen Sinn zu geben, musste ich nahezu zwei Stunden lang die Geduld der Zuhörer missbrauchen. Schwarz auf weiss (in Springer's Lecture Notes) nimmt's wohl an die 400, 500 Seiten. Ein packendes Beispiel dafür, wie unser Wissens- und Entdeckungsdrang sich immer mehr in einem lebensentrückten logischen Delirium auslebt, während das Leben selbst auf tausendfache Art zum Teufel geht – und mit endgültiger Vernichtung bedroht ist. Höchste Zeit, unsern Kurs zu ändern!

(6.12.1971) Alexander Grothendieck

京都・パリ・ハーバードで鍛えた発想と手法が、代数幾何の未解決難問の解決へと導いた

代数的多様体の研究
（昭和 45 年）

広中平祐

財団法人数理科学振興会理事長、
京都大学名誉教授、日本学士院会員
数学

Heisuke Hironaka

昭和6年山口県出身。京都大学理学部を卒業。ブランダイス大学助教授、コロンビア大学教授、ハーバード大学教授、京都大学数理解析研究所長、山口大学長を歴任。昭和45年にフィールズ賞受賞

総会議場での記念撮影。柏原正樹氏（左）、広中平祐氏（中央）、森重文氏（右）。2009年9月13日、日本学士院にて

どパリの代数幾何の連中がみんな来ていました。グロタンディークが『代数幾何原論』を13巻まで書くといって、講義をしながら書き始めた頃です。しばらくして、チャーン類（クラス）を定義して、リーマン・ロッホ定理を一般化した。それがK理論につながったのです。「できた！」と非常に喜んでいました。このころグロタンディークは本格的に注目されるようになりました。そのあとエタール・コホモロジーにいくのだけど、そのころは、僕はハーバードに帰っていました。

Why is GRR an extension of HRR?

Because [GRR for $a_X : X \to pt$ (a map to a point)] = HRR !!!

Indeed, let's consider the following commutative diagram!

$$[\text{GRR for } a_X : X \to pt] \quad === \quad \begin{array}{ccc} K^0(X) & \xrightarrow{ch(-)\cup td(TX)} & H^*(X)\otimes\mathbb{Q} \\ {\scriptstyle (a_X)_!}\downarrow & & \downarrow{\scriptstyle (a_X)_!} \\ K^0(pt) & \xrightarrow[ch(-)\cup td(pt)]{} & H^*(pt)\otimes\mathbb{Q}. \end{array}$$

Namely, for $E \in K^0(X)$

$$ch((a_X)_! E) \cup td(pt) = (a_X)_! \, (ch(E) \cup td(TX)).$$

$$ch((a_X)_! E) \cup td(pt) = \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots = \chi(X, E)$$

$$(a_X)_! \, (ch(E) \cup td(TX)) = \cdots\cdots\cdots = \int_X (td(TX) \cup ch(E)) \cap [X].$$

Thus we have HRR:

$$\chi(X, E) = \int_X (td(X) \cup ch(E)) \cap [X].$$

My guess: Probably Grothendieck thought as follows:
Note that for a vector space V, $ch(V) = \dim V$, so

$$\chi(X; E) = \sum_{i=0}^{\dim X} (-1)^i \dim_{\mathbb{C}} H^i(X, E) = \sum_{i=0}^{\dim X} (-1)^i ch(H^i(X, E))$$
$$= ch\left(\sum_{i=0}^{\dim X} (-1)^i H^i(X, E) \right)$$

$$\int_X (td(TX) \cup ch(E)) \cap [X] = (a_X)_* \left((td(TX) \cup ch(E)) \cap [X] \right)$$

$$\begin{array}{ccccc}
K^0(X) & \xrightarrow{td(TX)\cup ch(-)} & H^*(X) & \xrightarrow[\cong]{-\cap[X]} & H_*(X) \\
\Big\downarrow \text{“}(a_X)_!\text{”} & & \Big\downarrow (a_X)_! & & \Big\downarrow (a_X)_* \\
K^0(pt) & \xrightarrow[td(T_{pt})\cup ch(-) = ch(-)]{} & H^*(pt) & \xrightarrow[\cap[pt]]{\cong} & H_*(pt)
\end{array}$$

$$\begin{array}{ccc}
E & \xrightarrow{td(TX)\cup ch(-)} & td(TX) \cup ch(E) \\
\Big\downarrow \text{“}(a_X)_!\text{”} & & \Big\downarrow (a_X)_! \\
\sum_{i=0}^{\dim X} (-1)^i H^i(X, E) & \xrightarrow[ch(-)]{} & \chi(X; E)
\end{array}$$

$$
\begin{array}{ccccccc}
K^0(X) & & & \xrightarrow{\quad ch(-)\cup td(TX) \quad} & & & H^*(X)\otimes\mathbb{Q} \\
 & \searrow^{\Omega}_{\cong} & & & & \swarrow^{\cap[X]}_{\cong} & \\
 & & K_0(X) & \xrightarrow{ch(\Omega^{-1}(-))\cup td(TX)\cap[X]} & H_*(X)\otimes\mathbb{Q} & & \\
\downarrow f_! & & \downarrow f_* & & \downarrow f_* & & \downarrow f_! \\
 & & K_0(Y) & \xrightarrow[ch(\Omega^{-1}(-))\cup td(TY)\cap[X]]{} & H_*(Y)\otimes\mathbb{Q} & & \\
 & \nearrow^{\cong}_{\Omega} & & & & \nwarrow^{\cong}_{\cap[Y]} & \\
K_0(Y) & & & \xrightarrow[\quad ch(-)\cup td(TY) \quad]{} & & & H^*(Y)\otimes\mathbb{Q}
\end{array}
$$

The commutativity of the outer square follows from that of the inner square. $K_0(Z)$ is K-theory of coherent sheaves on Z. $f_* : K_0(X) \to K_0(Y)$ is defined by $f_*\mathcal{F} := \sum_{i=0}^{\dim X}(-1)^i R^i f_*\mathcal{F}$. For $X \xrightarrow{a_X} pt$, $(a_X)_*E = \sum_{i=0}^{\dim X}(-1)^i H^i(X, E)$. In fact,

$$
\begin{array}{ccc}
K^0(X) & \xrightarrow{ch(-)\cup td(TX)} & H^*(X)\otimes\mathbb{Q} \\
f_! \downarrow & & \downarrow f_! \\
K^0(Y) & \xrightarrow[ch(-)\cup td(TY)]{} & H^*(Y)\otimes\mathbb{Q}.
\end{array}
\quad \text{is expressed as} \quad
\begin{array}{ccc}
K^0(X) & \xrightarrow{ch} & H^*(X)\otimes\mathbb{Q} \\
f_! \downarrow & & \downarrow f_!\left(\mathbf{td(T_f)}\cup -\right) \\
K^0(Y) & \xrightarrow[ch]{} & H^*(Y)\otimes\mathbb{Q},
\end{array}
$$

Here $T_f := TX - f^*TY \in K^0(X)$ and $td(T_f) = \dfrac{td(TX)}{f^*td(TY)} \in H^*(X)\otimes\mathbb{Q}$

Indeed, the left diagram means: for $E \in K^0(X)$

$$td(TY) \cup ch(f_! E) = f_!(td(TX) \cup ch(E))$$

$f_! = \mathcal{P}_Y^{-1} \circ f_* \circ \mathcal{P}_X$. Here $\mathcal{P}_X = H^*(X) \xrightarrow[\cong]{\cap[X]} H_*(X)$ and

$\mathcal{P}_Y = H^*(Y) \xrightarrow[\cong]{\cap[Y]} H_*(Y)$ the Poincaré duality isomorphisms (since X and Y are smooth). So, $td(TY) \cup ch(f_! E) = \mathcal{P}_Y^{-1} \circ f_* \circ \mathcal{P}_X(td(TX) \cup ch(E))$ can be written as $\left(td(TY) \cup ch(f_! E)\right) \cap [Y] = f_*\Big(\left(td(TX) \cup ch(E)\right) \cap [X]\Big)$.

$$td(TY) \cap \left(ch(f_! E) \cap [Y]\right) = f_*\Big(td(TX) \cap \left(ch(E) \cap [X]\right)\Big).$$

$$ch(f_! E) \cap [Y] = \frac{1}{td(TY)} \cap f_*\Big(td(TX) \cap \left(ch(E) \cap [X]\right)\Big).$$

By the projection formula, the tight-hand-side becomes as follows:

$$ch(f_! E) \cap [Y] = f_*\left(f^*\left(\frac{1}{td(TY)}\right) \cap \Big(td(TX) \cap \left(ch(E) \cap [X]\right)\Big)\right).$$

$$ch(f_! E) \cap [Y] = f_*\left(\frac{1}{f^* td(TY)} \cap \Big(td(TX) \cap \left(ch(E) \cap [X]\right)\Big)\right).$$

$$ch(f_! E) \cap [Y] = f_*\left(\left(\frac{td(TX)}{f^* td(TY)} \cup ch(E)\right) \cap [X]\right)\Big)$$

$$ch(f_! E) \cap [Y] = f_*\left(\left(td(T_f) \cup ch(E)\right) \cap [X]\right)\Big).$$

$$ch(f_! E) = \mathcal{P}_Y^{-1} \circ f_* \circ \mathcal{P}_X\left(td(T_f) \cup ch(E)\right),$$

$$ch(f_! E) = f_!\left(td(T_f) \cup ch(E)\right).$$

GRR was extended to the following

"SGA 6", 1971: For a proper and local complete intersection morphism $f : X \to Y$

$$
\begin{array}{ccc}
K^0(X) & \xrightarrow{ch} & H^*(X) \otimes \mathbb{Q} \\
\downarrow{\scriptstyle f_!} & & \downarrow{\scriptstyle f_!\left(\mathbf{td}(\mathbf{T_f}) \cup -\right)} \\
K^0(Y) & \xrightarrow[ch]{} & H^*(Y) \otimes \mathbb{Q},
\end{array}
$$

Here $T_f \in K^0(X)$ is the **relative tangent bundle of** f. If $f : X \to Y$ is a map of smooth manifolds, then $T_f = TX - f^*TY \in K^0(X)$.

The inner commutative square was extended to singular varieties **"BFM–RR"(Baum–Fulton–MacPherson's Riemann–Roch), Publ.Math.IHES. 45 (1975), 101-145."**:

$\exists$ a natural transformation

$$\tau^{\mathrm{BFM}} : K_0(-) \to H_*(-) \otimes \mathbb{Q}$$

such that if X is non-singular, $\tau^{\mathrm{BFM}}(\mathcal{O}_X) = td(TX) \cap [X]$, the Poincaré dual of the Todd class $td(TX)$ of TX: i.e., for a proper map $f : X \to Y$

$$
\begin{array}{ccc}
K_0(X) & \xrightarrow{\tau^{\mathrm{BFM}}} & H_*(X) \otimes \mathbb{Q} \\
\downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} \\
K_0(Y) & \xrightarrow[\tau^{\mathrm{BFM}}]{} & H_*(Y) \otimes \mathbb{Q},
\end{array}
$$

“BFM–RR” is **motivated by MacPherson’s Chern class transformation (Ann. Math, 100 (1974),423-432)**

$$\exists!\, c_* : F(-) \to H_*(-)$$

such that if X is nonsingular $c_*(1\!\!1_X) = c(TX) \cap [X]$ the Poincaré dual of the total Chern class of TX.
Here $F(X)$ is the abelian group of constructible functions of X.)

(NOTE: MacPherson’s Chern class transformation $c_* : F(-) \to H_*(-)$ is a “Grothendieck-Riemann-Roch”-type theorem for Chern classes for singular varieties. However, in his paper **there was no word of “Riemann-Roch”!**)

“Verdier–RR”, Astérisque, 1983 (**conjectured in BFM’s paper; proved by Verdier**): For a l.c.i. morphism $f : X \to Y$ we have the commutative diagram:

$$\begin{array}{ccc}
K_0(Y) & \xrightarrow{\tau^{BFM}} & H_*(Y)\otimes\mathbb{Q} \\
{\scriptstyle f^!}\downarrow & & \downarrow{\scriptstyle \mathbf{td}(\mathbf{T_f})\cap f^!} \\
K_0(X) & \xrightarrow[\tau^{BFM}]{} & H_*(X)\otimes\mathbb{Q}.
\end{array}$$

§3. Fulton–MacPherson's bivariant theory

Fulton-MacPherson introduced **Bivariant Theory** [FM] in order to unify these "GRR"-type formulas, i.e.,"SGA6","BFM-RR","Verdier-RR".

NOTE (important!): "SGA6" and "Verdier-RR" deal with Gysin maps (wrong-way maps) for $f : X \to Y$.: $f_! : K^0(X) \to K^0(Y)$, $f^! : K_0(Y) \to K_0(X)$.

FM's theorem ([FM] Part II:Products in Riemann-Roch (p.119-161)): *Let $\mathbb{K}(X \to Y)$ be a bivariant K-theory such that*

(i) $\mathbb{K}(X \to pt) = K_0(X)$ Grothendieck group of coherent sheaves,

(ii) $\mathbb{K}(X \xrightarrow{\mathrm{id}_X} X) = K^0(X)$ Grothendieck group of complex vector bundles.

Let $\mathbb{H}(X \to Y)$ be a bivariant homology theory such that

(i) $\mathbb{H}(X \to pt) = H_*(X)$ homology , (ii) $\mathbb{H}(X \xrightarrow{\mathrm{id}_X} X) = H^*(X)$ cohomology.

Then, there exists a Grothendieck transformation

$$\gamma : \mathbb{K}(-) \to \mathbb{H}(-) \otimes \mathbb{Q}$$

such that

(i) $\gamma : \mathbb{K}(X \to pt) \to \mathbb{H}(X \to pt) \otimes \mathbb{Q}$ is BFM-RR $\tau^{BFM} : K_0(X) \to H_*(X) \otimes \mathbb{Q}$,

(ii) for a l.c.i. morphism $f : X \to Y$

$$\gamma(\theta_{\mathbb{K}}(f)) = td(T_f) \bullet \theta_{\mathbb{H}}(f) \quad (\textbf{Riemann–Roch formula}) \ (not\ \gamma(\theta_{\mathbb{K}}(f)) = \theta_{\mathbb{H}}(f))$$

$$\theta_{\mathbb{K}}(f) \in \mathbb{K}(X \xrightarrow{f} Y), \theta_{\mathbb{H}}(f) \in \mathbb{H}(X \xrightarrow{f} Y), td(T_f) \in \mathbb{H}(X \xrightarrow{\mathrm{id}_X} X) = H^*(X)$$

This RR-formula implies "SGA6","BFM-RR","Verdier-RR"!!!

§3.1. Ingredients of Fulton–MacPherson's bivariant theory

1. An underlying category $\mathcal{V}$,
2. A map $\mathbb{B}$ assigning to each map $f : X \to Y \in \mathcal{V}$ a graded abelian group $\mathbb{B}^i(X \xrightarrow{f} Y)$. (Note: sometimes it can be just a set (cf. §4.3 Differentiable RR of [FM]))

   an element $\alpha \in \mathbb{B}(X \xrightarrow{f} Y)$ is expressed as follows:

   $$X \xrightarrow[f]{\textcircled{\alpha}} Y$$

3. A class $\mathcal{C}$ of maps in $\mathcal{V}$, called "confined maps" (e.g., proper maps)
4. A class $\mathcal{I}nd$ of commutative squares in $\mathcal{V}$, called "independent squares" (e.g., fiber square)

   $$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow[g]{} & Y \end{array}$$

Conditions on the classes $\mathcal{C}$ and $\mathcal{I}nd$

1. The class $\mathcal{C}$ is closed under composition and base change and contain all the identity maps.
2. The class $\mathcal{I}nd$ satisfies the following:

   2.1 if the two inside squares in
   $$\begin{array}{ccccc} X'' & \xrightarrow{h'} & X' & \xrightarrow{g'} & X \\ \downarrow{\scriptstyle f''} & & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ Y'' & \xrightarrow[h]{} & Y' & \xrightarrow[g]{} & Y \end{array}$$
   are independent, then the outside square is also independent,

   2.2 for any $f : X \to Y$,
   $$\begin{array}{ccc} X & \xrightarrow{\mathrm{id}_X} & X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow[\mathrm{id}_Y]{} & Y \end{array} \quad \text{and} \quad \begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle \mathrm{id}_X}\downarrow & & \downarrow{\scriptstyle \mathrm{id}_Y} \\ X & \xrightarrow[f]{} & Y \end{array}$$
   are independent:

   2.3 In an independent square
   $$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow[g]{} & Y \end{array},$$
   if f (resp., g) is confined, then f' (resp., g') is confined.

A REMARK: Given an independent square

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow[g]{} & Y \end{array}$$

, its transpose

$$\begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow[f]{} & Y \end{array}$$

is not necessarily independent.

EXAMPLE: Consider the category of topological spaces and continuous maps. Let any map be confined, and allow a fiber square

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow[g]{} & Y \end{array}$$

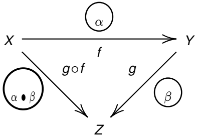
to be independent only if g is proper (hence g' is also proper). Then its transpose is not independent unless f is proper.

NOTE: The pullback of a proper map by any (continuous) map is proper, because “proper” is equivalent to “universally closed” (i.e., the pullback by any map is closed.)

§3.2. Bivariant operations on $\mathbb{B}$

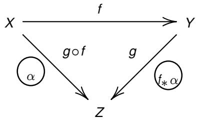
1. **Product**: For $f : X \to Y$ and $g : Y \to Z$ in $\mathcal{V}$, the homomorphism

$$\bullet : \mathbb{B}^i(X \xrightarrow{f} Y) \otimes \mathbb{B}^j(Y \xrightarrow{g} Z) \to \mathbb{B}^{i+j}(X \xrightarrow{g \circ f} Z),$$



2. **Pushforward**: For $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathcal{V}$ with f confined, the homomorphism

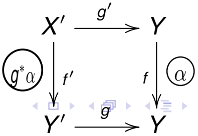
$$f_* : \mathbb{B}^i(X \xrightarrow{g \circ f} Z) \to \mathbb{B}^i(Y \xrightarrow{g} Z),$$



3. **Pullback** : For an independent square

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow[g]{} & Y, \end{array}$$

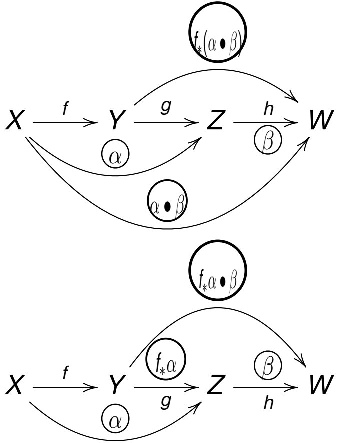
$$g^* : \mathbb{B}^i(X \xrightarrow{f} Y) \to \mathbb{B}^i(X' \xrightarrow{f'} Y'),$$



§3.3. Seven axioms required on these 3 operations

1. (A_1) **Product is associative**: for $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W$ with $\alpha \in \mathbb{B}(X \xrightarrow{f} Y), \beta \in \mathbb{B}(Y \xrightarrow{g} Z), \gamma \in \mathbb{B}(Z \xrightarrow{h} W)$,
$$(\alpha \bullet \beta) \bullet \gamma = \alpha \bullet (\beta \bullet \gamma).$$
2. (A_2) **Pushforward is functorial** : for $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W$ with f and g confined and $\alpha \in \mathbb{B}(X \xrightarrow{h\circ g\circ f} W)$
$$(g \circ f)_*(\alpha) = g_*(f_*(\alpha)).$$
3. (A_3) **Pullback is functorial**: given independent squares
$$\begin{array}{ccccc} X'' & \xrightarrow{h'} & X' & \xrightarrow{g'} & X \\ \downarrow{\scriptstyle f''} & & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ Y'' & \xrightarrow[h]{} & Y' & \xrightarrow[g]{} & Y \end{array}$$
$$(g \circ h)^* = h^* \circ g^*.$$
4. (A_{12}) **Product and pushforward commute**: for $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W$ with f confined and $\alpha \in \mathbb{B}(X \xrightarrow{g\circ f} Z), \beta \in \mathbb{B}(Z \xrightarrow{h} W)$,
$$f_*(\alpha \bullet \beta) = f_*(\alpha) \bullet \beta \in \mathbb{B}(Y \xrightarrow{h\circ g} W).$$

(A_{12}) means the following:

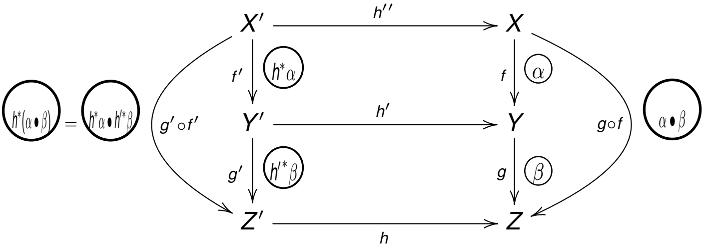


5. (A_{13}) **Product and pullback commute**: given independent squares

$$\begin{array}{ccc} X' & \xrightarrow{h''} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{h'} & Y \\ {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle g} \\ Z' & \xrightarrow[h]{} & Z \end{array}$$

with $\alpha \in \mathbb{B}(X \xrightarrow{f} Y), \beta \in \mathbb{B}(Y \xrightarrow{g} Z)$,

$$h^*(\alpha \bullet \beta) = h'^*(\alpha) \bullet h^*(\beta) \in \mathbb{B}(X' \xrightarrow{g' \circ f'} Z').$$

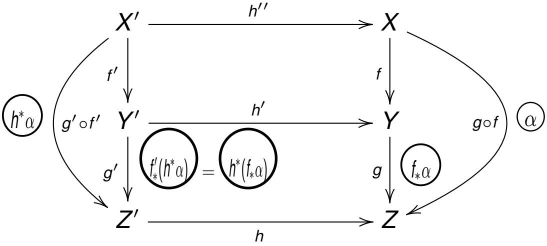


6. (A_{23}) **Pushforward and pullback commute**:for independent squares

$$\begin{array}{ccc} X' & \xrightarrow{h''} & X \\ f' \downarrow & & \downarrow f \\ Y' & \xrightarrow{h'} & Y \\ g' \downarrow & & \downarrow g \\ Z' & \xrightarrow[h]{} & Z \end{array}$$

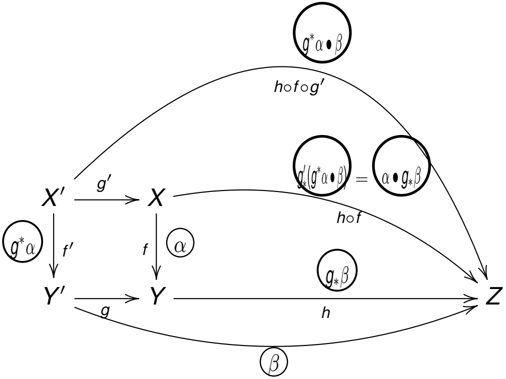
with f confined and $\alpha \in \mathbb{B}(X \xrightarrow{g \circ f} Z)$,

$$f'_*(h^*(\alpha)) = h^*(f_*(\alpha)) \in \mathbb{B}(Y' \xrightarrow{g'} Z').$$



7. (A_{123}) **Projection formula**: given an independent square with g confined and $\alpha \in \mathbb{B}(X \xrightarrow{f} Y), \beta \in \mathbb{B}(Y' \xrightarrow{h \circ g} Z)$, we have

$$g'_*(g^*\alpha \bullet \beta) = \alpha \bullet g_*\beta \in \mathbb{B}(X \xrightarrow{h \circ f} Z).$$



We also require the theory $\mathbb{B}$ to have multiplicative units:

(Units) For all $X \in \mathcal{V}$, there is an element $1_X \in \mathbb{B}^0(X \xrightarrow{\mathrm{id}_X} X)$ such that $\alpha \bullet 1_X = \alpha$ for all morphisms $W \to X$ and all $\alpha \in \mathbb{B}(W \to X)$, and such that $1_X \bullet \beta = \beta$ for all morphisms $X \to Y$ and all $\beta \in \mathbb{B}(X \to Y)$, and such that $g^* 1_X = 1_{X'}$ for all $g : X' \to X$.

§3.4. Grothendieck transformation

Let $\mathbb{B}, \mathbb{B}'$ be two bivariant theories on a category $\mathcal{V}$.
A *Grothendieck transformation* from $\mathbb{B}$ to $\mathbb{B}'$,

$$\gamma : \mathbb{B} \to \mathbb{B}'$$

is a collection of homomorphisms

$$\mathbb{B}(X \to Y) \to \mathbb{B}'(X \to Y)$$

which preserves the above three basic operations:

1. $\gamma(\alpha \bullet_{\mathbb{B}} \beta) = \gamma(\alpha) \bullet_{\mathbb{B}'} \gamma(\beta)$,
2. $\gamma(f_*\alpha) = f_*\gamma(\alpha)$,
3. $\gamma(g^*\alpha) = g^*\gamma(\alpha)$.

A remark

In FM's book,a Grothendieck transformation is defined as follows.
Let $- : \mathcal{V} \to \overline{\mathcal{V}}$ be a functor sending confined maps in $\mathcal{V}$ to confined maps in $\overline{\mathcal{V}}$, and independent squares in $\mathcal{V}$ to independent squares in $\overline{\mathcal{V}}$.
Write $\overline{X}$ and $\overline{f}$ for the image in $\overline{\mathcal{V}}$ of an object X and a map f in $\mathcal{V}$. Let T be a bivariant theory on $\mathcal{V}$ and U be a bivariant theory on $\overline{\mathcal{V}}$. Then a Grothendieck transformation

$$t : T \to U$$

is a collection of homomorphisms

$$t : T(X \xrightarrow{f} Y) \to U(\overline{X} \xrightarrow{\overline{f}} \overline{Y}),$$

which commutes with product, pushforward and pullback.
However, if we define

$$U(X \xrightarrow{f} Y) := U(\overline{X} \xrightarrow{\overline{f}} \overline{Y})$$

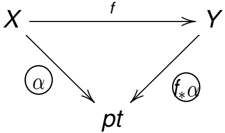
then the bivariant theory U on $\overline{\mathcal{V}}$ can be considered as a bivariant theory on $\mathcal{V}$, thus a Grothendieck transformation can be defined as above.

§4. Associated covariant & contravariant functors $\mathbb{B}_*, \mathbb{B}^*$

$\mathbb{B}$ unifies a covariant theory $\mathbb{B}_*$ and a contravariant theory $\mathbb{B}^*$:

1. $\mathbb{B}_i(X) := \mathbb{B}^{-i}(X \to pt)$ is covariant for confined maps:

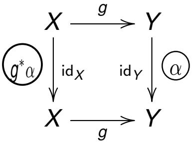
   $f_* : \mathbb{B}_i(X) \to \mathbb{B}_i(Y)$ (for a confined map $f : X \to Y$),

   $(g \circ f)_* = g_* \circ f_*$ follows from (A_2) (the functoriality of pushforward).

2. $\mathbb{B}^i(X) := \mathbb{B}^i(X \xrightarrow{\mathrm{id}_X} X)$ is contravariant for any morphisms: for $g : X \to Y$

   $g^* : \mathbb{B}^j(Y) \to \mathbb{B}^j(X)$,

   $(g \circ f)^* = f^* \circ g^*$ follows from (A_3) (the functoriality of pullback).

That is why $\mathbb{B}(X \to Y)$ is called a bivariant theory.
$\gamma : \mathbb{B} \to \mathbb{B}'$ induces natural transformations $\gamma : \mathbb{B}_* \to \mathbb{B}'_*$ and $\gamma : \mathbb{B}^* \to \mathbb{B}'^*$.
(Sometimes they are denoted γ_* and γ^* with $*$.)

§5. Canonical orientation

Let $\mathcal{S}'$ be another class of maps in $\mathcal{V}$, which is closed under compositions and containing all identity maps. (*We keep the symbol $\mathcal{S}$ for another class considered later*)

NOTE: For the class $\mathcal{C}$ of confined maps, we require the stability of pullback, i.e., the pullback of a confined map is confined. For this class $\mathcal{S}'$ we do not require the stability of pullback.

If for $f : X \to Y \in \mathcal{S}'$ there is assigned an element

$$\theta(f) \in \mathbb{B}(X \xrightarrow{f} Y)$$

satisfying

(i) $\theta(g \circ f) = \theta(f) \bullet \theta(g)$

(ii) $\theta(\mathrm{id}_X) = 1_X$ (the unit element).

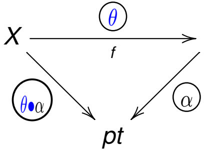
Then $\theta(f)$ is called a **canonical orientation** of f.

§6. Gysin maps induced by bivariant elements

Any bivariant element $\theta \in \mathbb{B}^i(X \xrightarrow{f} Y)$ gives rise to Gysin ("wrong-way") homomorphisms

1. $\theta^! : \mathbb{B}_j(Y) \to \mathbb{B}_{j-i}(X)$, *i.e.*, $\theta^! : \mathbb{B}^{-j}(Y \to pt) \to \mathbb{B}^{-j+i}(Y \to pt)$

   defined by $\theta^!(\alpha) := \theta \bullet \alpha$,

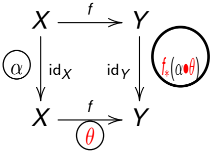
   

   For $\eta \in \mathbb{B}^j(Z \xrightarrow{g} X)$ and $\theta \in \mathbb{B}^i(X \xrightarrow{f} Y)$, $(\eta \bullet \theta)^! = \eta^! \circ \theta^!$. Because $(\eta \bullet \theta)^!(\alpha) := (\eta \bullet \theta) \bullet \alpha = \eta \bullet (\theta \bullet \alpha) = \eta^!(\theta^!(\alpha)) = (\eta^! \circ \theta^!)(\alpha)$.

2. $\theta_! : \mathbb{B}^j(X) \to \mathbb{B}^{j+i}(Y)$, *i.e.*, $\theta_! : \mathbb{B}^j(X \xrightarrow{\mathrm{id}_X} X) \to \mathbb{B}^j(Y \xrightarrow{\mathrm{id}_Y} Y)$

   defined by ($f : X \to Y$ is a confined map)

   $\theta_!(\alpha) := f_*(\alpha \bullet \theta)$,

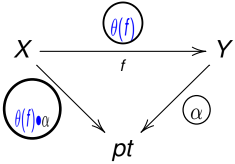
   $(\eta \bullet \theta)_! = \theta_! \circ \eta_!$. Because $(\eta \bullet \theta)_!(\alpha) := (f \circ g)_*(\alpha \bullet (\eta \bullet \theta)) = f_*(g_*((\alpha \bullet \eta) \bullet \theta)) = f_*(\eta_!(\alpha) \bullet \theta) = \theta_!(\eta_!(\alpha)) = (\theta_! \circ \eta_!)(\alpha)$.

§7. Gysin maps induced by canonical orientations

In particular, a canonical orientation $\theta(f)$ ($f \in \mathcal{S}'$) makes

1. the covariant functor $\mathbb{B}_*(X)$ contravariant for maps in $\mathcal{S}'$:
   For $f : X \to Y \in \mathcal{S}'$, $f^! : \mathbb{B}_*(Y) \to \mathbb{B}_*(X)$ defined by $\theta(f)^!$,i.e,

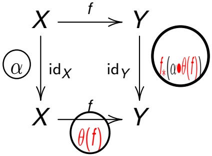
   $$f^!(\alpha) := \theta(f)^!(\alpha) = \theta(f) \bullet \alpha,$$

   $$(g \circ f)^! = \theta(g \circ f)^! = (\theta(f) \bullet \theta(g))^! = \theta(f)^! \circ \theta(g)^! = f^! \circ g^!.$$

2. the contravariant functor $\mathbb{B}^*$ covariant for maps in $\mathcal{C} \cap \mathcal{S}$.
   For $f : X \to Y \in \mathcal{C} \cap \mathcal{S}'$, $f_! : \mathbb{B}^*(X) \to \mathbb{B}^*(Y)$ defined by

   $$f_!(\alpha) := f_*(\alpha \bullet \theta(f)),$$

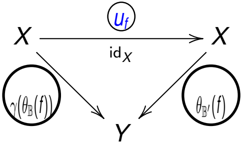
   $$(g \circ f)_! = \theta(g \circ f)_! = (\theta(f) \bullet \theta(g))_! = \theta(g)_! \circ \theta(f)_! = g_! \circ f_!.$$

$f^!$ and $f_!$ should carry the data $\mathcal{S}'$ and θ, but usually omitted.

§8. Riemann–Roch formula by Fulton-MacPherson

Let $\mathbb{B}, \mathbb{B}'$ be bivariant theories and let $\theta_{\mathbb{B}}, \theta_{\mathbb{B}'}$ be canonical orientations on $\mathbb{B}, \mathbb{B}'$ for a class $\mathcal{S}'$. Let $\gamma : \mathbb{B} \to \mathbb{B}'$ be a Grothendieck transformation. If there exists a bivariant element $u_f \in \mathbb{B}'(X \xrightarrow{\mathrm{id}_X} X)$ for $f : X \to Y \in \mathcal{S}$ such that

$$\gamma(\theta_{\mathbb{B}}(f)) = u_f \bullet \theta_{\mathbb{B}'}(f),$$



it is called a Riemann–Roch formula for $\gamma : \mathbb{B} \to \mathbb{B}'$ with respect to $\theta_{\mathbb{B}}$ and $\theta_{\mathbb{B}'}$.
In fact this RR-formula gives rise to the formulas of the following types

"BFM-RR", "SGA6", "Verdier-RR".

Indeed
(1) The Grothendieck transformation $\gamma : \mathbb{B} \to \mathbb{B}'$ gives us:
"BFM-RR" type formula: for a proper map $f : X \to Y$

$$\begin{array}{ccc} \mathbb{B}_*(X) & \xrightarrow{\gamma} & \mathbb{B}'_*(X) \\ {\scriptstyle f_*}\downarrow & & \downarrow{\scriptstyle f_*} \\ \mathbb{B}_*(Y) & \xrightarrow[\gamma]{} & \mathbb{B}'_*(Y), \end{array}$$

This is due to $\gamma(f_*\alpha) = f_*\gamma(\alpha)$.

(2) "SGA6" type formula: for a map $f : X \to Y \in \mathcal{C} \cap S'$

$$\begin{array}{ccc} \mathbb{B}^*(X) & \xrightarrow{\gamma} & \mathbb{B}'^*(X) \\ {\scriptstyle f_!}\downarrow & & \downarrow{\scriptstyle f_!(-\bullet u_f)} \\ \mathbb{B}^*(Y) & \xrightarrow[\gamma]{} & \mathbb{B}'^*(Y), \end{array}$$

$$\begin{aligned}
\gamma(f_!\alpha) &= \gamma(f_*(\alpha \bullet \theta_{\mathbb{B}}(f))) \quad \text{(by the definition of } f_!\text{)} \\
&= f_*\gamma(\alpha \bullet \theta_{\mathbb{B}}(f)) \\
&= f_*\Big(\gamma(\alpha) \bullet \gamma(\theta_{\mathbb{B}}(f))\Big) \\
&= f_*\Big(\gamma(\alpha) \bullet \Big(u_f \bullet \theta_{\mathbb{B}'}(f)\Big)\Big) \quad \text{(by RR-formula } \gamma(\theta_{\mathbb{B}}(f)) = u_f \bullet \theta_{\mathbb{B}'}(f)\text{)} \\
&= f_*\Big(\Big(\gamma(\alpha) \bullet u_f\Big) \bullet \theta_{\mathbb{B}'}(f)\Big) \\
&= f_!(\gamma(\alpha) \bullet u_f) \quad \text{(by the definition of } f_!(-) := f_*(- \bullet \theta_{\mathbb{B}'}(f))\text{)}
\end{aligned}$$

(3) "Verdier-RR" type formula: for a map $f : X \to Y \in \mathcal{S}'$

$$\begin{array}{ccc} \mathbb{B}_*(Y) & \xrightarrow{\gamma} & \mathbb{B}'_*(Y) \\ {\scriptstyle f^!}\downarrow & & \downarrow{\scriptstyle u_f \bullet f^!} \\ \mathbb{B}_*(X) & \xrightarrow[\gamma]{} & \mathbb{B}'_*(X), \end{array}$$

$$\begin{aligned} \gamma(f^!\alpha) &= \gamma(\theta_{\mathbb{B}}(f) \bullet \alpha) \\ &= \gamma(\theta_{\mathbb{B}}(f)) \bullet \gamma(\alpha) \\ &= \Big(u_f \bullet \theta_{\mathbb{B}'}(f)\Big) \bullet \gamma(\alpha) \quad \text{(by RR-formula } \gamma(\theta_{\mathbb{B}}(f)) = u_f \bullet \theta_{\mathbb{B}'}(f)\text{)} \\ &= u_f \bullet \Big(\theta_{\mathbb{B}'}(f) \bullet \gamma(\alpha)\Big) \\ &= u_f \bullet f^!(\gamma(\alpha)). \end{aligned}$$

So Fulton-MacPherson's Grothendieck transformation

$$\gamma : \mathbb{K}(-) \to \mathbb{H}(-) \otimes \mathbb{Q}$$

with **Riemann–Roch formula** $\gamma(\theta_{\mathbb{K}}(f)) = td(T_f) \bullet \theta_{\mathbb{H}}(f)$ implies

(1) BFM-RR:

$$\begin{array}{ccc} \mathbb{B}_*(X) & \xrightarrow{\gamma} & \mathbb{B}'_*(X) \\ f_* \downarrow & & \downarrow f_* \\ \mathbb{B}_*(Y) & \xrightarrow[\gamma]{} & \mathbb{B}'_*(Y), \end{array} \quad ==> \quad \begin{array}{ccc} K_0(X) & \xrightarrow{\tau^{BFM}} & H_*(X)\otimes\mathbb{Q} \\ f_* \downarrow & & \downarrow f_* \\ K_0(Y) & \xrightarrow[\tau^{BFM}]{} & H_*(Y)\otimes\mathbb{Q}, \end{array}$$

(2) "SGA 6":

$$\begin{array}{ccc} \mathbb{B}^*(X) & \xrightarrow{\gamma} & \mathbb{B}'^*(X) \\ f_! \downarrow & & \downarrow f_!(-\bullet u_f) \\ \mathbb{B}^*(Y) & \xrightarrow[\gamma]{} & \mathbb{B}'^*(Y), \end{array} \quad ==> \quad \begin{array}{ccc} K^0(X) & \xrightarrow{ch} & H^*(X)\otimes\mathbb{Q} \\ f_! \downarrow & & \downarrow f_!(-\cup td(T_f)) \\ K^0(Y) & \xrightarrow[ch]{} & H^*(Y)\otimes\mathbb{Q}, \end{array}$$

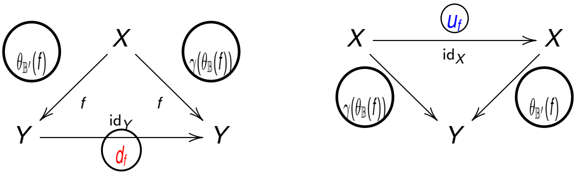
(3) "Verdier-RR"

$$\begin{array}{ccc} \mathbb{B}_*(Y) & \xrightarrow{\gamma} & \mathbb{B}'_*(Y) \\ f^! \downarrow & & \downarrow u_f \bullet f^! \\ \mathbb{B}_*(X) & \xrightarrow[\gamma]{} & \mathbb{B}'_*(X), \end{array} \quad ==> \quad \begin{array}{ccc} K_0(Y) & \xrightarrow{\tau^{BFM}} & H_*(Y)\otimes\mathbb{Q} \\ f^! \downarrow & & \downarrow td(T_f)\cap f^! \\ K_0(X) & \xrightarrow[\tau^{BFM}]{} & H_*(X)\otimes\mathbb{Q}, \end{array}$$

§9. A remark on RR-formulas

1. **“downstairs” Riemann–Roch formula** (by S.Y.):

$$\gamma(\theta_{\mathbb{B}}(f)) = \theta_{\mathbb{B}'}(f) \bullet d_f, \quad d_f \in \mathbb{B}'(Y \xrightarrow{\mathrm{id}_Y} Y).$$



(MEMO:I suppose Fulton-MacPherson use “u” for “u_f”, indicating “unit”, not “upstairs”.)

$\gamma(\theta_{\mathbb{B}}(f)) = \theta_{\mathbb{B}'}(f) \bullet d_f$ implies the corresponding
“SGA6” (for $f : X \to Y \in \mathcal{C} \cap \mathcal{S}'$) and “Verdier–RR” (for $f : X \to Y \in \mathcal{S}'$)

(i) "downstairs" ""SGA6" type formula: for a map $f : X \to Y \in \mathcal{C} \cap S'$

$$\begin{array}{ccc} \mathbb{B}^*(X) & \xrightarrow{\gamma} & \mathbb{B}'^*(X) \\ {\scriptstyle f_!}\downarrow & & \downarrow{\scriptstyle f_!(-)\bullet d_f} \\ \mathbb{B}^*(Y) & \xrightarrow[\gamma]{} & \mathbb{B}'^*(Y), \end{array}$$

$$\begin{aligned}
\gamma(f_!\alpha) &= \gamma(f_*(\alpha \bullet \theta_{\mathbb{B}}(f))) \\
&= f_*\gamma(\alpha \bullet \theta_{\mathbb{B}}(f)) \\
&= f_*\Big(\gamma(\alpha) \bullet \gamma(\theta_{\mathbb{B}}(f))\Big) \\
&= f_*\Big(\gamma(\alpha) \bullet \big(\theta_{\mathbb{B}'}(f) \bullet d_f\big)\Big) \quad \text{(by d-RR-formula } \gamma(\theta_{\mathbb{B}}(f)) = \theta_{\mathbb{B}'}(f) \bullet d_f \text{)} \\
&= f_*\Big(\big(\gamma(\alpha) \bullet \theta_{\mathbb{B}'}(f)\big) \bullet d_f\Big) \\
&= f_*\big(\gamma(\alpha) \bullet \theta_{\mathbb{B}'}(f)\big) \bullet d_f \quad \text{(by } (A_{12}\text{:product and pushforward commutes)} \\
&= f_!(\gamma(\alpha)) \bullet d_f \quad \text{(by the definition of } f_!(-) := f_*(- \bullet \theta_{\mathbb{B}'}(f)))
\end{aligned}$$

(ii)“downstairs” “Verdier-RR” type formula: for a map $f : X \to Y \in \mathcal{S}'$

$$\begin{array}{ccc} \mathbb{B}_*(Y) & \xrightarrow{\gamma} & \mathbb{B}'_*(Y) \\ {\scriptstyle f^!}\downarrow & & \downarrow{\scriptstyle f^!(d_f \bullet -)} \\ \mathbb{B}_*(X) & \xrightarrow[\gamma]{} & \mathbb{B}'_*(X), \end{array}$$

$$\begin{aligned} \gamma(f^!\alpha) &= \gamma(\theta_{\mathbb{B}}(f) \bullet \alpha) \\ &= \gamma(\theta_{\mathbb{B}}(f)) \bullet \gamma(\alpha) \\ &= \left(\theta_{\mathbb{B}'}(f) \bullet d_f\right) \bullet \gamma(\alpha) \quad \text{(by d-RR-formula } \gamma(\theta_{\mathbb{B}}(f)) = \theta_{\mathbb{B}'}(f) \bullet d_f \text{)} \\ &= \theta_{\mathbb{B}'}(f) \bullet \left(d_f \bullet \gamma(\alpha)\right) \\ &= f^!(d_f \bullet \gamma(\alpha)). \end{aligned}$$

Summing up:

““SGA 6” type formulas (“upstairs” and “downstairs”)

$$\begin{array}{ccc} \mathbb{B}^*(X) & \xrightarrow{\gamma} & \mathbb{B}'^*(X) \\ {\scriptstyle f_!}\downarrow & & \downarrow{\scriptstyle f_!(-\bullet u_f)} \\ \mathbb{B}^*(Y) & \xrightarrow[\gamma]{} & \mathbb{B}'^*(Y), \end{array} \qquad \begin{array}{ccc} \mathbb{B}^*(X) & \xrightarrow{\gamma} & \mathbb{B}'^*(X) \\ {\scriptstyle f_!}\downarrow & & \downarrow{\scriptstyle f_!(-)\bullet d_f} \\ \mathbb{B}^*(Y) & \xrightarrow[\gamma]{} & \mathbb{B}'^*(Y), \end{array}$$

“Verdier-RR” type formulas (“upstairs” and “downstairs”)

$$\begin{array}{ccc} \mathbb{B}_*(Y) & \xrightarrow{\gamma} & \mathbb{B}'_*(Y) \\ {\scriptstyle f^!}\downarrow & & \downarrow{\scriptstyle u_f\bullet f^!(-)} \\ \mathbb{B}_*(X) & \xrightarrow[\gamma]{} & \mathbb{B}'_*(X), \end{array} \qquad \begin{array}{ccc} \mathbb{B}_*(Y) & \xrightarrow{\gamma} & \mathbb{B}'_*(Y) \\ {\scriptstyle f^!}\downarrow & & \downarrow{\scriptstyle f^!(d_f\bullet -)} \\ \mathbb{B}_*(X) & \xrightarrow[\gamma]{} & \mathbb{B}'_*(X), \end{array}$$

2. **Riemann–Roch "self" formula**: Let $\mathbb{B}$ be a bivariant theory and θ, θ' be two canonical orientations of $\mathbb{B}$ for a class $\mathcal{S}'$:

1. ("upstairs" Riemann-Roch "self" formula (by S.Y.))

$$\theta(f) = u_f \bullet \theta'(f), \quad u_f \in \mathbb{B}(X \xrightarrow{\mathrm{id}_X} X),$$

Letting $f_{!!} := \theta'(f)_!, f^{!!} := \theta'(f)^!$, we have $f_! = f_{!!}(- \bullet u_f)$ and $f^! = u_f \bullet f^{!!}$.

2. ("downstairs" Riemann-Roch "self" formula (by S.Y.))

$$\theta(f) = \theta'(f) \bullet d_f, \quad d_f \in \mathbb{B}(Y \xrightarrow{\mathrm{id}_Y} Y).$$

As above, we have $f_! = f_{!!}(-) \bullet d_f$ and $f^! = f^{!!}(d_f \bullet -)$

In other words, we think
$\gamma = \mathrm{id} : \mathbb{B} \to \mathbb{B}$, $\mathrm{id}(\theta(f)) = u_f \bullet \theta'(f)$, $\mathrm{id}(\theta(f)) = \theta'(f) \bullet d_f$. Thus, we have
""SGA 6" type formulas ("upstairs" and "downstairs")

$$\begin{array}{ccc} \mathbb{B}^*(X) & \xrightarrow{\mathrm{id}} & \mathbb{B}^*(X) \\ {\scriptstyle f_!}\downarrow & & \downarrow{\scriptstyle f_{!!}(-\bullet u_f)} \\ \mathbb{B}^*(Y) & \xrightarrow[\mathrm{id}]{} & \mathbb{B}^*(Y), \end{array} \qquad \begin{array}{ccc} \mathbb{B}^*(X) & \xrightarrow{\mathrm{id}} & \mathbb{B}^*(X) \\ {\scriptstyle f_!}\downarrow & & \downarrow{\scriptstyle f_{!!}(-)\bullet d_f} \\ \mathbb{B}^*(Y) & \xrightarrow[\mathrm{id}]{} & \mathbb{B}^*(Y), \end{array}$$

"Verdier-RR" type formulas ("upstairs" and "downstairs")

$$\begin{array}{ccc} \mathbb{B}_*(Y) & \xrightarrow{\mathrm{id}} & \mathbb{B}_*(Y) \\ {\scriptstyle f^!}\downarrow & & \downarrow{\scriptstyle u_f \bullet f^!(-)} \\ \mathbb{B}_*(X) & \xrightarrow[\mathrm{id}]{} & \mathbb{B}_*(X), \end{array} \qquad \begin{array}{ccc} \mathbb{B}_*(Y) & \xrightarrow{\mathrm{id}} & \mathbb{B}_*(Y) \\ {\scriptstyle f^!}\downarrow & & \downarrow{\scriptstyle f^!(d_f \bullet -)} \\ \mathbb{B}_*(X) & \xrightarrow[\mathrm{id}]{} & \mathbb{B}_*(X), \end{array}$$

Thank you very much for your attention!