# Lecture 2:
# A universal bivariant theory and Riemann-Roch formulas

Shoji Yokura

Kagoshima University

December 5, 2020

# Menu

# §1. A universal bivariant theory and Riemann-Roch formulas

1. Let $\mathcal{S}$ be another class of maps called "specialized maps" (e.g., smooth maps in algebraic geometry), which is

# §1. A universal bivariant theory and Riemann-Roch formulas

1. Let $\mathcal{S}$ be another class of maps called "specialized maps" (e.g., smooth maps in algebraic geometry), which is
   1.1 **closed under composition**, $X \xrightarrow{f} Y, Y \xrightarrow{g} Z \in \mathcal{S} \Longrightarrow g \circ f \in \mathcal{S}$

# §1. A universal bivariant theory and Riemann-Roch formulas

1. Let $\mathcal{S}$ be another class of maps called "specialized maps" (e.g., smooth maps in algebraic geometry), which is
   1.1 **closed under composition**, $X \xrightarrow{f} Y, Y \xrightarrow{g} Z \in \mathcal{S} \Longrightarrow g \circ f \in \mathcal{S}$
   1.2 **closed under base change (i.e., stable by pullback)**
   (**NOTE**:this condition is **not required on the above class** $\mathcal{S}'$.)

# §1. A universal bivariant theory and Riemann-Roch formulas

1. Let $\mathcal{S}$ be another class of maps called "specialized maps" (e.g., smooth maps in algebraic geometry), which is

   1.1 **closed under composition**, $X \xrightarrow{f} Y, Y \xrightarrow{g} Z \in \mathcal{S} \Longrightarrow g \circ f \in \mathcal{S}$

   1.2 **closed under base change (i.e., stable by pullback)**
   (**NOTE**:this condition is **not required on the above class** $\mathcal{S}'$.)

   1.3 and **containing all identity maps**, $X \xrightarrow{\mathrm{id}_X} X \in \mathcal{S}$.

# §1. A universal bivariant theory and Riemann-Roch formulas

1. Let $\mathcal{S}$ be another class of maps called "specialized maps" (e.g., smooth maps in algebraic geometry), which is
   1.1 **closed under composition**, $X \xrightarrow{f} Y, Y \xrightarrow{g} Z \in \mathcal{S} \Longrightarrow g \circ f \in \mathcal{S}$
   1.2 **closed under base change (i.e., stable by pullback)** (**NOTE**:this condition is **not required on the above class** $\mathcal{S}'$.)
   1.3 and **containing all identity maps**, $X \xrightarrow{\mathrm{id}_X} X \in \mathcal{S}$.
2. Let $\mathcal{S}$ be as above. If the orientation $\theta$ on $\mathcal{S}$ is **stable by pullback**,

# §1. A universal bivariant theory and Riemann-Roch formulas

1. Let $\mathcal{S}$ be another class of maps called "specialized maps" (e.g., smooth maps in algebraic geometry), which is
   1.1 **closed under composition**, $X \xrightarrow{f} Y, Y \xrightarrow{g} Z \in \mathcal{S} \Longrightarrow g \circ f \in \mathcal{S}$
   1.2 **closed under base change (i.e., stable by pullback)** (**NOTE**:this condition is **not required on the above class** $\mathcal{S}'$.)
   1.3 and **containing all identity maps**, $X \xrightarrow{\mathrm{id}_X} X \in \mathcal{S}$.
2. Let $\mathcal{S}$ be as above. If the orientation $\theta$ on $\mathcal{S}$ is **stable by pullback**,

# §1. A universal bivariant theory and Riemann-Roch formulas

1. Let $\mathcal{S}$ be another class of maps called "specialized maps" (e.g., smooth maps in algebraic geometry), which is
   1.1 **closed under composition**, $X \xrightarrow{f} Y, Y \xrightarrow{g} Z \in \mathcal{S} \Longrightarrow g \circ f \in \mathcal{S}$
   1.2 **closed under base change (i.e., stable by pullback)**
   (**NOTE**:this condition is **not required on the above class** $\mathcal{S}'$.)
   1.3 and **containing all identity maps**, $X \xrightarrow{\mathrm{id}_X} X \in \mathcal{S}$.
2. Let $\mathcal{S}$ be as above. If the orientation $\theta$ on $\mathcal{S}$ is **stable by pullback**, i.e., satisfies

$$\theta(f') = g^*\theta(f)$$

for an independent square with $f \in \mathcal{S}$

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow[g]{} & Y \end{array}$$

# §1. A universal bivariant theory and Riemann-Roch formulas

1. Let $\mathcal{S}$ be another class of maps called "specialized maps" (e.g., smooth maps in algebraic geometry), which is
   1.1 **closed under composition**, $X \xrightarrow{f} Y, Y \xrightarrow{g} Z \in \mathcal{S} \Longrightarrow g \circ f \in \mathcal{S}$
   1.2 **closed under base change (i.e., stable by pullback)** (**NOTE**:this condition is **not required on the above class** $\mathcal{S}'$.)
   1.3 and **containing all identity maps**, $X \xrightarrow{\mathrm{id}_X} X \in \mathcal{S}$.
2. Let $\mathcal{S}$ be as above. If the orientation $\theta$ on $\mathcal{S}$ is **stable by pullback**, i.e., satisfies

$$\theta(f') = g^*\theta(f)$$

for an independent square with $f \in \mathcal{S}$

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow[g]{} & Y \end{array}$$

$\theta$ is called a **nice canonical orientation** of $\mathbb{B}$.

# §1.1. A universal bivariant theory

From now on we assume that our category $\mathcal{V}$ satisfies that **any fiber square**

$$\begin{array}{ccc} P' & \longrightarrow & P \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Q' & \longrightarrow & Q \end{array}$$

**with $f$ being confined, i.e., $f \in \mathcal{C}$, is an independent square.**

# §1.1. A universal bivariant theory

From now on we assume that our category $\mathcal{V}$ satisfies that **any fiber square**

$$\begin{array}{ccc} P' & \longrightarrow & P \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Q' & \longrightarrow & Q \end{array}$$

**with $f$ being confined, i.e., $f \in \mathcal{C}$, is an independent square**.

## Theorem

*(S.Y., "Oriented bivariant theory, I", Int. J. Math. 20 (2009), 1305–1334), in fact a work of 2005-2006.)*

# §1.1. A universal bivariant theory

From now on we assume that our category $\mathcal{V}$ satisfies that **any fiber square**

$$\begin{array}{ccc} P' & \longrightarrow & P \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Q' & \longrightarrow & Q \end{array}$$

**with $f$ being confined, i.e., $f \in \mathcal{C}$, is an independent square**.

## Theorem

*(S.Y., "Oriented bivariant theory, I", Int. J. Math. 20 (2009), 1305–1334), in fact a work of 2005-2006.)*

# §1.1. A universal bivariant theory

From now on we assume that our category $\mathcal{V}$ satisfies that **any fiber square**

$$\begin{array}{ccc} P' & \longrightarrow & P \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Q' & \longrightarrow & Q \end{array}$$

**with $f$ being confined, i.e., $f \in \mathcal{C}$, is an independent square**.

## Theorem

*(S.Y., "Oriented bivariant theory, I", Int. J. Math. 20 (2009), 1305–1334), in fact a work of 2005-2006.)*

*Let $\mathcal{V}$ be a category with a class $\mathcal{C}$* **of confined maps***, a class $\mathcal{I}nd$* **of independent squares** *and a class $\mathcal{S}$* **of specialized maps***.*

# §1.1. A universal bivariant theory

From now on we assume that our category $\mathcal{V}$ satisfies that **any fiber square**

$$\begin{array}{ccc} P' & \longrightarrow & P \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Q' & \longrightarrow & Q \end{array}$$

**with $f$ being confined, i.e., $f \in \mathcal{C}$, is an independent square**.

## Theorem

*(S.Y., "Oriented bivariant theory, I", Int. J. Math. 20 (2009), 1305–1334), in fact a work of 2005-2006.)*

*Let $\mathcal{V}$ be a category with a class $\mathcal{C}$* **of confined maps**, *a class $\mathcal{I}nd$* **of independent squares** *and a class $\mathcal{S}$* **of specialized maps**. *Define*

$$\mathbb{M}^{\mathcal{C}}_{\mathcal{S}}(X \xrightarrow{f} Y)$$

*to be the* **free abelian group** *generated by the set of isomorphism classes of confined maps $h : W \to X$*

# §1.1. A universal bivariant theory

From now on we assume that our category $\mathcal{V}$ satisfies that **any fiber square**

$$\begin{array}{ccc} P' & \longrightarrow & P \\ {\scriptstyle f'}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ Q' & \longrightarrow & Q \end{array}$$

**with $f$ being confined, i.e., $f \in \mathcal{C}$, is an independent square**.

## Theorem

*(S.Y., "Oriented bivariant theory, I", Int. J. Math. 20 (2009), 1305–1334), in fact a work of 2005-2006.)*

*Let $\mathcal{V}$ be a category with a class $\mathcal{C}$* **of confined maps**, *a class $\mathcal{I}nd$* **of independent squares** *and a class $\mathcal{S}$* **of specialized maps**. *Define*

$$\mathbb{M}^{\mathcal{C}}_{\mathcal{S}}(X \xrightarrow{f} Y)$$

*to be the* **free abelian group** *generated by the set of isomorphism classes of confined maps $h : W \to X$ such that*

$$f \circ h : W \to Y \in \mathcal{S}.$$

# §1.1. A universal bivariant theory

From now on we assume that our category $\mathcal{V}$ satisfies that **any fiber square**

$$\begin{array}{ccc} P' & \longrightarrow & P \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Q' & \longrightarrow & Q \end{array}$$

**with** $f$ **being confined, i.e.,** $f \in \mathcal{C}$**, is an independent square**.

## Theorem

*(S.Y., "Oriented bivariant theory, I", Int. J. Math. 20 (2009), 1305–1334), in fact a work of 2005-2006.)*

*Let* $\mathcal{V}$ *be a category with a class* $\mathcal{C}$ **of confined maps**, *a class* $\mathcal{I}nd$ **of independent squares** *and a class* $\mathcal{S}$ **of specialized maps**. *Define*

$$\mathbb{M}^{\mathcal{C}}_{\mathcal{S}}(X \xrightarrow{f} Y)$$

*to be the* **free abelian group** *generated by the set of isomorphism classes of confined maps* $h: W \to X$ *such that*

$$f \circ h : W \to Y \in \mathcal{S}.$$

*(1) The association* $\mathbb{M}^{\mathcal{C}}_{\mathcal{S}}$ *is a bivariant theory, i.e., satisfies 7 axioms, if the bivariant operations are defined as follows:*

*(i).* **Product***:*

$$\bullet : \mathbb{M}_{\mathcal{S}}^{\mathcal{C}}(X \xrightarrow{f} Y) \otimes \mathbb{M}_{\mathcal{S}}^{\mathcal{C}}(Y \xrightarrow{g} Z) \to \mathbb{M}_{\mathcal{S}}^{\mathcal{C}}(X \xrightarrow{g \circ f} Z)$$

*is defined by*

$$[V \xrightarrow{h} X] \bullet [W \xrightarrow{k} Y] := [V' \xrightarrow{h \circ k''} X]$$

*and extended linearly, where*

$$\begin{array}{ccccccc}
V' & \xrightarrow{h'} & X' & \xrightarrow{f'} & W & & \\
{\scriptstyle k''}\downarrow & & {\scriptstyle k'}\downarrow & & {\scriptstyle k}\downarrow & & \\
V & \xrightarrow[h]{} & X & \xrightarrow[f]{} & Y & \xrightarrow[g]{} & Z.
\end{array}$$

*(i).* **Product***:*

$$\bullet : \mathbb{M}^{\mathcal{C}}_{\mathcal{S}}(X \xrightarrow{f} Y) \otimes \mathbb{M}^{\mathcal{C}}_{\mathcal{S}}(Y \xrightarrow{g} Z) \to \mathbb{M}^{\mathcal{C}}_{\mathcal{S}}(X \xrightarrow{g \circ f} Z)$$

*is defined by*

$$[V \xrightarrow{h} X] \bullet [W \xrightarrow{k} Y] := [V' \xrightarrow{h \circ k''} X]$$

*and extended linearly, where*

$$\begin{array}{ccccccc} V' & \xrightarrow{h'} & X' & \xrightarrow{f'} & W & & \\ {\scriptstyle k''}\downarrow & & {\scriptstyle k'}\downarrow & & {\scriptstyle k}\downarrow & & \\ V & \xrightarrow[h]{} & X & \xrightarrow[f]{} & Y & \xrightarrow[g]{} & Z. \end{array}$$

*(ii).* **Pushforward***: For $f : X \to Y$ and $g : Y \to Z$ with $f$ confined,*

$f_* : \mathbb{M}^{\mathcal{C}}_{\mathcal{S}}(X \xrightarrow{g \circ f} Z) \to \mathbb{M}^{\mathcal{C}}_{\mathcal{S}}(Y \xrightarrow{g} Z)$ *is defined by* $f_*\left([V \xrightarrow{h} X]\right) := [V \xrightarrow{f \circ h} Y]$

*and extended linearly.*

*(i).* **Product***:*

$$\bullet : \mathbb{M}^{\mathcal{C}}_{\mathcal{S}}(X \xrightarrow{f} Y) \otimes \mathbb{M}^{\mathcal{C}}_{\mathcal{S}}(Y \xrightarrow{g} Z) \to \mathbb{M}^{\mathcal{C}}_{\mathcal{S}}(X \xrightarrow{g\circ f} Z)$$

*is defined by*

$$[V \xrightarrow{h} X] \bullet [W \xrightarrow{k} Y] := [V' \xrightarrow{h\circ k''} X]$$

*and extended linearly, where*

$$\begin{array}{ccccccc} V' & \xrightarrow{h'} & X' & \xrightarrow{f'} & W & & \\ {\scriptstyle k''}\downarrow & & {\scriptstyle k'}\downarrow & & {\scriptstyle k}\downarrow & & \\ V & \xrightarrow[h]{} & X & \xrightarrow[f]{} & Y & \xrightarrow[g]{} & Z. \end{array}$$

*(ii).* **Pushforward***: For $f : X \to Y$ and $g : Y \to Z$ with $f$ <u>confined</u>,*

$f_* : \mathbb{M}^{\mathcal{C}}_{\mathcal{S}}(X \xrightarrow{g\circ f} Z) \to \mathbb{M}^{\mathcal{C}}_{\mathcal{S}}(Y \xrightarrow{g} Z)$ *is defined by* $f_*\left([V \xrightarrow{h} X]\right) := [V \xrightarrow{f\circ h} Y]$

*and extended linearly.* $V \xrightarrow{h} X \xrightarrow{f} Y \xrightarrow{g} Z \Longrightarrow V \xrightarrow{h} X \xrightarrow{f} Y \xrightarrow{g} Z$

*(i).* **Product***:*

$$\bullet : \mathbb{M}^{\mathcal{C}}_{\mathcal{S}}(X \xrightarrow{f} Y) \otimes \mathbb{M}^{\mathcal{C}}_{\mathcal{S}}(Y \xrightarrow{g} Z) \to \mathbb{M}^{\mathcal{C}}_{\mathcal{S}}(X \xrightarrow{g\circ f} Z)$$

*is defined by*

$$[V \xrightarrow{h} X] \bullet [W \xrightarrow{k} Y] := [V' \xrightarrow{h\circ k''} X]$$

*and extended linearly, where*

$$\begin{array}{ccccccc} V' & \xrightarrow{h'} & X' & \xrightarrow{f'} & W & & \\ {\scriptstyle k''}\downarrow & & {\scriptstyle k'}\downarrow & & {\scriptstyle k}\downarrow & & \\ V & \xrightarrow[h]{} & X & \xrightarrow[f]{} & Y & \xrightarrow[g]{} & Z. \end{array}$$

*(ii).* **Pushforward***: For $f : X \to Y$ and $g : Y \to Z$ with $f$ <u>confined</u>,*

$f_* : \mathbb{M}^{\mathcal{C}}_{\mathcal{S}}(X \xrightarrow{g\circ f} Z) \to \mathbb{M}^{\mathcal{C}}_{\mathcal{S}}(Y \xrightarrow{g} Z)$ *is defined by* $f_*\left([V \xrightarrow{h} X]\right) := [V \xrightarrow{f\circ h} Y]$

*and extended linearly.* $V \xrightarrow{h} X \xrightarrow{f} Y \xrightarrow{g} Z \Longrightarrow V \xrightarrow{h} X \xrightarrow{f} Y \xrightarrow{g} Z$

*(iii).***Pullback***: For an independent square*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow[g]{} & Y, \end{array}$$

$g^* : \mathbb{M}^{\mathcal{C}}_{\mathcal{S}}(X \xrightarrow{f} Y) \to \mathbb{M}^{\mathcal{C}}_{\mathcal{S}}(X' \xrightarrow{f'} Y')$ *is defined by* $g^*\left([V \xrightarrow{h} X]\right) := [V' \xrightarrow{h'} X']$

*and extended linearly, where*

$$\begin{array}{ccc} V' & \xrightarrow{g''} & V \\ {\scriptstyle h'}\downarrow & & \downarrow{\scriptstyle h} \\ X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow[g]{} & Y. \end{array}$$

*and extended linearly, where*

$$\begin{array}{ccc} V' & \xrightarrow{g''} & V \\ {\scriptstyle h'}\downarrow & & \downarrow{\scriptstyle h} \\ X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow[g]{} & Y. \end{array}$$

*(2) For a specialized map* $f : X \to Y \in \mathcal{S}$

$$\theta_{\mathbb{M}^{\mathcal{C}}_{\mathcal{S}}}(f) = [X \xrightarrow{\mathrm{id}_X} X] \in \mathbb{M}^{\mathcal{C}}_{\mathcal{S}}(X \xrightarrow{f} Y)$$

*is a nice canonical orientation of* $\mathbb{M}^{\mathcal{C}}_{\mathcal{S}}$ *for* $\mathcal{S}$.

*and extended linearly, where*

$$\begin{array}{ccc} V' & \xrightarrow{g''} & V \\ {\scriptstyle h'}\downarrow & & \downarrow{\scriptstyle h} \\ X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow[g]{} & Y. \end{array}$$

*(2) For a specialized map* $f : X \to Y \in \mathcal{S}$

$$\theta_{\mathbb{M}^{\mathcal{C}}_{\mathcal{S}}}(f) = [X \xrightarrow{\mathrm{id}_X} X] \in \mathbb{M}^{\mathcal{C}}_{\mathcal{S}}(X \xrightarrow{f} Y)$$

*is a nice canonical orientation of* $\mathbb{M}^{\mathcal{C}}_{\mathcal{S}}$ *for* $\mathcal{S}$. $X \xrightarrow{\mathrm{id}_X} X \xrightarrow{f} Y$

*and extended linearly, where*

$$\begin{array}{ccc} V' & \xrightarrow{g''} & V \\ {\scriptstyle h'}\downarrow & & \downarrow{\scriptstyle h} \\ X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow[g]{} & Y. \end{array}$$

*(2) For a specialized map* $f : X \to Y \in \mathcal{S}$

$$\theta_{\mathbb{M}^{\mathcal{C}}_{\mathcal{S}}}(f) = [X \xrightarrow{\mathrm{id}_X} X] \in \mathbb{M}^{\mathcal{C}}_{\mathcal{S}}(X \xrightarrow{f} Y)$$

*is a nice canonical orientation of* $\mathbb{M}^{\mathcal{C}}_{\mathcal{S}}$ *for* $\mathcal{S}$. $X \xrightarrow{\mathrm{id}_X} X \xrightarrow{f} Y$

*(3) (**A universality of** $\mathbb{M}^{\mathcal{C}}_{\mathcal{S}}$) Let* $\mathbb{B}$ *be a bivariant theory on the same* $\mathcal{V}$ *with the same* $\mathcal{C}, \mathcal{I}nd$ *and* $\mathcal{S}$, *and* $\theta_{\mathbb{B}}$ *a nice canonical orientation of* $\mathbb{B}$ *for* $\mathcal{S}$.

*and extended linearly, where*

$$\begin{array}{ccc} V' & \xrightarrow{g''} & V \\ {\scriptstyle h'}\downarrow & & \downarrow{\scriptstyle h} \\ X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow[g]{} & Y. \end{array}$$

*(2) For a specialized map* $f : X \to Y \in \mathcal{S}$

$$\theta_{\mathbb{M}^{\mathcal{C}}_{\mathcal{S}}}(f) = [X \xrightarrow{\mathrm{id}_X} X] \in \mathbb{M}^{\mathcal{C}}_{\mathcal{S}}(X \xrightarrow{f} Y)$$

*is a nice canonical orientation of* $\mathbb{M}^{\mathcal{C}}_{\mathcal{S}}$ *for* $\mathcal{S}$. $X \xrightarrow{\mathrm{id}_X} X \xrightarrow{f} Y$

*(3) (**A universality of** $\mathbb{M}^{\mathcal{C}}_{\mathcal{S}}$) Let* $\mathbb{B}$ *be a bivariant theory on the same* $\mathcal{V}$ *with the same* $\mathcal{C}$, $\mathcal{I}nd$ *and* $\mathcal{S}$, *and* $\theta_{\mathbb{B}}$ *a nice canonical orientation of* $\mathbb{B}$ *for* $\mathcal{S}$. *Then there exists a unique Grothendieck transformation*

$$\gamma_{\mathbb{B}} : \mathbb{M}^{\mathcal{C}}_{\mathcal{S}} \to \mathbb{B}$$

*and extended linearly, where*

$$\begin{array}{ccc} V' & \xrightarrow{g''} & V \\ {\scriptstyle h'}\downarrow & & \downarrow{\scriptstyle h} \\ X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow[g]{} & Y. \end{array}$$

*(2) For a specialized map $f : X \to Y \in \mathcal{S}$*

$$\theta_{\mathbb{M}^{\mathcal{C}}_{\mathcal{S}}}(f) = [X \xrightarrow{\mathrm{id}_X} X] \in \mathbb{M}^{\mathcal{C}}_{\mathcal{S}}(X \xrightarrow{f} Y)$$

*is a nice canonical orientation of $\mathbb{M}^{\mathcal{C}}_{\mathcal{S}}$ for $\mathcal{S}$.* $X \xrightarrow{\mathrm{id}_X} X \xrightarrow{f} Y$

*(3)* (**A universality of** $\mathbb{M}^{\mathcal{C}}_{\mathcal{S}}$) *Let $\mathbb{B}$ be a bivariant theory on the same $\mathcal{V}$ with the same $\mathcal{C}, \mathcal{I}nd$ and $\mathcal{S}$, and $\theta_{\mathbb{B}}$ a nice canonical orientation of $\mathbb{B}$ for $\mathcal{S}$. Then there exists a unique Grothendieck transformation*

$$\gamma_{\mathbb{B}} : \mathbb{M}^{\mathcal{C}}_{\mathcal{S}} \to \mathbb{B}$$

*such that for $X \xrightarrow{f} Y \in \mathcal{S}$, $\gamma_{\mathbb{B}} : \mathbb{M}^{\mathcal{C}}_{\mathcal{S}}(X \xrightarrow{f} Y) \to \mathbb{B}(X \xrightarrow{f} Y)$ satisfies*

$$\gamma_{\mathbb{B}}(\theta_{\mathbb{M}^{\mathcal{C}}_{\mathcal{S}}}(f)) = \theta_{\mathbb{B}}(f).$$

*and extended linearly, where*

$$\begin{array}{ccc} V' & \xrightarrow{g''} & V \\ {\scriptstyle h'}\downarrow & & \downarrow{\scriptstyle h} \\ X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow[g]{} & Y. \end{array}$$

*(2) For a specialized map* $f : X \to Y \in \mathcal{S}$

$$\theta_{\mathbb{M}^{\mathcal{C}}_{\mathcal{S}}}(f) = [X \xrightarrow{\mathrm{id}_X} X] \in \mathbb{M}^{\mathcal{C}}_{\mathcal{S}}(X \xrightarrow{f} Y)$$

*is a nice canonical orientation of* $\mathbb{M}^{\mathcal{C}}_{\mathcal{S}}$ *for* $\mathcal{S}$. $X \xrightarrow{\mathrm{id}_X} X \xrightarrow{f} Y$
*(3) (**A universality of** $\mathbb{M}^{\mathcal{C}}_{\mathcal{S}}$) Let* $\mathbb{B}$ *be a bivariant theory on the same* $\mathcal{V}$ *with the same* $\mathcal{C}, \mathcal{I}nd$ *and* $\mathcal{S}$*, and* $\theta_{\mathbb{B}}$ *a nice canonical orientation of* $\mathbb{B}$ *for* $\mathcal{S}$*. Then there exists a unique Grothendieck transformation*

$$\gamma_{\mathbb{B}} : \mathbb{M}^{\mathcal{C}}_{\mathcal{S}} \to \mathbb{B}$$

*such that for* $X \xrightarrow{f} Y \in \mathcal{S}$, $\gamma_{\mathbb{B}} : \mathbb{M}^{\mathcal{C}}_{\mathcal{S}}(X \xrightarrow{f} Y) \to \mathbb{B}(X \xrightarrow{f} Y)$ *satisfies*

$$\gamma_{\mathbb{B}}(\theta_{\mathbb{M}^{\mathcal{C}}_{\mathcal{S}}}(f)) = \theta_{\mathbb{B}}(f).$$

*(In a sense, this is a RR-formula with* $u_f = 1_X$ *or* $d_f = 1_Y$*.)*

# Commutativity

$\mathbb{M}_{\mathcal{S}}^{\mathcal{C}}$ is commutative in the following sense: for the fiber square

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Z' & \xrightarrow[g]{} & Z, \end{array}$$

and $\forall \alpha \in \mathbb{M}_{\mathcal{S}}^{\mathcal{C}}(X \xrightarrow{f} Z), \forall \beta \in \mathbb{M}_{\mathcal{S}}^{\mathcal{C}}(Z' \xrightarrow{g} Z)$ we have

$$g^*(\alpha) \bullet \beta = f^*(\beta) \bullet \alpha.$$

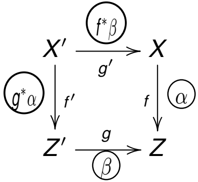
# Commutativity

$\mathbb{M}_{\mathcal{S}}^{\mathcal{C}}$ is commutative in the following sense: for the fiber square

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Z' & \xrightarrow[g]{} & Z, \end{array}$$

and $\forall \alpha \in \mathbb{M}_{\mathcal{S}}^{\mathcal{C}}(X \xrightarrow{f} Z), \forall \beta \in \mathbb{M}_{\mathcal{S}}^{\mathcal{C}}(Z' \xrightarrow{g} Z)$ we have

$$g^*(\alpha) \bullet \beta = f^*(\beta) \bullet \alpha.$$



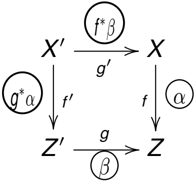
# Commutativity

$\mathbb{M}^{\mathcal{C}}_{\mathcal{S}}$ is commutative in the following sense: for the fiber square

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Z' & \xrightarrow[g]{} & Z, \end{array}$$

and $\forall \alpha \in \mathbb{M}^{\mathcal{C}}_{\mathcal{S}}(X \xrightarrow{f} Z), \forall \beta \in \mathbb{M}^{\mathcal{C}}_{\mathcal{S}}(Z' \xrightarrow{g} Z)$ we have

$$g^*(\alpha) \bullet \beta = f^*(\beta) \bullet \alpha.$$




If $g^*(\alpha) \bullet \beta = (-1)^{\deg(\alpha)\deg(\beta)} f^*(\beta) \bullet \alpha$, it is called *skew-commutative* (see [Part I:Bivariant Theories] of Fulton-MacPherson's book).

# Remark

The main purpose of introducing the UBT $\mathbb{M}_{\mathcal{S}}^{\mathcal{C}}(X \to Y)$

# Remark

The main purpose of introducing the UBT $\mathbb{M}^{\mathcal{C}}_{\mathcal{S}}(X \to Y)$ was constructing a bivariant analogue

$$\mathbb{B}\Omega(X \xrightarrow{f} Y)$$

of Levine–Morel's algebraic cobordism $\Omega^*(X)$:

• Levine-Morel, "Algebraic Cobordism", Springer-Verlag (2007).

• Levine-Pandharipande, "Algebraic cobordism revisited", Invent. Math., 176 (2009), 63-130.

# Remark

The main purpose of introducing the UBT $\mathbb{M}^{\mathcal{C}}_{\mathcal{S}}(X \to Y)$ was constructing a bivariant analogue

$$\mathbb{B}\Omega(X \xrightarrow{f} Y)$$

of Levine–Morel's algebraic cobordism $\Omega^*(X)$:

• Levine-Morel, "Algebraic Cobordism", Springer-Verlag (2007).

• Levine-Pandharipande, "Algebraic cobordism revisited", Invent. Math., 176 (2009), 63-130.

so that

1. $\mathbb{B}\Omega(X \to pt) \cong \Omega_*(X)$ Levine-Morel's "algebraic cobrodism"

# Remark

The main purpose of introducing the UBT $\mathbb{M}^{\mathcal{C}}_{\mathcal{S}}(X \to Y)$ was constructing a bivariant analogue

$$\mathbb{B}\Omega(X \xrightarrow{f} Y)$$

of Levine–Morel's algebraic cobordism $\Omega^*(X)$:

• Levine-Morel, "Algebraic Cobordism", Springer-Verlag (2007).

• Levine-Pandharipande, "Algebraic cobordism revisited", Invent. Math., 176 (2009), 63-130.

so that

1. $\mathbb{B}\Omega(X \to pt) \cong \Omega_*(X)$ Levine-Morel's "algebraic cobrodism"

# Remark

The main purpose of introducing the UBT $\mathbb{M}^{\mathcal{C}}_{\mathcal{S}}(X \to Y)$ was constructing a bivariant analogue

$$\mathbb{B}\Omega(X \xrightarrow{f} Y)$$

of Levine–Morel's algebraic cobordism $\Omega^*(X)$:

• Levine-Morel, "Algebraic Cobordism", Springer-Verlag (2007).

• Levine-Pandharipande, "Algebraic cobordism revisited", Invent. Math., 176 (2009), 63-130.

so that

1. $\mathbb{B}\Omega(X \to pt) \cong \Omega_*(X)$ Levine-Morel's "algebraic cobrodism" (NOTE:Levine-Morel's algebraic cobordism is **in fact not "cobordism", but a "bordism" theory**.)

# Remark

The main purpose of introducing the UBT $\mathbb{M}^{\mathcal{C}}_{\mathcal{S}}(X \to Y)$ was constructing a bivariant analogue

$$\mathbb{B}\Omega(X \xrightarrow{f} Y)$$

of Levine–Morel's algebraic cobordism $\Omega^*(X)$:

• Levine-Morel, "Algebraic Cobordism", Springer-Verlag (2007).

• Levine-Pandharipande, "Algebraic cobordism revisited", Invent. Math., 176 (2009), 63-130.

so that

1. $\mathbb{B}\Omega(X \to pt) \cong \Omega_*(X)$ Levine-Morel's "algebraic cobrodism" (NOTE:Levine-Morel's algebraic cobordism is **in fact not "cobordism", but a "bordism" theory**.)
2. $\mathbb{B}\Omega(X \xrightarrow{\mathrm{id}_X} X) =: \mathbb{B}\Omega^*(X)$ is a new "algebraic cobrodism"

# Remark

The main purpose of introducing the UBT $\mathbb{M}^{\mathcal{C}}_{\mathcal{S}}(X \to Y)$ was constructing a bivariant analogue

$$\mathbb{B}\Omega(X \xrightarrow{f} Y)$$

of Levine–Morel's algebraic cobordism $\Omega^*(X)$:

• Levine-Morel, "Algebraic Cobordism", Springer-Verlag (2007).

• Levine-Pandharipande, "Algebraic cobordism revisited", Invent. Math., 176 (2009), 63-130.

so that

1. $\mathbb{B}\Omega(X \to pt) \cong \Omega_*(X)$ Levine-Morel's "algebraic cobrodism" (NOTE:Levine-Morel's algebraic cobordism is **in fact not "cobordism", but a "bordism" theory**.)
2. $\mathbb{B}\Omega(X \xrightarrow{\mathrm{id}_X} X) =: \mathbb{B}\Omega^*(X)$ is a new "algebraic cobrodism"

# Remark

The main purpose of introducing the UBT $\mathbb{M}^{\mathcal{C}}_{\mathcal{S}}(X \to Y)$ was constructing a bivariant analogue

$$\mathbb{B}\Omega(X \xrightarrow{f} Y)$$

of Levine–Morel's algebraic cobordism $\Omega^*(X)$:

• Levine-Morel, "Algebraic Cobordism", Springer-Verlag (2007).

• Levine-Pandharipande, "Algebraic cobordism revisited", Invent. Math., 176 (2009), 63-130.

so that

1. $\mathbb{B}\Omega(X \to pt) \cong \Omega_*(X)$ Levine-Morel's "algebraic cobrodism" (NOTE:Levine-Morel's algebraic cobordism is **in fact not "cobordism", but a "bordism" theory**.)
2. $\mathbb{B}\Omega(X \xrightarrow{\mathrm{id}_X} X) =: \mathbb{B}\Omega^*(X)$ is a new "algebraic cobrodism"

My plan was to mod out $\mathbb{M}^{\mathcal{C}}_{\mathcal{S}}(X \to Y)$ by some subgroups $\mathcal{R}^{\mathcal{C}}_{\mathcal{S}}(X \to Y)$ obtained by a "bivariant version" of Levine–Morel's relations: :

$$\mathbb{B}\Omega(X \xrightarrow{f} Y) := \frac{\mathbb{M}^{\mathcal{C}}_{\mathcal{S}}(X \to Y)}{\mathcal{R}^{\mathcal{C}}_{\mathcal{S}}(X \to Y)}.$$

# Remark

The main purpose of introducing the UBT $\mathbb{M}^{\mathcal{C}}_{\mathcal{S}}(X \to Y)$ was constructing a bivariant analogue

$$\mathbb{B}\Omega(X \xrightarrow{f} Y)$$

of Levine–Morel's algebraic cobordism $\Omega^*(X)$:

• Levine-Morel, "Algebraic Cobordism", Springer-Verlag (2007).

• Levine-Pandharipande, "Algebraic cobordism revisited", Invent. Math., 176 (2009), 63-130.

so that

1. $\mathbb{B}\Omega(X \to pt) \cong \Omega_*(X)$ Levine-Morel's "algebraic cobrodism" (NOTE:Levine-Morel's algebraic cobordism is **in fact not "cobordism", but a "bordism" theory**.)
2. $\mathbb{B}\Omega(X \xrightarrow{\mathrm{id}_X} X) =: \mathbb{B}\Omega^*(X)$ is a new "algebraic cobrodism"

My plan was to mod out $\mathbb{M}^{\mathcal{C}}_{\mathcal{S}}(X \to Y)$ by some subgroups $\mathcal{R}^{\mathcal{C}}_{\mathcal{S}}(X \to Y)$ obtained by a "bivariant version" of Levine–Morel's relations: :

$$\mathbb{B}\Omega(X \xrightarrow{f} Y) := \frac{\mathbb{M}^{\mathcal{C}}_{\mathcal{S}}(X \to Y)}{\mathcal{R}^{\mathcal{C}}_{\mathcal{S}}(X \to Y)}.$$

Toni Annala (Univ.British Columbia) succeeded in doing this:
"Bivariant derived algebraic cobordism", J. Algebraic Geometry, 2020. (50 pp)

# Remark

The main purpose of introducing the UBT $\mathbb{M}^{\mathcal{C}}_{\mathcal{S}}(X \to Y)$ was constructing a bivariant analogue

$$\mathbb{B}\Omega(X \xrightarrow{f} Y)$$

of Levine–Morel's algebraic cobordism $\Omega^*(X)$:
• Levine-Morel, "Algebraic Cobordism", Springer-Verlag (2007).
• Levine-Pandharipande, "Algebraic cobordism revisited", Invent. Math., 176 (2009), 63-130.
so that

1. $\mathbb{B}\Omega(X \to pt) \cong \Omega_*(X)$ Levine-Morel's "algebraic cobrodism" (NOTE:Levine-Morel's algebraic cobordism is **in fact not "cobordism", but a "bordism" theory**.)
2. $\mathbb{B}\Omega(X \xrightarrow{\mathrm{id}_X} X) =: \mathbb{B}\Omega^*(X)$ is a new "algebraic cobrodism"

My plan was to mod out $\mathbb{M}^{\mathcal{C}}_{\mathcal{S}}(X \to Y)$ by some subgroups $\mathcal{R}^{\mathcal{C}}_{\mathcal{S}}(X \to Y)$ obtained by a "bivariant version" of Levine–Morel's relations: :

$$\mathbb{B}\Omega(X \xrightarrow{f} Y) := \frac{\mathbb{M}^{\mathcal{C}}_{\mathcal{S}}(X \to Y)}{\mathcal{R}^{\mathcal{C}}_{\mathcal{S}}(X \to Y)}.$$

Toni Annala (Univ.British Columbia) succeeded in doing this:
"Bivariant derived algebraic cobordism", J. Algebraic Geometry, 2020. (50 pp)
using Lowrey-Schürg's "Derived algebraic cobordism", J. Inst. Math. Jussieu, 15 (2016),407-443" and UBT $\mathbb{M}^{\mathcal{C}}_{\mathcal{S}}$.

# §1.2. Riemann–Roch formulas for the universal BT $\mathbb{M}_{S}^{\mathcal{C}}$

We can define many (in fact, infinitely many) nice canonical orientations.

# §1.2. Riemann–Roch formulas for the universal BT $\mathbb{M}_{\mathcal{S}}^{\mathcal{C}}$

We can define many (in fact, infinitely many) nice canonical orientations. First we define the following, motivated by the relative dim. $\mathrm{reldim}(f)$ of a smooth map $f : X \to Y$ in algebraic geometry:

# §1.2. Riemann–Roch formulas for the universal BT $\mathbb{M}_{\mathcal{S}}^{\mathcal{C}}$

We can define many (in fact, infinitely many) nice canonical orientations. First we define the following, motivated by the relative dim. $\mathrm{reldim}(f)$ of a smooth map $f : X \to Y$ in algebraic geometry:

## Definition

If there is an assignment $\mathrm{reldim}(f)$ for a specialized map $f : X \to Y$ such that

# §1.2. Riemann–Roch formulas for the universal BT $\mathbb{M}_{\mathcal{S}}^{\mathcal{C}}$

We can define many (in fact, infinitely many) nice canonical orientations. First we define the following, motivated by the relative dim. $\mathrm{reldim}(f)$ of a smooth map $f : X \to Y$ in algebraic geometry:

## Definition

If there is an assignment $\mathrm{reldim}(f)$ for a specialized map $f : X \to Y$ such that

1. $\mathrm{reldim}(f)$ is a non-negative integer and

# §1.2. Riemann–Roch formulas for the universal BT $\mathbb{M}_{\mathcal{S}}^{\mathcal{C}}$

We can define many (in fact, infinitely many) nice canonical orientations. First we define the following, motivated by the relative dim. $\mathrm{reldim}(f)$ of a smooth map $f : X \to Y$ in algebraic geometry:

## Definition

If there is an assignment $\mathrm{reldim}(f)$ for a specialized map $f : X \to Y$ such that

1. $\mathrm{reldim}(f)$ is a non-negative integer and
2. it satisfies the following conditions

# §1.2. Riemann–Roch formulas for the universal BT $\mathbb{M}^{\mathcal{C}}_{\mathcal{S}}$

We can define many (in fact, infinitely many) nice canonical orientations. First we define the following, motivated by the relative dim. $\mathrm{reldim}(f)$ of a smooth map $f : X \to Y$ in algebraic geometry:

## Definition

If there is an assignment $\mathrm{reldim}(f)$ for a specialized map $f : X \to Y$ such that

1. $\mathrm{reldim}(f)$ is a non-negative integer and
2. it satisfies the following conditions
   2.1 for $f : X \to Y, g : Y \to Z \in \mathcal{S}$

$$\mathrm{reldim}(g \circ f) = \mathrm{reldim}(g) + \mathrm{reldim}(f),$$

# §1.2. Riemann–Roch formulas for the universal BT $\mathbb{M}^{\mathcal{C}}_{\mathcal{S}}$

We can define many (in fact, infinitely many) nice canonical orientations. First we define the following, motivated by the relative dim. $\mathrm{reldim}(f)$ of a smooth map $f : X \to Y$ in algebraic geometry:

## Definition

If there is an assignment $\mathrm{reldim}(f)$ for a specialized map $f : X \to Y$ such that

1. $\mathrm{reldim}(f)$ is a non-negative integer and
2. it satisfies the following conditions
   2.1 for $f : X \to Y, g : Y \to Z \in \mathcal{S}$
   $$\mathrm{reldim}(g \circ f) = \mathrm{reldim}(g) + \mathrm{reldim}(f),$$
   2.2 $\mathrm{reldim}(\mathrm{id}_X) = 0$.

# §1.2. Riemann–Roch formulas for the universal BT $\mathbb{M}^{\mathcal{C}}_{\mathcal{S}}$

We can define many (in fact, infinitely many) nice canonical orientations. First we define the following, motivated by the relative dim. $\mathrm{reldim}(f)$ of a smooth map $f : X \to Y$ in algebraic geometry:

## Definition

If there is an assignment $\mathrm{reldim}(f)$ for a specialized map $f : X \to Y$ such that

1. $\mathrm{reldim}(f)$ is a non-negative integer and
2. it satisfies the following conditions
   - 2.1 for $f : X \to Y, g : Y \to Z \in \mathcal{S}$
   $$\mathrm{reldim}(g \circ f) = \mathrm{reldim}(g) + \mathrm{reldim}(f),$$
   - 2.2 $\mathrm{reldim}(\mathrm{id}_X) = 0$.
   - 2.3 for an independent square where $f, f' \in \mathcal{S}$, $\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow[g]{} & Y \end{array}$
   $$\mathrm{reldim}(f) = \mathrm{reldim}(f').$$

# §1.2. Riemann–Roch formulas for the universal BT $\mathbb{M}^{\mathcal{C}}_{\mathcal{S}}$

We can define many (in fact, infinitely many) nice canonical orientations. First we define the following, motivated by the relative dim. $\mathrm{reldim}(f)$ of a smooth map $f : X \to Y$ in algebraic geometry:

## Definition

If there is an assignment $\mathrm{reldim}(f)$ for a specialized map $f : X \to Y$ such that

1. $\mathrm{reldim}(f)$ is a non-negative integer and
2. it satisfies the following conditions
   - 2.1 for $f : X \to Y, g : Y \to Z \in \mathcal{S}$
   $$\mathrm{reldim}(g \circ f) = \mathrm{reldim}(g) + \mathrm{reldim}(f),$$
   - 2.2 $\mathrm{reldim}(\mathrm{id}_X) = 0.$
   - 2.3 for an independent square where $f, f' \in \mathcal{S}$, $\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow[g]{} & Y \end{array}$
   $\mathrm{reldim}(f) = \mathrm{reldim}(f').$

# §1.2. Riemann–Roch formulas for the universal BT $\mathbb{M}^{\mathcal{C}}_{\mathcal{S}}$

We can define many (in fact, infinitely many) nice canonical orientations. First we define the following, motivated by the relative dim. $\mathrm{reldim}(f)$ of a smooth map $f : X \to Y$ in algebraic geometry:

## Definition

If there is an assignment $\mathrm{reldim}(f)$ for a specialized map $f : X \to Y$ such that

1. $\mathrm{reldim}(f)$ is a non-negative integer and
2. it satisfies the following conditions
    - 2.1 for $f : X \to Y, g : Y \to Z \in \mathcal{S}$

$$\mathrm{reldim}(g \circ f) = \mathrm{reldim}(g) + \mathrm{reldim}(f),$$

    - 2.2 $\mathrm{reldim}(\mathrm{id}_X) = 0$.
    - 2.3 for an independent square where $f, f' \in \mathcal{S}$,

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow[g]{} & Y \end{array}$$

$$\mathrm{reldim}(f) = \mathrm{reldim}(f').$$

then the integer $\mathrm{reldim}(f)$ is called **a relative dimension** of $f : X \to Y$.

# §1.2. Riemann–Roch formulas for the universal BT $\mathbb{M}^{\mathcal{C}}_{\mathcal{S}}$

We can define many (in fact, infinitely many) nice canonical orientations. First we define the following, motivated by the relative dim. $\mathrm{reldim}(f)$ of a smooth map $f : X \to Y$ in algebraic geometry:

## Definition

If there is an assignment $\mathrm{reldim}(f)$ for a specialized map $f : X \to Y$ such that

1. $\mathrm{reldim}(f)$ is a non-negative integer and
2. it satisfies the following conditions
   2.1 for $f : X \to Y, g : Y \to Z \in \mathcal{S}$
   $$\mathrm{reldim}(g \circ f) = \mathrm{reldim}(g) + \mathrm{reldim}(f),$$
   2.2 $\mathrm{reldim}(\mathrm{id}_X) = 0$.
   2.3 for an independent square where $f, f' \in \mathcal{S}$,
   $$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow[g]{} & Y \end{array}$$
   $\mathrm{reldim}(f) = \mathrm{reldim}(f')$.

then the integer $\mathrm{reldim}(f)$ is called **a relative dimension** of $f : X \to Y$.

**REMARK:** Cleary there is a very trivial one: $\mathrm{reldim}(f) := 0$ for $\forall f \in \mathcal{S}$.

1. If $a_F : F \to pt$ is confined (e.g., a proper map), $F$ is called *confined* (e.g., a compact space).

1. If $a_F : F \to pt$ is confined (e.g., a proper map), $F$ is called *confined* (e.g., a compact space).
2. If $a_F : F \to pt$ is specialized (e.g., a smooth map), $F$ is called *specialized* (e.g., a smooth variety).

1. If $a_F : F \to pt$ is confined (e.g., a proper map), $F$ is called *confined* (e.g., a compact space).
2. If $a_F : F \to pt$ is specialized (e.g., a smooth map), $F$ is called *specialized* (e.g., a smooth variety).
3. If $a_F : F \to pt$ is confined and specialized, $F$ is called *confined and specialized* (e.g., a compact smooth variety).

1. If $a_F : F \to pt$ is confined (e.g., a proper map), $F$ is called *confined* (e.g., a compact space).
2. If $a_F : F \to pt$ is specialized (e.g., a smooth map), $F$ is called *specialized* (e.g., a smooth variety).
3. If $a_F : F \to pt$ is confined and specialized, $F$ is called *confined and specialized* (e.g., a compact smooth variety).

1. If $a_F : F \to pt$ is confined (e.g., a proper map), $F$ is called *confined* (e.g., a compact space).
2. If $a_F : F \to pt$ is specialized (e.g., a smooth map), $F$ is called *specialized* (e.g., a smooth variety).
3. If $a_F : F \to pt$ is confined and specialized, $F$ is called *confined and specialized* (e.g., a compact smooth variety).

Let $F$ be confined and specialized. Hence $[F \xrightarrow{a_F} pt] \in \mathbb{M}^{\mathcal{C}}_{\mathcal{S}}(pt \to pt)$. For $a_X : X \to pt$,

$$(a_X)^*[F \xrightarrow{a_F} pt] = [X \times F \xrightarrow{pr_1} X] \in \mathbb{M}^{\mathcal{C}}_{\mathcal{S}}(X \xrightarrow{\mathrm{id}_X} X)$$

$$\begin{array}{ccc}
X \times F & \longrightarrow & F \\
{\scriptstyle pr_1}\downarrow & & \downarrow{\scriptstyle a_F} \\
X & \xrightarrow{a_X} & pt \\
{\scriptstyle \mathrm{id}_X}\downarrow & & \downarrow \\
X & \underset{a_X}{\longrightarrow} & pt.
\end{array}$$

NOTE:for $[F \xrightarrow{a_F} pt] \in \mathbb{M}^{\mathcal{C}}_{\mathcal{S}}(pt \to pt)$

$$[F \xrightarrow{a_F} pt] \bullet [F \xrightarrow{a_F} pt] = [F \times F \xrightarrow{a_{F \times F}} pt] \in \mathbb{M}^{\mathcal{C}}_{\mathcal{S}}(pt \to pt)$$

by the definition of the bivariant product $\bullet$:

$$\begin{array}{ccccccc}
F \times F & \xrightarrow{pr_2} & F & \xrightarrow{\mathrm{id}_F} & F & & \\
{\scriptstyle pr_1}\downarrow & & {\scriptstyle a_F}\downarrow & & {\scriptstyle a_F}\downarrow & & \\
F & \xrightarrow[a_F]{} & pt & \longrightarrow & pt & \longrightarrow & pt.
\end{array}$$

$$\begin{array}{ccccccc}
F \times F & \xrightarrow{pr_2} & F & \xrightarrow{\mathrm{id}_F} & F & & \\
{\scriptstyle pr_1}\downarrow & \searrow^{a_{F \times F}} & \downarrow{\scriptstyle a_F} & & \downarrow{\scriptstyle a_F} & & \\
F & \xrightarrow[a_F]{} & pt & \longrightarrow & pt & \longrightarrow & pt
\end{array}$$

Then we have

$$
\begin{aligned}
([F \xrightarrow{a_F} pt] \bullet [F \xrightarrow{a_F} pt]) \bullet [F \xrightarrow{a_F} pt] &= [F \times F \xrightarrow{a_{F \times F}} pt] \bullet [F \xrightarrow{a_F} pt] \\
&= [F \times F \times F \xrightarrow{a_{F \times F \times F}} pt] \\
&= [F^3 \xrightarrow{a_{F^3}} pt]
\end{aligned}
$$

Then we have

$$
\begin{aligned}
([F \xrightarrow{a_F} pt] \bullet [F \xrightarrow{a_F} pt]) \bullet [F \xrightarrow{a_F} pt] \\
&= [F \times F \xrightarrow{a_{F\times F}} pt] \bullet [F \xrightarrow{a_F} pt] \\
&= [F \times F \times F \xrightarrow{a_{F\times F\times F}} pt] \\
&= [F^3 \xrightarrow{a_{F^3}} pt]
\end{aligned}
$$

By induction, for $n$ we have

$$
\begin{aligned}
[F \xrightarrow{a_F} pt]^n &:= \overbrace{[F \xrightarrow{a_F} pt] \bullet \cdots \bullet [F \xrightarrow{a_F} pt]}^{n} \\
&= [F^n \xrightarrow{a_{F^n}} pt]
\end{aligned}
$$

Then we have

$$
\begin{aligned}
([F \xrightarrow{a_F} pt] \bullet [F \xrightarrow{a_F} pt]) \bullet [F \xrightarrow{a_F} pt] & \\
&= [F \times F \xrightarrow{a_{F\times F}} pt] \bullet [F \xrightarrow{a_F} pt] \\
&= [F \times F \times F \xrightarrow{a_{F\times F\times F}} pt] \\
&= [F^3 \xrightarrow{a_{F^3}} pt]
\end{aligned}
$$

By induction, for $n$ we have

$$
\begin{aligned}
[F \xrightarrow{a_F} pt]^n &:= \overbrace{[F \xrightarrow{a_F} pt] \bullet \cdots \bullet [F \xrightarrow{a_F} pt]}^{n} \\
&= [F^n \xrightarrow{a_{F^n}} pt]
\end{aligned}
$$

Therefore we get

$$
(a_X)^*([F \xrightarrow{a_F} pt]^n) = [X \times F^n \xrightarrow{pr_1} X] \in \mathbb{M}_{\mathcal{S}}^{\mathcal{C}}(X \xrightarrow{\mathrm{id}_X} X)
$$

Then we have

$$\begin{aligned}([F \xrightarrow{a_F} pt] \bullet [F \xrightarrow{a_F} pt]) \bullet [F \xrightarrow{a_F} pt] &= [F \times F \xrightarrow{a_{F\times F}} pt] \bullet [F \xrightarrow{a_F} pt] \\ &= [F \times F \times F \xrightarrow{a_{F\times F\times F}} pt] \\ &= [F^3 \xrightarrow{a_{F^3}} pt]\end{aligned}$$

By induction, for $n$ we have

$$\begin{aligned}[F \xrightarrow{a_F} pt]^n := \overbrace{[F \xrightarrow{a_F} pt] \bullet \cdots \bullet [F \xrightarrow{a_F} pt]}^{n} \\ = [F^n \xrightarrow{a_{F^n}} pt]\end{aligned}$$

Therefore we get

$$(a_X)^*([F \xrightarrow{a_F} pt]^n) = [X \times F^n \xrightarrow{pr_1} X] \in \mathbb{M}^{\mathcal{C}}_{\mathcal{S}}(X \xrightarrow{\mathrm{id}_X} X)$$

**Convention**: For $n = 0$ we define $[F \xrightarrow{a_F} pt]^0 := [pt \to pt]$ and $F^0 := pt$.

By the above construction $X \times F^n \xrightarrow{pr_1} X$ is a confined and specialized map.

By the above construction $X \times F^n \xrightarrow{pr_1} X$ is a confined and specialized map. Hence, for a specialized map $f : X \to Y$,

$$f \circ pr_1 : X \times F^n \xrightarrow{pr_1} X \xrightarrow{f} Y$$

becomes a specialized map,

By the above construction $X \times F^n \xrightarrow{pr_1} X$ is a confined and specialized map. Hence, for a specialized map $f : X \to Y$,

$$f \circ pr_1 : X \times F^n \xrightarrow{pr_1} X \xrightarrow{f} Y$$

becomes a specialized map, hence we have

$$[X \times F^n \xrightarrow{pr_1} X] \in \mathbb{M}^{\mathcal{C}}_{\mathcal{S}}(X \xrightarrow{f} Y).$$

By the above construction $X \times F^n \xrightarrow{pr_1} X$ is a confined and specialized map. Hence, for a specialized map $f : X \to Y$,

$$f \circ pr_1 : X \times F^n \xrightarrow{pr_1} X \xrightarrow{f} Y$$

becomes a specialized map, hence we have

$$[X \times F^n \xrightarrow{pr_1} X] \in \mathbb{M}^{\mathcal{C}}_{\mathcal{S}}(X \xrightarrow{f} Y).$$

Thus $[X \times F^n \xrightarrow{pr_1} X] \in \mathbb{M}^{\mathcal{C}}_{\mathcal{S}}(X \xrightarrow{\mathrm{id}_X} X)$ and $[X \times F^n \xrightarrow{pr_1} X] \in \mathbb{M}^{\mathcal{C}}_{\mathcal{S}}(X \xrightarrow{f} Y)$.

By the above construction $X \times F^n \xrightarrow{pr_1} X$ is a confined and specialized map. Hence, for a specialized map $f : X \to Y$,

$$f \circ pr_1 : X \times F^n \xrightarrow{pr_1} X \xrightarrow{f} Y$$

becomes a specialized map, hence we have

$$[X \times F^n \xrightarrow{pr_1} X] \in \mathbb{M}^{\mathcal{C}}_{\mathcal{S}}(X \xrightarrow{f} Y).$$

Thus $[X \times F^n \xrightarrow{pr_1} X] \in \mathbb{M}^{\mathcal{C}}_{\mathcal{S}}(X \xrightarrow{\mathrm{id}_X} X)$ and $[X \times F^n \xrightarrow{pr_1} X] \in \mathbb{M}^{\mathcal{C}}_{\mathcal{S}}(X \xrightarrow{f} Y)$. This difference can be captured as the following bivariant product:

$$[X \times F^n \xrightarrow{pr_1} X] = [X \times F^n \xrightarrow{pr_1} X] \bullet [X \xrightarrow{\mathrm{id}_X} X] \tag{1.1}$$

By the above construction $X \times F^n \xrightarrow{pr_1} X$ is a confined and specialized map. Hence, for a specialized map $f : X \to Y$,

$$f \circ pr_1 : X \times F^n \xrightarrow{pr_1} X \xrightarrow{f} Y$$

becomes a specialized map, hence we have

$$[X \times F^n \xrightarrow{pr_1} X] \in \mathbb{M}^{\mathcal{C}}_{\mathcal{S}}(X \xrightarrow{f} Y).$$

Thus $[X \times F^n \xrightarrow{pr_1} X] \in \mathbb{M}^{\mathcal{C}}_{\mathcal{S}}(X \xrightarrow{\mathrm{id}_X} X)$ and $[X \times F^n \xrightarrow{pr_1} X] \in \mathbb{M}^{\mathcal{C}}_{\mathcal{S}}(X \xrightarrow{f} Y)$. This difference can be captured as the following bivariant product:

$$[X \times F^n \xrightarrow{pr_1} X] = [X \times F^n \xrightarrow{pr_1} X] \bullet [X \xrightarrow{\mathrm{id}_X} X] \tag{1.1}$$

Here $[X \xrightarrow{\mathrm{id}_X} X] \in \mathbb{M}^{\mathcal{C}}_{\mathcal{S}}(X \xrightarrow{f} Y)$ is a canonical orientation $\theta_{\mathbb{M}^{\mathcal{C}}_{\mathcal{S}}}(f)$. So, we can express the bivariant element $[X \times F^n \xrightarrow{pr_1} X] \in \mathbb{M}^{\mathcal{C}}_{\mathcal{S}}(X \xrightarrow{f} Y)$ by

$$[X \times F^n \xrightarrow{pr_1} X] = [X \times F^n \xrightarrow{pr_1} X] \bullet \theta_{\mathbb{M}^{\mathcal{C}}_{\mathcal{S}}}(f) = (a_X)^*([F \xrightarrow{a_F} pt]^n) \bullet \theta_{\mathbb{M}^{\mathcal{C}}_{\mathcal{S}}}(f).$$

## Theorem

*Assume that we can define the integer* reldim *on a class* $\mathcal{S}$ *of specialized maps.Let* $F$ *be confined and specialized. For a specialized map* $f : X \to Y$ *we define*

## Theorem

*Assume that we can define the integer* reldim *on a class* $\mathcal{S}$ *of specialized maps.Let* $F$ *be confined and specialized. For a specialized map* $f : X \to Y$ *we define*

## Theorem

*Assume that we can define the integer* reldim *on a class* $\mathcal{S}$ *of specialized maps.Let* $F$ *be confined and specialized. For a specialized map* $f : X \to Y$ *we define*

$$\theta^F_{\mathbb{M}^{\mathcal{C}}_{\mathcal{S}}}(f) := (a_X)^*([F \xrightarrow{a_F} pt]^{\mathrm{reldim}(f)}) \bullet \theta_{\mathbb{M}^{\mathcal{C}}_{\mathcal{S}}}(f) \in \mathbb{M}^{\mathcal{C}}_{\mathcal{S}}(X \xrightarrow{f} Y).$$

# Theorem

*Assume that we can define the integer* reldim *on a class $\mathcal{S}$ of specialized maps.Let $F$ be confined and specialized. For a specialized map $f : X \to Y$ we define*

$$\theta^F_{\mathbb{M}^{\mathcal{C}}_{\mathcal{S}}}(f) := (a_X)^*([F \xrightarrow{a_F} pt]^{\mathrm{reldim}(f)}) \bullet \theta_{\mathbb{M}^{\mathcal{C}}_{\mathcal{S}}}(f) \in \mathbb{M}^{\mathcal{C}}_{\mathcal{S}}(X \xrightarrow{f} Y).$$

$\theta^F_{\mathbb{M}^{\mathcal{C}}_{\mathcal{S}}}$ *is a nice canonical orientation,*

## Theorem

*Assume that we can define the integer* reldim *on a class $\mathcal{S}$ of specialized maps.Let $F$ be confined and specialized. For a specialized map $f : X \to Y$ we define*

$$\theta^F_{\mathbb{M}^{\mathcal{C}}_{\mathcal{S}}}(f) := (a_X)^*([F \xrightarrow{a_F} pt]^{\mathrm{reldim}(f)}) \bullet \theta_{\mathbb{M}^{\mathcal{C}}_{\mathcal{S}}}(f) \in \mathbb{M}^{\mathcal{C}}_{\mathcal{S}}(X \xrightarrow{f} Y).$$

$\theta^F_{\mathbb{M}^{\mathcal{C}}_{\mathcal{S}}}$ *is a nice canonical orientation, i.e., it satisfies the following:*

1. $\theta^F_{\mathbb{M}^{\mathcal{C}}_{\mathcal{S}}}(g \circ f) = \theta^F_{\mathbb{M}^{\mathcal{C}}_{\mathcal{S}}}(f) \bullet \theta^F_{\mathbb{M}^{\mathcal{C}}_{\mathcal{S}}}(g)$ *for $f : X \to Y, g : Y \to Z \in \mathcal{S}$.*

## Theorem

*Assume that we can define the integer* reldim *on a class* $\mathcal{S}$ *of specialized maps.Let* $F$ *be confined and specialized. For a specialized map* $f : X \to Y$ *we define*

$$\theta^F_{\mathbb{M}^{\mathcal{C}}_{\mathcal{S}}}(f) := (a_X)^*([F \xrightarrow{a_F} pt]^{\mathrm{reldim}(f)}) \bullet \theta_{\mathbb{M}^{\mathcal{C}}_{\mathcal{S}}}(f) \in \mathbb{M}^{\mathcal{C}}_{\mathcal{S}}(X \xrightarrow{f} Y).$$

$\theta^F_{\mathbb{M}^{\mathcal{C}}_{\mathcal{S}}}$ *is a nice canonical orientation, i.e., it satisfies the following:*

1. $\theta^F_{\mathbb{M}^{\mathcal{C}}_{\mathcal{S}}}(g \circ f) = \theta^F_{\mathbb{M}^{\mathcal{C}}_{\mathcal{S}}}(f) \bullet \theta^F_{\mathbb{M}^{\mathcal{C}}_{\mathcal{S}}}(g)$ *for* $f : X \to Y, g : Y \to Z \in \mathcal{S}$.
2. $\theta^F_{\mathbb{M}^{\mathcal{C}}_{\mathcal{S}}}(\mathrm{id}_X) = 1_X$.

## Theorem

*Assume that we can define the integer* reldim *on a class $\mathcal{S}$ of specialized maps.Let $F$ be confined and specialized. For a specialized map $f : X \to Y$ we define*

$$\theta^F_{\mathbb{M}^{\mathcal{C}}_{\mathcal{S}}}(f) := (a_X)^*([F \xrightarrow{a_F} pt]^{\mathrm{reldim}(f)}) \bullet \theta_{\mathbb{M}^{\mathcal{C}}_{\mathcal{S}}}(f) \in \mathbb{M}^{\mathcal{C}}_{\mathcal{S}}(X \xrightarrow{f} Y).$$

$\theta^F_{\mathbb{M}^{\mathcal{C}}_{\mathcal{S}}}$ *is a nice canonical orientation, i.e., it satisfies the following:*

1. $\theta^F_{\mathbb{M}^{\mathcal{C}}_{\mathcal{S}}}(g \circ f) = \theta^F_{\mathbb{M}^{\mathcal{C}}_{\mathcal{S}}}(f) \bullet \theta^F_{\mathbb{M}^{\mathcal{C}}_{\mathcal{S}}}(g)$ *for $f : X \to Y, g : Y \to Z \in \mathcal{S}$.*
2. $\theta^F_{\mathbb{M}^{\mathcal{C}}_{\mathcal{S}}}(\mathrm{id}_X) = 1_X$.
3. *for an independent square where $f, f' \in \mathcal{S}$,*
$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow[g]{} & Y, \end{array}$$
$$g^*(\theta^F_{\mathbb{M}^{\mathcal{C}}_{\mathcal{S}}}(f)) = \theta^F_{\mathbb{M}^{\mathcal{C}}_{\mathcal{S}}}(f').$$

## Theorem

*Assume that we can define the integer* reldim *on a class $\mathcal{S}$ of specialized maps.Let $F$ be confined and specialized. For a specialized map $f : X \to Y$ we define*

$$\theta^F_{\mathbb{M}^{\mathcal{C}}_{\mathcal{S}}}(f) := (a_X)^*([F \xrightarrow{a_F} pt]^{\mathrm{reldim}(f)}) \bullet \theta_{\mathbb{M}^{\mathcal{C}}_{\mathcal{S}}}(f) \in \mathbb{M}^{\mathcal{C}}_{\mathcal{S}}(X \xrightarrow{f} Y).$$

$\theta^F_{\mathbb{M}^{\mathcal{C}}_{\mathcal{S}}}$ *is a nice canonical orientation, i.e., it satisfies the following:*

1. $\theta^F_{\mathbb{M}^{\mathcal{C}}_{\mathcal{S}}}(g \circ f) = \theta^F_{\mathbb{M}^{\mathcal{C}}_{\mathcal{S}}}(f) \bullet \theta^F_{\mathbb{M}^{\mathcal{C}}_{\mathcal{S}}}(g)$ *for* $f : X \to Y, g : Y \to Z \in \mathcal{S}$.
2. $\theta^F_{\mathbb{M}^{\mathcal{C}}_{\mathcal{S}}}(\mathrm{id}_X) = 1_X$.
3. *for an independent square where* $f, f' \in \mathcal{S}$, $\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow[g]{} & Y, \end{array}$

$$g^*(\theta^F_{\mathbb{M}^{\mathcal{C}}_{\mathcal{S}}}(f)) = \theta^F_{\mathbb{M}^{\mathcal{C}}_{\mathcal{S}}}(f').$$

## Theorem

*Assume that we can define the integer* reldim *on a class* $\mathcal{S}$ *of specialized maps.Let F be confined and specialized. For a specialized map* $f : X \to Y$ *we define*

$$\theta^F_{\mathbb{M}^{\mathcal{C}}_{\mathcal{S}}}(f) := (a_X)^*([F \xrightarrow{a_F} pt]^{\mathrm{reldim}(f)}) \bullet \theta_{\mathbb{M}^{\mathcal{C}}_{\mathcal{S}}}(f) \in \mathbb{M}^{\mathcal{C}}_{\mathcal{S}}(X \xrightarrow{f} Y).$$

$\theta^F_{\mathbb{M}^{\mathcal{C}}_{\mathcal{S}}}$ *is a nice canonical orientation, i.e., it satisfies the following:*

1. $\theta^F_{\mathbb{M}^{\mathcal{C}}_{\mathcal{S}}}(g \circ f) = \theta^F_{\mathbb{M}^{\mathcal{C}}_{\mathcal{S}}}(f) \bullet \theta^F_{\mathbb{M}^{\mathcal{C}}_{\mathcal{S}}}(g)$ *for* $f : X \to Y, g : Y \to Z \in \mathcal{S}$.
2. $\theta^F_{\mathbb{M}^{\mathcal{C}}_{\mathcal{S}}}(\mathrm{id}_X) = 1_X$.
3. *for an independent square where* $f, f' \in \mathcal{S}$,

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow[g]{} & Y, \end{array}$$

$$g^*(\theta^F_{\mathbb{M}^{\mathcal{C}}_{\mathcal{S}}}(f)) = \theta^F_{\mathbb{M}^{\mathcal{C}}_{\mathcal{S}}}(f').$$

*In particular, when* $F = pt$, *we have that* $\theta^{pt}_{\mathbb{M}^{\mathcal{C}}_{\mathcal{S}}}(f) = \theta_{\mathbb{M}^{\mathcal{C}}_{\mathcal{S}}}(f)$.

In fact, the above bivariant element $[F \to pt] \in \mathbb{M}^{\mathcal{C}}_{\mathcal{S}}(pt \to pt)$ **can be replaced by any element**

$$\Delta \in \mathbb{M}^{\mathcal{C}}_{\mathcal{S}}(pt \to pt).$$

In fact, the above bivariant element $[F \to pt] \in \mathbb{M}^{\mathcal{C}}_{\mathcal{S}}(pt \to pt)$ **can be replaced by any element**

$$\Delta \in \mathbb{M}^{\mathcal{C}}_{\mathcal{S}}(pt \to pt).$$

Hence we can show the following formula.

In fact, the above bivariant element $[F \to pt] \in \mathbb{M}^{\mathcal{C}}_{\mathcal{S}}(pt \to pt)$ **can be replaced by any element**

$$\Delta \in \mathbb{M}^{\mathcal{C}}_{\mathcal{S}}(pt \to pt).$$

Hence we can show the following formula.
**COROLLARY**: *Let* $\Delta \in \mathbb{M}^{\mathcal{C}}_{\mathcal{S}}(pt \to pt)$*.*

1. *("upstairs" Riemann–Roch "self" formula)*

$$\theta^{\Delta}_{\mathbb{M}^{\mathcal{C}}_{\mathcal{S}}}(f) = (a_X)^*(\Delta^{\mathrm{reldim}(f)}) \bullet \theta_{\mathbb{M}^{\mathcal{C}}_{\mathcal{S}}}(f)$$

*is a nice canonical orientation, where* $(a_X)^*(\Delta^{\mathrm{reldim}(f)}) \in \mathbb{M}^{\mathcal{C}}_{\mathcal{S}}(X \xrightarrow{\mathrm{id}_X} X)$*.*

In fact, the above bivariant element $[F \to pt] \in \mathbb{M}^{\mathcal{C}}_{\mathcal{S}}(pt \to pt)$ **can be replaced by any element**

$$\Delta \in \mathbb{M}^{\mathcal{C}}_{\mathcal{S}}(pt \to pt).$$

Hence we can show the following formula.
**COROLLARY**: *Let* $\Delta \in \mathbb{M}^{\mathcal{C}}_{\mathcal{S}}(pt \to pt)$.

1. *("upstairs" Riemann–Roch "self" formula)*

$$\theta^{\Delta}_{\mathbb{M}^{\mathcal{C}}_{\mathcal{S}}}(f) = (a_X)^*(\Delta^{\mathrm{reldim}(f)}) \bullet \theta_{\mathbb{M}^{\mathcal{C}}_{\mathcal{S}}}(f)$$

*is a nice canonical orientation, where* $(a_X)^*(\Delta^{\mathrm{reldim}(f)}) \in \mathbb{M}^{\mathcal{C}}_{\mathcal{S}}(X \xrightarrow{\mathrm{id}_X} X)$.

2. *("downstairs" Riemann–Roch "self" formula)*

$${}_{\Delta}\theta_{\mathbb{M}^{\mathcal{C}}_{\mathcal{S}}}(f) = \theta_{\mathbb{M}^{\mathcal{C}}_{\mathcal{S}}}(f) \bullet (a_Y)^*(\Delta^{\mathrm{reldim}(f)})$$

*is a nice canonical orientation, where* $(a_Y)^*(\Delta^{\mathrm{reldim}(f)}) \in \mathbb{M}^{\mathcal{C}}_{\mathcal{S}}(Y \xrightarrow{\mathrm{id}_Y} Y)$.

In fact, the above bivariant element $[F \to pt] \in \mathbb{M}^{\mathcal{C}}_{\mathcal{S}}(pt \to pt)$ **can be replaced by any element**

$$\Delta \in \mathbb{M}^{\mathcal{C}}_{\mathcal{S}}(pt \to pt).$$

Hence we can show the following formula.
**COROLLARY**: *Let* $\Delta \in \mathbb{M}^{\mathcal{C}}_{\mathcal{S}}(pt \to pt)$.

1. *("upstairs" Riemann–Roch "self" formula)*

$$\theta^{\Delta}_{\mathbb{M}^{\mathcal{C}}_{\mathcal{S}}}(f) = (a_X)^*(\Delta^{\mathrm{reldim}(f)}) \bullet \theta_{\mathbb{M}^{\mathcal{C}}_{\mathcal{S}}}(f)$$

*is a nice canonical orientation, where* $(a_X)^*(\Delta^{\mathrm{reldim}(f)}) \in \mathbb{M}^{\mathcal{C}}_{\mathcal{S}}(X \xrightarrow{\mathrm{id}_X} X)$.

2. *("downstairs" Riemann–Roch "self" formula)*

$$_{\Delta}\theta_{\mathbb{M}^{\mathcal{C}}_{\mathcal{S}}}(f) = \theta_{\mathbb{M}^{\mathcal{C}}_{\mathcal{S}}}(f) \bullet (a_Y)^*(\Delta^{\mathrm{reldim}(f)})$$

*is a nice canonical orientation, where* $(a_Y)^*(\Delta^{\mathrm{reldim}(f)}) \in \mathbb{M}^{\mathcal{C}}_{\mathcal{S}}(Y \xrightarrow{\mathrm{id}_Y} Y)$.

In fact, the above bivariant element $[F \to pt] \in \mathbb{M}^{\mathcal{C}}_{\mathcal{S}}(pt \to pt)$ **can be replaced by any element**

$$\Delta \in \mathbb{M}^{\mathcal{C}}_{\mathcal{S}}(pt \to pt).$$

Hence we can show the following formula.

**COROLLARY**: *Let* $\Delta \in \mathbb{M}^{\mathcal{C}}_{\mathcal{S}}(pt \to pt)$*.*

1. *("upstairs" Riemann–Roch "self" formula)*

$$\theta^{\Delta}_{\mathbb{M}^{\mathcal{C}}_{\mathcal{S}}}(f) = (a_X)^*(\Delta^{\mathrm{reldim(f)}}) \bullet \theta_{\mathbb{M}^{\mathcal{C}}_{\mathcal{S}}}(f)$$

   *is a nice canonical orientation, where* $(a_X)^*(\Delta^{\mathrm{reldim(f)}}) \in \mathbb{M}^{\mathcal{C}}_{\mathcal{S}}(X \xrightarrow{\mathrm{id}_X} X)$*.*

2. *("downstairs" Riemann–Roch "self" formula)*

$${}_{\Delta}\theta_{\mathbb{M}^{\mathcal{C}}_{\mathcal{S}}}(f) = \theta_{\mathbb{M}^{\mathcal{C}}_{\mathcal{S}}}(f) \bullet (a_Y)^*(\Delta^{\mathrm{reldim(f)}})$$

   *is a nice canonical orientation, where* $(a_Y)^*(\Delta^{\mathrm{reldim(f)}}) \in \mathbb{M}^{\mathcal{C}}_{\mathcal{S}}(Y \xrightarrow{\mathrm{id}_Y} Y)$*.*

It turns out that in this case

$$(a_X)^*(\Delta^{\mathrm{reldim(f)}}) \bullet \theta_{\mathbb{M}^{\mathcal{C}}_{\mathcal{S}}}(f) = \theta_{\mathbb{M}^{\mathcal{C}}_{\mathcal{S}}}(f) \bullet (a_Y)^*(\Delta^{\mathrm{reldim(f)}}),$$

which follows from Commutativity!

In fact, the above bivariant element $[F \to pt] \in \mathbb{M}^{\mathcal{C}}_{\mathcal{S}}(pt \to pt)$ **can be replaced by any element**

$$\Delta \in \mathbb{M}^{\mathcal{C}}_{\mathcal{S}}(pt \to pt).$$

Hence we can show the following formula.
**COROLLARY**: *Let* $\Delta \in \mathbb{M}^{\mathcal{C}}_{\mathcal{S}}(pt \to pt)$*.*

1. *("upstairs" Riemann–Roch "self" formula)*

$$\theta^{\Delta}_{\mathbb{M}^{\mathcal{C}}_{\mathcal{S}}}(f) = (a_X)^*(\Delta^{\mathrm{reldim(f)}}) \bullet \theta_{\mathbb{M}^{\mathcal{C}}_{\mathcal{S}}}(f)$$

   *is a nice canonical orientation, where* $(a_X)^*(\Delta^{\mathrm{reldim(f)}}) \in \mathbb{M}^{\mathcal{C}}_{\mathcal{S}}(X \xrightarrow{\mathrm{id}_X} X)$*.*

2. *("downstairs" Riemann–Roch "self" formula)*

$${}_{\Delta}\theta_{\mathbb{M}^{\mathcal{C}}_{\mathcal{S}}}(f) = \theta_{\mathbb{M}^{\mathcal{C}}_{\mathcal{S}}}(f) \bullet (a_Y)^*(\Delta^{\mathrm{reldim(f)}})$$

   *is a nice canonical orientation, where* $(a_Y)^*(\Delta^{\mathrm{reldim(f)}}) \in \mathbb{M}^{\mathcal{C}}_{\mathcal{S}}(Y \xrightarrow{\mathrm{id}_Y} Y)$*.*

It turns out that in this case

$$(a_X)^*(\Delta^{\mathrm{reldim(f)}}) \bullet \theta_{\mathbb{M}^{\mathcal{C}}_{\mathcal{S}}}(f) = \theta_{\mathbb{M}^{\mathcal{C}}_{\mathcal{S}}}(f) \bullet (a_Y)^*(\Delta^{\mathrm{reldim(f)}}),$$

which follows from Commutativity! Thus $\theta^{\Delta}_{\mathbb{M}^{\mathcal{C}}_{\mathcal{S}}}(f) = {}_{\Delta}\theta_{\mathbb{M}^{\mathcal{C}}_{\mathcal{S}}}(f)$.

We also get the following, which follows from the universality of UBT:

## Corollary

*Let* $\Delta \in \mathbb{M}^{\mathcal{C}}_{\mathcal{S}}(pt \to pt)$*.* $\exists$ *a unique Grothendieck (auto-)transformation*

$$\gamma^{\Delta} : \mathbb{M}^{\mathcal{C}}_{\mathcal{S}} \to \mathbb{M}^{\mathcal{C}}_{\mathcal{S}}$$

*such that* $\gamma^{\Delta}(\theta_{\mathbb{M}^{\mathcal{C}}_{\mathcal{S}}}(f)) = \theta^{\Delta}_{\mathbb{M}^{\mathcal{C}}_{\mathcal{S}}}(f) = (a_X)^*(\Delta^{\mathrm{reldim}(f)}) \bullet \theta_{\mathbb{M}^{\mathcal{C}}_{\mathcal{S}}}(f)$ *for a specialized map* $f : X \to Y$*. (Because* $\theta^{\Delta}_{\mathbb{M}^{\mathcal{C}}_{\mathcal{S}}}(f)$ *is a nice canonical orientation.)*

We also get the following, which follows from the universality of UBT:

## Corollary

*Let* $\Delta \in \mathbb{M}_{\mathcal{S}}^{\mathcal{C}}(pt \to pt)$*.* $\exists$ *a unique Grothendieck (auto-)transformation*

$$\gamma^{\Delta} : \mathbb{M}_{\mathcal{S}}^{\mathcal{C}} \to \mathbb{M}_{\mathcal{S}}^{\mathcal{C}}$$

*such that* $\gamma^{\Delta}(\theta_{\mathbb{M}_{\mathcal{S}}^{\mathcal{C}}}(f)) = \theta^{\Delta}_{\mathbb{M}_{\mathcal{S}}^{\mathcal{C}}}(f) = (a_X)^*(\Delta^{\mathrm{reldim}(f)}) \bullet \theta_{\mathbb{M}_{\mathcal{S}}^{\mathcal{C}}}(f)$ *for a specialized map* $f : X \to Y$*. (Because* $\theta^{\Delta}_{\mathbb{M}_{\mathcal{S}}^{\mathcal{C}}}(f)$ *is a nice canonical orientation.)*

We also get the following, which follows from the universality of UBT:

## Corollary

*Let* $\Delta \in \mathbb{M}^{\mathcal{C}}_{\mathcal{S}}(pt \to pt)$. $\exists$ *a unique Grothendieck (auto-)transformation*

$$\gamma^{\Delta} : \mathbb{M}^{\mathcal{C}}_{\mathcal{S}} \to \mathbb{M}^{\mathcal{C}}_{\mathcal{S}}$$

*such that* $\gamma^{\Delta}(\theta_{\mathbb{M}^{\mathcal{C}}_{\mathcal{S}}}(f)) = \theta^{\Delta}_{\mathbb{M}^{\mathcal{C}}_{\mathcal{S}}}(f) = (a_X)^*(\Delta^{\mathrm{reldim}(f)}) \bullet \theta_{\mathbb{M}^{\mathcal{C}}_{\mathcal{S}}}(f)$ *for a specialized map* $f : X \to Y$. *(Because* $\theta^{\Delta}_{\mathbb{M}^{\mathcal{C}}_{\mathcal{S}}}(f)$ *is a nice canonical orientation.)*
*In particular, we have "SGA6", "BFM–RR" and "Verdier–RR": (Note:* $\mathbb{M}^{\mathcal{C}}_{\mathcal{S}}{}^*(Z) := \mathbb{M}^{\mathcal{C}}_{\mathcal{S}}(Z \xrightarrow{\mathrm{id}_Z} Z)$ *and* $\mathbb{M}^{\mathcal{C}}_{\mathcal{S}}{}_*(Z) := \mathbb{M}^{\mathcal{C}}_{\mathcal{S}}(Z \to pt)$

We also get the following, which follows from the universality of UBT:

## Corollary

*Let* $\Delta \in \mathbb{M}^{\mathcal{C}}_{\mathcal{S}}(pt \to pt)$. $\exists$ *a unique Grothendieck (auto-)transformation*

$$\gamma^{\Delta} : \mathbb{M}^{\mathcal{C}}_{\mathcal{S}} \to \mathbb{M}^{\mathcal{C}}_{\mathcal{S}}$$

*such that* $\gamma^{\Delta}(\theta_{\mathbb{M}^{\mathcal{C}}_{\mathcal{S}}}(f)) = \theta^{\Delta}_{\mathbb{M}^{\mathcal{C}}_{\mathcal{S}}}(f) = (a_X)^*(\Delta^{\mathrm{reldim}(f)}) \bullet \theta_{\mathbb{M}^{\mathcal{C}}_{\mathcal{S}}}(f)$ *for a specialized map* $f : X \to Y$. *(Because* $\theta^{\Delta}_{\mathbb{M}^{\mathcal{C}}_{\mathcal{S}}}(f)$ *is a nice canonical orientation.)*
*In particular, we have "SGA6", "BFM–RR" and "Verdier–RR": (Note:* $\mathbb{M}^{\mathcal{C}}_{\mathcal{S}}{}^*(Z) := \mathbb{M}^{\mathcal{C}}_{\mathcal{S}}(Z \xrightarrow{\mathrm{id}_Z} Z)$ *and* $\mathbb{M}^{\mathcal{C}}_{\mathcal{S}\,*}(Z) := \mathbb{M}^{\mathcal{C}}_{\mathcal{S}}(Z \to pt)$)

1. *"SGA6":For a confined and specialized map* $f : X \to Y$ *we have the following commutative diagram:*

$$\begin{array}{ccc} \mathbb{M}^{\mathcal{C}}_{\mathcal{S}}{}^*(X) & \xrightarrow{\gamma^{\Delta}} & \mathbb{M}^{\mathcal{C}}_{\mathcal{S}}{}^*(X) \\ {\scriptstyle f_!}\downarrow & & \downarrow {\scriptstyle f_!\left((a_X)^*(\Delta^{\mathrm{reldim}(f)})\bullet -\right) = f_!\left(-\right)\bullet(a_Y)^*(\Delta^{\mathrm{reldim}(f)})} \\ \mathbb{M}^{\mathcal{C}}_{\mathcal{S}}{}^*(Y) & \xrightarrow[\gamma^{\Delta}]{} & \mathbb{M}^{\mathcal{C}}_{\mathcal{S}}{}^*(Y), \end{array}$$

1. *“BFM–RR”: For a confined map $f : X \to Y$ we have the following commutative diagram:*

$$
\begin{array}{ccc}
\mathbb{M}^{\mathcal{C}}_{\mathcal{S}\,*}(X) & \xrightarrow{\gamma^{\Delta}} & \mathbb{M}^{\mathcal{C}}_{\mathcal{S}\,*}(X) \\
{\scriptstyle f_*}\big\downarrow & & \big\downarrow{\scriptstyle f_*} \\
\mathbb{M}^{\mathcal{C}}_{\mathcal{S}\,*}(Y) & \xrightarrow[\gamma^{\Delta}]{} & \mathbb{M}^{\mathcal{C}}_{\mathcal{S}\,*}(Y)
\end{array}
$$

1. *"BFM–RR": For a confined map $f : X \to Y$ we have the following commutative diagram:*

$$\begin{array}{ccc} \mathbb{M}^{\mathcal{C}}_{\mathcal{S}*}(X) & \xrightarrow{\gamma^{\Delta}} & \mathbb{M}^{\mathcal{C}}_{\mathcal{S}*}(X) \\ {\scriptstyle f_*}\downarrow & & \downarrow{\scriptstyle f_*} \\ \mathbb{M}^{\mathcal{C}}_{\mathcal{S}*}(Y) & \xrightarrow[\gamma^{\Delta}]{} & \mathbb{M}^{\mathcal{C}}_{\mathcal{S}*}(Y) \end{array}$$

2. *"Verdier–RR": For a specialized map $f : X \to Y$ the following diagram commute:*

$$\begin{array}{ccc} \mathbb{M}^{\mathcal{C}}_{\mathcal{S}*}(Y) & \xrightarrow{\gamma^{\Delta}} & \mathbb{M}^{\mathcal{C}}_{\mathcal{S}*}(Y) \\ {\scriptstyle f^!}\downarrow & & \downarrow{\scriptstyle (a_X)^*(\Delta^{\mathrm{reldim(f)}})\bullet f^! = f^!\left((a_Y)^*(\Delta^{\mathrm{reldim(f)}})\bullet -\right)} \\ \mathbb{M}^{\mathcal{C}}_{\mathcal{S}*}(X) & \xrightarrow[\gamma^{\Delta}]{} & \mathbb{M}^{\mathcal{C}}_{\mathcal{S}*}(X), \end{array}$$

# §1.3. A very naive universal bivariant theory

We give another **very naive and simple** universal bivariant theory $\mathbb{M}^{\mathcal{C}}$ **without using the class $\mathcal{S}$ of specialized maps**.

**THEOREM** (A very naive universal bivariant theory) *Let $\mathcal{V}$ be a category with a class $\mathcal{C}$ of confined maps and a class of independent squares. Define*

$$\mathbb{M}^{\mathcal{C}}(X \xrightarrow{f} Y)$$

*to be the free abelian group generated by the set of isomorphism classes of confined morphisms $h : W \to X$.*

*(1) The association $\mathbb{M}^{\mathcal{C}}$ is a bivariant theory if the three bivariant operations are defined exactly in the same way as in UBT above.*

*(i).* **Product***: For morphisms $f : X \to Y$ and $g : Y \to Z$, the product operation*

$$\bullet : \mathbb{M}^{\mathcal{C}}(X \xrightarrow{f} Y) \otimes \mathbb{M}^{\mathcal{C}}(Y \xrightarrow{g} Z) \to \mathbb{M}^{\mathcal{C}}(X \xrightarrow{gf} Z).$$

*(ii).* **Pushforward***: For morphisms $f : X \to Y$ and $g : Y \to Z$ with $f$ confined, the pushforward operation*

$$f_* : \mathbb{M}^{\mathcal{C}}(X \xrightarrow{gf} Z) \to \mathbb{M}^{\mathcal{C}}(Y \xrightarrow{g} Z).$$

*(iii).* **Pullback***: For an independent square*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ Y' & \xrightarrow[g]{} & Y, \end{array}$$

*the pullback operation*

$$g^* : \mathbb{M}^{\mathcal{C}}(X \xrightarrow{f} Y) \to \mathbb{M}^{\mathcal{C}}(X' \xrightarrow{f'} Y').$$

*(2). (A very naive universality of $\mathbb{M}^{\mathcal{C}}$) Let $\mathbb{B}$ be a bivariant theory on the same category $\mathcal{V}$ with the same class $\mathcal{C}$ of confined morphisms and the same lass of independent squares. Let $\theta_{\mathbb{B}}$ be a canonical orientation for all maps in $\mathcal{V}$. Then there exists a unique Grothendieck transformation*

$$\gamma_{\mathbb{B}} : \mathbb{M}^{\mathcal{C}} \to \mathbb{B}$$

*such that $\gamma_{\mathbb{B}} : \mathbb{M}^{\mathcal{C}}(X \xrightarrow{f} Y) \to \mathbb{B}(X \xrightarrow{f} Y)$ satisfies the normalization condition that for any map $f : X \to Y$ in $\mathcal{C}$*

$$\gamma_{\mathbb{B}}([X \xrightarrow{\mathrm{id}_X} X]) = \theta_{\mathbb{B}}(f).$$

# §2 Simple examples of bivariant theories and Riemann–Roch formulas

# §2 Simple examples of bivariant theories and Riemann–Roch formulas

Let's consider the category $\mathcal{F}$ of finite sets as an easy example.

# §2 Simple examples of bivariant theories and Riemann–Roch formulas

Let's consider the category $\mathcal{F}$ of finite sets as an easy example.

**DEFINITION** (cf.[§6.1 The bivariant theory $\mathbb{F}$ and §10.1.2 The Frobenius] of FM.) For $\mathcal{F}$, let **any map be confined** and **any fiber square be independent**.

# §2 Simple examples of bivariant theories and Riemann–Roch formulas

Let's consider the category $\mathcal{F}$ of finite sets as an easy example.

**DEFINITION** (cf.[§6.1 The bivariant theory $\mathbb{F}$ and §10.1.2 The Frobenius] of FM.) For $\mathcal{F}$, let **any map be confined** and **any fiber square be independent**. For a map $f : X \to Y$ we define

$$\mathbb{F}(X \xrightarrow{f} Y) := \mathbb{F}^0(X \xrightarrow{f} Y)$$

to be the abelian group of $\mathbb{R}$-valued functions on $X$.

# §2 Simple examples of bivariant theories and Riemann–Roch formulas

Let's consider the category $\mathcal{F}$ of finite sets as an easy example.

**DEFINITION** (cf.[§6.1 The bivariant theory $\mathbb{F}$ and §10.1.2 The Frobenius] of FM.) For $\mathcal{F}$, let **any map be confined** and **any fiber square be independent**. For a map $f : X \to Y$ we define

$$\mathbb{F}(X \xrightarrow{f} Y) := \mathbb{F}^0(X \xrightarrow{f} Y)$$

to be the abelian group of $\mathbb{R}$-valued functions on $X$.

1. (**product**)

$$\bullet : \mathbb{F}(X \xrightarrow{f} Y) \otimes \mathbb{F}(Y \xrightarrow{g} Z) \to \mathbb{F}(X \xrightarrow{g \circ f} Z)$$

for $\alpha \in \mathbb{F}(X \xrightarrow{f} Y), \beta \in \mathbb{F}(Y \xrightarrow{g} Z)$ and for $x \in X$

$$(\alpha \bullet \beta)(x) := \alpha(x) \cdot \beta(f(x)).$$

## §2 Simple examples of bivariant theories and Riemann–Roch formulas

Let's consider the category $\mathcal{F}$ of finite sets as an easy example.
**DEFINITION** (cf.[§6.1 The bivariant theory $\mathbb{F}$ and §10.1.2 The Frobenius] of FM.) For $\mathcal{F}$, let **any map be confined** and **any fiber square be independent**. For a map $f : X \to Y$ we define

$$\mathbb{F}(X \xrightarrow{f} Y) := \mathbb{F}^0(X \xrightarrow{f} Y)$$

to be the abelian group of $\mathbb{R}$-valued functions on $X$.

1. (**product**)
$$\bullet : \mathbb{F}(X \xrightarrow{f} Y) \otimes \mathbb{F}(Y \xrightarrow{g} Z) \to \mathbb{F}(X \xrightarrow{g\circ f} Z)$$
for $\alpha \in \mathbb{F}(X \xrightarrow{f} Y), \beta \in \mathbb{F}(Y \xrightarrow{g} Z)$ and for $x \in X$
$$(\alpha \bullet \beta)(x) := \alpha(x) \cdot \beta(f(x)).$$
2. (**pushforward**) for any map $f : X \to Y$ (note that any map is confined)
$$f_* : \mathbb{F}(X \xrightarrow{g\circ f} Z) \to \mathbb{F}(Y \xrightarrow{g} Z)$$
for $y \in Y$
$$f_*(\alpha)(y) := \sum_{x\in f^{-1}(y)} \alpha(x).$$

1. (**pullback**) For a fiber square

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow[g]{} & Y, \end{array}$$

$$g^* : \mathbb{F}(X \xrightarrow{f} Y) \to \mathbb{F}(X' \xrightarrow{f'} Y')$$

is defined by (the usual functional pullback)

$$(g^*\alpha)(x') := \alpha(g'(x')).$$

The following is obvious:

The following is obvious:
**Lemma** *For any map $f : X \to Y$ we define*

$$\theta_1(f) := 1_X \in \mathbb{F}(X \xrightarrow{f} Y)$$

*the characteristic function, i.e.,* $1_X(x) = 1$*.*

The following is obvious:

**Lemma** *For any map* $f : X \to Y$ *we define*

$$\theta_1(f) := 1_X \in \mathbb{F}(X \xrightarrow{f} Y)$$

*the characteristic function, i.e.,* $1_X(x) = 1$*. Then* $\theta_1$ *is a canonical orientation for all maps and it is a nice canonical orientation,*

The following is obvious:

**Lemma** *For any map $f : X \to Y$ we define*

$$\theta_1(f) := 1_X \in \mathbb{F}(X \xrightarrow{f} Y)$$

*the characteristic function, i.e., $1_X(x) = 1$. Then $\theta_1$ is a canonical orientation for all maps and it is a nice canonical orientation, i.e., we have*

1. $\theta_1(g \circ f) = \theta_1(f) \bullet \theta_1(g)$ *for ay maps $f : X \to Y$ and $g : Y \to Z$,*

The following is obvious:

**Lemma** *For any map $f : X \to Y$ we define*

$$\theta_1(f) := 1_X \in \mathbb{F}(X \xrightarrow{f} Y)$$

*the characteristic function, i.e., $1_X(x) = 1$. Then $\theta_1$ is a canonical orientation for all maps and it is a nice canonical orientation, i.e., we have*

1. $\theta_1(g \circ f) = \theta_1(f) \bullet \theta_1(g)$ *for ay maps $f : X \to Y$ and $g : Y \to Z$,*
2. $\theta_1(\mathrm{id}_X) = 1_X$.

The following is obvious:
**Lemma** *For any map $f : X \to Y$ we define*

$$\theta_1(f) := 1_X \in \mathbb{F}(X \xrightarrow{f} Y)$$

*the characteristic function, i.e., $1_X(x) = 1$. Then $\theta_1$ is a canonical orientation for all maps and it is a nice canonical orientation, i.e., we have*

1. $\theta_1(g \circ f) = \theta_1(f) \bullet \theta_1(g)$ *for ay maps $f : X \to Y$ and $g : Y \to Z$,*
2. $\theta_1(\mathrm{id}_X) = 1_X$.
3. $\theta_1$ *is a nice canonical orientation.*

The following is obvious:

**Lemma** *For any map* $f : X \to Y$ *we define*

$$\theta_1(f) := 1_X \in \mathbb{F}(X \xrightarrow{f} Y)$$

*the characteristic function, i.e.,* $1_X(x) = 1$*. Then* $\theta_1$ *is a canonical orientation for all maps and it is a nice canonical orientation, i.e., we have*

1. $\theta_1(g \circ f) = \theta_1(f) \bullet \theta_1(g)$ *for ay maps* $f : X \to Y$ *and* $g : Y \to Z$*,*
2. $\theta_1(\mathrm{id}_X) = 1_X$.
3. $\theta_1$ *is a nice canonical orientation.*

The following is obvious:

**Lemma** *For any map $f : X \to Y$ we define*

$$\theta_1(f) := 1_X \in \mathbb{F}(X \xrightarrow{f} Y)$$

*the characteristic function, i.e., $1_X(x) = 1$. Then $\theta_1$ is a canonical orientation for all maps and it is a nice canonical orientation, i.e., we have*

1. $\theta_1(g \circ f) = \theta_1(f) \bullet \theta_1(g)$ *for ay maps $f : X \to Y$ and $g : Y \to Z$,*
2. $\theta_1(\mathrm{id}_X) = 1_X$.
3. $\theta_1$ *is a nice canonical orientation.*

**REMARK**: *Let $r \in \mathbb{R} \setminus \{0\}$. For any map $f : X \to Y$ we can define*

$$\theta_r(f) := r^{|X|-|Y|}\theta_1(f) = r^{|X|-|Y|}1_X.$$

*Here $|Z|$ denotes the number of element of a finite set.*

The following is obvious:

**Lemma** *For any map $f : X \to Y$ we define*

$$\theta_1(f) := 1_X \in \mathbb{F}(X \xrightarrow{f} Y)$$

*the characteristic function, i.e., $1_X(x) = 1$. Then $\theta_1$ is a canonical orientation for all maps and it is a nice canonical orientation, i.e., we have*

1. $\theta_1(g \circ f) = \theta_1(f) \bullet \theta_1(g)$ *for ay maps $f : X \to Y$ and $g : Y \to Z$,*
2. $\theta_1(\mathrm{id}_X) = 1_X$.
3. $\theta_1$ *is a nice canonical orientation.*

**REMARK**: *Let $r \in \mathbb{R} \setminus \{0\}$. For any map $f : X \to Y$ we can define*

$$\theta_r(f) := r^{|X|-|Y|}\theta_1(f) = r^{|X|-|Y|}1_X.$$

*Here $|Z|$ denotes the number of element of a finite set. Then we have*

1. $\theta_r(g \circ f) = \theta_r(f) \bullet \theta_r(g)$ *for $f : X \to Y$ and $g : Y \to Z$,*

The following is obvious:
**Lemma** *For any map $f : X \to Y$ we define*

$$\theta_1(f) := 1_X \in \mathbb{F}(X \xrightarrow{f} Y)$$

*the characteristic function, i.e., $1_X(x) = 1$. Then $\theta_1$ is a canonical orientation for all maps and it is a nice canonical orientation, i.e., we have*

1. $\theta_1(g \circ f) = \theta_1(f) \bullet \theta_1(g)$ *for ay maps $f : X \to Y$ and $g : Y \to Z$,*
2. $\theta_1(\mathrm{id}_X) = 1_X$.
3. $\theta_1$ *is a nice canonical orientation.*

**REMARK**: *Let $r \in \mathbb{R} \setminus \{0\}$. For any map $f : X \to Y$ we can define*

$$\theta_r(f) := r^{|X|-|Y|}\theta_1(f) = r^{|X|-|Y|}1_X.$$

*Here $|Z|$ denotes the number of element of a finite set. Then we have*

1. $\theta_r(g \circ f) = \theta_r(f) \bullet \theta_r(g)$ *for $f : X \to Y$ and $g : Y \to Z$,*
2. $\theta_r(\mathrm{id}_X) = 1_X$.

The following is obvious:

**Lemma** *For any map $f : X \to Y$ we define*

$$\theta_1(f) := 1_X \in \mathbb{F}(X \xrightarrow{f} Y)$$

*the characteristic function, i.e., $1_X(x) = 1$. Then $\theta_1$ is a canonical orientation for all maps and it is a nice canonical orientation, i.e., we have*

1. $\theta_1(g \circ f) = \theta_1(f) \bullet \theta_1(g)$ *for ay maps $f : X \to Y$ and $g : Y \to Z$,*
2. $\theta_1(\mathrm{id}_X) = 1_X$.
3. $\theta_1$ *is a nice canonical orientation.*

**REMARK**: *Let $r \in \mathbb{R} \setminus \{0\}$. For any map $f : X \to Y$ we can define*

$$\theta_r(f) := r^{|X|-|Y|}\theta_1(f) = r^{|X|-|Y|}1_X.$$

*Here $|Z|$ denotes the number of element of a finite set. Then we have*

1. $\theta_r(g \circ f) = \theta_r(f) \bullet \theta_r(g)$ *for $f : X \to Y$ and $g : Y \to Z$,*
2. $\theta_r(\mathrm{id}_X) = 1_X$.

The following is obvious:

**Lemma** *For any map $f : X \to Y$ we define*

$$\theta_1(f) := 1_X \in \mathbb{F}(X \xrightarrow{f} Y)$$

*the characteristic function, i.e., $1_X(x) = 1$. Then $\theta_1$ is a canonical orientation for all maps and it is a nice canonical orientation, i.e., we have*

1. $\theta_1(g \circ f) = \theta_1(f) \bullet \theta_1(g)$ *for ay maps $f : X \to Y$ and $g : Y \to Z$,*
2. $\theta_1(\mathrm{id}_X) = 1_X$.
3. $\theta_1$ *is a nice canonical orientation.*

**REMARK**: *Let $r \in \mathbb{R} \setminus \{0\}$. For any map $f : X \to Y$ we can define*

$$\theta_r(f) := r^{|X|-|Y|}\theta_1(f) = r^{|X|-|Y|}1_X.$$

*Here $|Z|$ denotes the number of element of a finite set. Then we have*

1. $\theta_r(g \circ f) = \theta_r(f) \bullet \theta_r(g)$ *for $f : X \to Y$ and $g : Y \to Z$,*
2. $\theta_r(\mathrm{id}_X) = 1_X$.
3. *if $r \neq 1$, then $\theta_r$ is not a nice canonical orientation.*

The following theorem follows from the theorems of UBT:

The following theorem follows from the theorems of UBT:

## Theorem

*For the category $\mathcal{F}$ of finite sets there exists a unique Grothendieck transformation*

$$\gamma_{\mathbb{F}} : \mathbb{M}^{\mathcal{C}} \to \mathbb{F}$$

*such that for any map $f : X \to Y$*

$$\gamma_{\mathbb{F}}(\theta_{\mathbb{M}^{\mathcal{C}}}(f)) = \theta_1(f).$$

*Note* $\theta_{\mathbb{M}^{\mathcal{C}}}(f) = [X \xrightarrow{\mathrm{id}_X} X] \in \mathbb{M}^{\mathcal{C}}(X \xrightarrow{f} Y)$ *and* $\theta_1(f) = 1_X \in \mathbb{F}(X \xrightarrow{f} Y)$.

The following theorem follows from the theorems of UBT:

## Theorem

*For the category $\mathcal{F}$ of finite sets there exists a unique Grothendieck transformation*

$$\gamma_{\mathbb{F}} : \mathbb{M}^{\mathcal{C}} \to \mathbb{F}$$

*such that for any map $f : X \to Y$*

$$\gamma_{\mathbb{F}}(\theta_{\mathbb{M}^{\mathcal{C}}}(f)) = \theta_1(f).$$

*Note* $\theta_{\mathbb{M}^{\mathcal{C}}}(f) = [X \xrightarrow{\mathrm{id}_X} X] \in \mathbb{M}^{\mathcal{C}}(X \xrightarrow{f} Y)$ *and* $\theta_1(f) = 1_X \in \mathbb{F}(X \xrightarrow{f} Y)$.

The following theorem follows from the theorems of UBT:

## Theorem

*For the category $\mathcal{F}$ of finite sets there exists a unique Grothendieck transformation*

$$\gamma_{\mathbb{F}} : \mathbb{M}^{\mathcal{C}} \to \mathbb{F}$$

*such that for any map $f : X \to Y$*

$$\gamma_{\mathbb{F}}(\theta_{\mathbb{M}^{\mathcal{C}}}(f)) = \theta_1(f).$$

*Note $\theta_{\mathbb{M}^{\mathcal{C}}}(f) = [X \xrightarrow{\mathrm{id}_X} X] \in \mathbb{M}^{\mathcal{C}}(X \xrightarrow{f} Y)$ and $\theta_1(f) = 1_X \in \mathbb{F}(X \xrightarrow{f} Y)$.*

**REMARK**: The “SGA 6”, “BFM–RR” and “Verdier–RR” of $\gamma_F : \mathbb{M}^{\mathcal{C}} \to \mathbb{F}$ become the following respectively:
(1) “SGA 6” (for any map $f : X \to Y$):

$$\begin{array}{ccc}
\mathbb{M}^{\mathcal{C}}(X \xrightarrow{\mathrm{id}_X} X) & \xrightarrow{\gamma_{\mathbb{F}}} & \mathbb{F}(X \xrightarrow{\mathrm{id}_X} X) \\
{\scriptstyle f_*=f_!}\downarrow & & \downarrow{\scriptstyle f_!=f_*} \\
\mathbb{M}^{\mathcal{C}}(Y \xrightarrow{\mathrm{id}_Y} Y) & \xrightarrow[\gamma_{\mathbb{F}}]{} & \mathbb{F}(Y \xrightarrow{\mathrm{id}_Y} Y),
\end{array}$$

$f_!([V \xrightarrow{h} X]) = [V \xrightarrow{f \circ h} Y]$, $\gamma_{\mathbb{F}}([V \xrightarrow{h} X]) = h_* 1_V$ for $[V \xrightarrow{h} X] \in \mathbb{M}^{\mathcal{C}}(X \xrightarrow{\mathrm{id}_X} X)$.
For $\alpha \in \mathbb{F}(X \xrightarrow{\mathrm{id}_X} X)$ $f_! \alpha = f_* \alpha$.

(2) "BFM–RR" (for any map $f : X \to Y$):

$$\begin{array}{ccc} \mathbb{M}^{\mathcal{C}}(X \to pt) & \xrightarrow{\gamma_{\mathbb{F}}} & \mathbb{F}(X \to pt) \\ {\scriptstyle f_*}\downarrow & & \downarrow{\scriptstyle f_*} \\ \mathbb{M}^{\mathcal{C}}(Y \to pt) & \xrightarrow[\gamma_{\mathbb{F}}]{} & \mathbb{F}(Y \to pt), \end{array}$$

$f_*([V \xrightarrow{h} X]) = [V \xrightarrow{f \circ h} Y]$, $\gamma_{\mathbb{F}}([V \xrightarrow{h} X]) = h_* 1_V$ for $[V \xrightarrow{h} X] \in \mathbb{M}^{\mathcal{C}}(X \xrightarrow{\mathrm{id}_X} X)$.

(2) “BFM–RR” (for any map $f : X \to Y$):

$$\begin{array}{ccc} \mathbb{M}^{\mathcal{C}}(X \to pt) & \xrightarrow{\gamma_{\mathbb{F}}} & \mathbb{F}(X \to pt) \\ {\scriptstyle f_*}\downarrow & & \downarrow{\scriptstyle f_*} \\ \mathbb{M}^{\mathcal{C}}(Y \to pt) & \xrightarrow[\gamma_{\mathbb{F}}]{} & \mathbb{F}(Y \to pt), \end{array}$$

$f_*([V \xrightarrow{h} X]) = [V \xrightarrow{f \circ h} Y]$, $\gamma_{\mathbb{F}}([V \xrightarrow{h} X]) = h_* 1_V$ for $[V \xrightarrow{h} X] \in \mathbb{M}^{\mathcal{C}}(X \xrightarrow{\mathrm{id}_X} X)$.
So, “SGA6” and “BFM–RR” are the same.

(2) "BFM–RR" (for any map $f : X \to Y$):

$$\begin{array}{ccc} \mathbb{M}^{\mathcal{C}}(X \to pt) & \xrightarrow{\gamma_{\mathbb{F}}} & \mathbb{F}(X \to pt) \\ {\scriptstyle f_*}\big\downarrow & & \big\downarrow{\scriptstyle f_*} \\ \mathbb{M}^{\mathcal{C}}(Y \to pt) & \xrightarrow[\gamma_{\mathbb{F}}]{} & \mathbb{F}(Y \to pt), \end{array}$$

$f_*([V \xrightarrow{h} X]) = [V \xrightarrow{f\circ h} Y]$, $\gamma_{\mathbb{F}}([V \xrightarrow{h} X]) = h_* 1_V$ for $[V \xrightarrow{h} X] \in \mathbb{M}^{\mathcal{C}}(X \xrightarrow{\mathrm{id}_X} X)$.
So, "SGA6" and "BFM–RR" are the same.

(3) "Verdier–RR" (for any map $f : X \to Y$):

$$\begin{array}{ccc} \mathbb{M}^{\mathcal{C}}(Y \to pt) & \xrightarrow{\gamma_{\mathbb{F}}} & \mathbb{F}(Y \to pt) \\ {\scriptstyle f^!}\big\downarrow & & \big\downarrow{\scriptstyle f^!=f^*} \\ \mathbb{M}^{\mathcal{C}}(X \to pt) & \xrightarrow[\gamma_{\mathbb{F}}]{} & \mathbb{F}(X \to pt), \end{array}$$

**DEFINITION**: If a map $f : X \to Y$ satisfies that each fiber has the same finite number of points, namely, $X \cong Y \times F$ with a finite set $F$, then $f$ is called a **“specialized” map**.

**DEFINITION**: If a map $f : X \to Y$ satisfies that each fiber has the same finite number of points, namely, $X \cong Y \times F$ with a finite set $F$, then $f$ is called a **"specialized" map**. The finite number $|F|$ is called the **"Euler number"** of $f$, denoted $\chi(f)$.

**DEFINITION**: If a map $f : X \to Y$ satisfies that each fiber has the same finite number of points, namely, $X \cong Y \times F$ with a finite set $F$, then $f$ is called a **"specialized" map**. The finite number $|F|$ is called the **"Euler number"** of $f$, denoted $\chi(f)$. (NOTE:$f : X \to Y$ is isomorphic to the projection $pr_1 : Y \times F \to Y$, thus a specialized map is a projection, but we stick to the above naming. Also $\chi(f) = |F| = \frac{|X|}{|Y|}$.)

**DEFINITION**: If a map $f : X \to Y$ satisfies that each fiber has the same finite number of points, namely, $X \cong Y \times F$ with a finite set $F$, then $f$ is called a **"specialized" map**. The finite number $|F|$ is called the **"Euler number"** of $f$, denoted $\chi(f)$. (NOTE:$f : X \to Y$ is isomorphic to the projection $pr_1 : Y \times F \to Y$, thus a specialized map is a projection, but we stick to the above naming. Also $\chi(f) = |F| = \frac{|X|}{|Y|}$.)

**LEMMA**: For two specialized maps $f : X \to Y$ and $g : Y \to Z$ we have

$$\chi(g \circ f) = \chi(g)\chi(f).$$

**DEFINITION**: If a map $f : X \to Y$ satisfies that each fiber has the same finite number of points, namely, $X \cong Y \times F$ with a finite set $F$, then $f$ is called a **"specialized" map**. The finite number $|F|$ is called the **"Euler number"** of $f$, denoted $\chi(f)$. (NOTE:$f : X \to Y$ is isomorphic to the projection $pr_1 : Y \times F \to Y$, thus a specialized map is a projection, but we stick to the above naming. Also $\chi(f) = |F| = \frac{|X|}{|Y|}$.)

**LEMMA**: For two specialized maps $f : X \to Y$ and $g : Y \to Z$ we have

$$\chi(g \circ f) = \chi(g)\chi(f).$$

**LEMMA**: Let $\mathcal{S}$ be the set of all specialized maps. For a specialized map $f : X \to Y \in \mathcal{S}$, we define

$$\theta_{\mathcal{S}}(f) := \chi(f)1_X = \chi(f)\theta_1(f).$$

**DEFINITION**: If a map $f : X \to Y$ satisfies that each fiber has the same finite number of points, namely, $X \cong Y \times F$ with a finite set $F$, then $f$ is called a **"specialized" map**. The finite number $|F|$ is called the **"Euler number"** of $f$, denoted $\chi(f)$. (NOTE:$f : X \to Y$ is isomorphic to the projection $pr_1 : Y \times F \to Y$, thus a specialized map is a projection, but we stick to the above naming. Also $\chi(f) = |F| = \frac{|X|}{|Y|}$.)

**LEMMA**: For two specialized maps $f : X \to Y$ and $g : Y \to Z$ we have

$$\chi(g \circ f) = \chi(g)\chi(f).$$

**LEMMA**: Let $\mathcal{S}$ be the set of all specialized maps. For a specialized map $f : X \to Y \in \mathcal{S}$, we define

$$\theta_{\mathcal{S}}(f) := \chi(f)1_X = \chi(f)\theta_1(f).$$

Then $\theta_{\mathcal{S}}$ is a nice canonical orientation, i.e., we have

1. $\theta_{\mathcal{S}}(g \circ f) = \theta_{\mathcal{S}}(f) \bullet \theta_{\mathcal{S}}(g)$ for $f : X \to Y, g : Y \to Z \in \mathcal{S}$,

**DEFINITION**: If a map $f : X \to Y$ satisfies that each fiber has the same finite number of points, namely, $X \cong Y \times F$ with a finite set $F$, then $f$ is called a **"specialized" map**. The finite number $|F|$ is called the **"Euler number"** of $f$, denoted $\chi(f)$. (NOTE:$f : X \to Y$ is isomorphic to the projection $pr_1 : Y \times F \to Y$, thus a specialized map is a projection, but we stick to the above naming. Also $\chi(f) = |F| = \frac{|X|}{|Y|}$.)

**LEMMA**: For two specialized maps $f : X \to Y$ and $g : Y \to Z$ we have

$$\chi(g \circ f) = \chi(g)\chi(f).$$

**LEMMA**: Let $\mathcal{S}$ be the set of all specialized maps. For a specialized map $f : X \to Y \in \mathcal{S}$, we define

$$\theta_{\mathcal{S}}(f) := \chi(f)1_X = \chi(f)\theta_1(f).$$

Then $\theta_{\mathcal{S}}$ is a nice canonical orientation, i.e., we have

1. $\theta_{\mathcal{S}}(g \circ f) = \theta_{\mathcal{S}}(f) \bullet \theta_{\mathcal{S}}(g)$ for $f : X \to Y, g : Y \to Z \in \mathcal{S}$,
2. $\theta_{\mathcal{S}}(\mathrm{id}_X) = 1_X$,

**DEFINITION**: If a map $f : X \to Y$ satisfies that each fiber has the same finite number of points, namely, $X \cong Y \times F$ with a finite set $F$, then $f$ is called a **"specialized" map**. The finite number $|F|$ is called the **"Euler number"** of $f$, denoted $\chi(f)$. (NOTE:$f : X \to Y$ is isomorphic to the projection $pr_1 : Y \times F \to Y$, thus a specialized map is a projection, but we stick to the above naming. Also $\chi(f) = |F| = \frac{|X|}{|Y|}$.)

**LEMMA**: For two specialized maps $f : X \to Y$ and $g : Y \to Z$ we have

$$\chi(g \circ f) = \chi(g)\chi(f).$$

**LEMMA**: Let $\mathcal{S}$ be the set of all specialized maps. For a specialized map $f : X \to Y \in \mathcal{S}$, we define

$$\theta_{\mathcal{S}}(f) := \chi(f)1_X = \chi(f)\theta_1(f).$$

Then $\theta_{\mathcal{S}}$ is a nice canonical orientation, i.e., we have

1. $\theta_{\mathcal{S}}(g \circ f) = \theta_{\mathcal{S}}(f) \bullet \theta_{\mathcal{S}}(g)$ for $f : X \to Y, g : Y \to Z \in \mathcal{S}$,
2. $\theta_{\mathcal{S}}(\mathrm{id}_X) = 1_X$,
3. for the following fiber square

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow[g]{} & Y, \end{array}$$

$$g^*(\theta_{\mathcal{S}}(f)) := \theta_{\mathcal{S}}(f').$$

The following theorem follows from the theorems of UBT:
**THEOREM**: *Let the situation be as above. On the category $\mathcal{F}$ of finite sets there exists a unique Grothendieck transformation*

$$\gamma_F : \mathbb{M}^{\mathcal{C}}_{\mathcal{S}} \to \mathbb{F}$$

*such that for any* $f : X \to Y \in \mathcal{S}$

$$\gamma_{\mathbb{F}}(\theta_{\mathbb{M}^{\mathcal{C}}_{\mathcal{S}}}(f)) = \theta_{\mathcal{S}}(f) = \chi(f)\theta_1(f).$$

The following theorem follows from the theorems of UBT:
**THEOREM**: *Let the situation be as above. On the category $\mathcal{F}$ of finite sets there exists a unique Grothendieck transformation*

$$\gamma_F : \mathbb{M}^{\mathcal{C}}_{\mathcal{S}} \to \mathbb{F}$$

*such that for any $f : X \to Y \in \mathcal{S}$*

$$\gamma_{\mathbb{F}}(\theta_{\mathbb{M}^{\mathcal{C}}_{\mathcal{S}}}(f)) = \theta_{\mathcal{S}}(f) = \chi(f)\theta_1(f).$$

**REMARK**: As to the abelian group $\mathbb{M}^{\mathcal{C}}_{\mathcal{S}}(X \xrightarrow{f} Y)$, we note that if $f : X \to Y$ is not surjective, then $\mathbb{M}^{\mathcal{C}}_{\mathcal{S}}(X \xrightarrow{f} Y) = 0$.

The following theorem follows from the theorems of UBT:
**THEOREM**: *Let the situation be as above. On the category $\mathcal{F}$ of finite sets there exists a unique Grothendieck transformation*

$$\gamma_F : \mathbb{M}^{\mathcal{C}}_{\mathcal{S}} \to \mathbb{F}$$

*such that for any $f : X \to Y \in \mathcal{S}$*

$$\gamma_{\mathbb{F}}(\theta_{\mathbb{M}^{\mathcal{C}}_{\mathcal{S}}}(f)) = \theta_{\mathcal{S}}(f) = \chi(f)\theta_1(f).$$

**REMARK**: As to the abelian group $\mathbb{M}^{\mathcal{C}}_{\mathcal{S}}(X \xrightarrow{f} Y)$, we note that if $f : X \to Y$ is not surjective, then $\mathbb{M}^{\mathcal{C}}_{\mathcal{S}}(X \xrightarrow{f} Y) = 0$. Note that this kind of thing happens,

The following theorem follows from the theorems of UBT:
**THEOREM**: *Let the situation be as above. On the category $\mathcal{F}$ of finite sets there exists a unique Grothendieck transformation*

$$\gamma_F : \mathbb{M}^{\mathcal{C}}_{\mathcal{S}} \to \mathbb{F}$$

*such that for any* $f : X \to Y \in \mathcal{S}$

$$\gamma_{\mathbb{F}}(\theta_{\mathbb{M}^{\mathcal{C}}_{\mathcal{S}}}(f)) = \theta_{\mathcal{S}}(f) = \chi(f)\theta_1(f).$$

**REMARK**: As to the abelian group $\mathbb{M}^{\mathcal{C}}_{\mathcal{S}}(X \xrightarrow{f} Y)$, we note that if $f : X \to Y$ is not surjective, then $\mathbb{M}^{\mathcal{C}}_{\mathcal{S}}(X \xrightarrow{f} Y) = 0$. Note that this kind of thing happens, for example, in the case of bivariant theory $\mathbb{F}$ of constructible functions (see [§6.1.2 Definition of $\mathbb{F}$, Remarks., p.62] of FM).

**REMARK**: The above equality

$$\gamma_{\mathbb{F}}(\theta_{\mathbb{M}_{\mathcal{S}}^{\mathcal{C}}}(f)) = \chi(f)\theta_1(f).$$

is *a Riemann–Roch formula* for the Grothendieck transformation

$$\gamma_{\mathbb{F}} : \mathbb{M}_{\mathcal{S}}^{\mathcal{C}} \to \mathbb{F}$$

with respect to the canonical orientations $\theta_{\mathbb{M}_{\mathcal{S}}^{\mathcal{C}}}$ on $\mathbb{M}_{\mathcal{S}}^{\mathcal{C}}$ and $\theta_1$ on $\mathbb{F}$.

**REMARK**: The above equality

$$\gamma_{\mathbb{F}}(\theta_{\mathbb{M}^{\mathcal{C}}_{\mathcal{S}}}(f)) = \chi(f)\theta_1(f).$$

is *a Riemann–Roch formula* for the Grothendieck transformation

$$\gamma_{\mathbb{F}} : \mathbb{M}^{\mathcal{C}}_{\mathcal{S}} \to \mathbb{F}$$

with respect to the canonical orientations $\theta_{\mathbb{M}^{\mathcal{C}}_{\mathcal{S}}}$ on $\mathbb{M}^{\mathcal{C}}_{\mathcal{S}}$ and $\theta_1$ on $\mathbb{F}$.
The "modifier" $\chi(f)$ can be considered as a bivariant element $u_f := \chi(f)1_X \in \mathbb{F}(X \xrightarrow{\mathrm{id}_X} X)$ and we have $\chi(f)\theta_1(f) = u_f \bullet \theta_1(f)$.

**REMARK**: The above equality

$$\gamma_{\mathbb{F}}(\theta_{\mathbb{M}^{\mathcal{C}}_{\mathcal{S}}}(f)) = \chi(f)\theta_1(f).$$

is *a Riemann–Roch formula* for the Grothendieck transformation

$$\gamma_{\mathbb{F}} : \mathbb{M}^{\mathcal{C}}_{\mathcal{S}} \to \mathbb{F}$$

with respect to the canonical orientations $\theta_{\mathbb{M}^{\mathcal{C}}_{\mathcal{S}}}$ on $\mathbb{M}^{\mathcal{C}}_{\mathcal{S}}$ and $\theta_1$ on $\mathbb{F}$.
The "modifier" $\chi(f)$ can be considered as a bivariant element
$u_f := \chi(f)1_X \in \mathbb{F}(X \xrightarrow{\mathrm{id}_X} X)$ and we have $\chi(f)\theta_1(f) = u_f \bullet \theta_1(f)$.
Or, it can be also considered as a bivariant element
$d_f := \chi(f)1_Y \in \mathbb{F}(Y \xrightarrow{\mathrm{id}_Y} Y)$ and we have $\chi(f)\theta_1(f) = \theta_1(f) \bullet d_f$.

**REMARK**: The above equality

$$\gamma_{\mathbb{F}}(\theta_{\mathbb{M}^{\mathcal{C}}_{\mathcal{S}}}(f)) = \chi(f)\theta_1(f).$$

is *a Riemann–Roch formula* for the Grothendieck transformation

$$\gamma_{\mathbb{F}} : \mathbb{M}^{\mathcal{C}}_{\mathcal{S}} \to \mathbb{F}$$

with respect to the canonical orientations $\theta_{\mathbb{M}^{\mathcal{C}}_{\mathcal{S}}}$ on $\mathbb{M}^{\mathcal{C}}_{\mathcal{S}}$ and $\theta_1$ on $\mathbb{F}$.
The "modifier" $\chi(f)$ can be considered as a bivariant element $u_f := \chi(f)1_X \in \mathbb{F}(X \xrightarrow{\mathrm{id}_X} X)$ and we have $\chi(f)\theta_1(f) = u_f \bullet \theta_1(f)$.
Or, it can be also considered as a bivariant element $d_f := \chi(f)1_Y \in \mathbb{F}(Y \xrightarrow{\mathrm{id}_Y} Y)$ and we have $\chi(f)\theta_1(f) = \theta_1(f) \bullet d_f$.

Furthermore from this Riemann–Roch formula we can get the following "SGA 6", "BFM–RR" and "Verdier–RR":
(1) "SGA 6" (for a specialized map $f : X \to Y$):

$$\begin{array}{ccc} {\mathbb{M}^{\mathcal{C}}_{\mathcal{S}}}^*(X) & \xrightarrow{\gamma_{\mathbb{F}}} & \mathbb{F}^*(X) \\ {\scriptstyle f_!}\downarrow & & \downarrow{\scriptstyle f_!\left(\chi(f)1_X \bullet -\right) = \chi(f) f_*} \\ {\mathbb{M}^{\mathcal{C}}_{\mathcal{S}}}^*(Y) & \xrightarrow[\gamma_{\mathbb{F}}]{} & \mathbb{F}^*(X), \end{array}$$

(2) “BFM–RR” (for any map $f : X \to Y$):

$$\begin{array}{ccc} \mathbb{M}^{\mathcal{C}}_{\mathcal{S}*}(X) & \xrightarrow{\gamma_{\mathbb{F}}} & \mathbb{F}_*(X) \\ {\scriptstyle f_*}\downarrow & & \downarrow{\scriptstyle f_*} \\ \mathbb{M}^{\mathcal{C}}_{\mathcal{S}*}(Y) & \xrightarrow[\gamma_{\mathbb{F}}]{} & \mathbb{F}_*(Y), \end{array}$$

(3) “Verdier–RR” (for a specialized map $f : X \to Y$):

$$\begin{array}{ccc} \mathbb{M}^{\mathcal{C}}_{\mathcal{S}*}(Y) & \xrightarrow{\gamma_{\mathbb{F}}} & \mathbb{F}_*(Y) \\ {\scriptstyle f^!}\downarrow & & \downarrow{\scriptstyle \chi(f)1_X \bullet f^! = \chi(f) f^*} \\ \mathbb{M}^{\mathcal{C}}_{\mathcal{S}*}(X) & \xrightarrow[\gamma_{\mathbb{F}}]{} & \mathbb{F}_*(X), \end{array}$$

**REMARK**: The above bivariant theory $\mathbb{F}$ is considered as the target of a Grohendieck transformation

$$tr : K(X \xrightarrow{f} Y) \to \mathbb{F}(X \xrightarrow{f} Y)$$

in [Part I: Bivariant Theories, §10.1 Fixed point theorems for coherent sheaves] of [FM].

**REMARK**: The above bivariant theory $\mathbb{F}$ is considered as the target of a Grohendieck transformation

$$tr : K(X \xrightarrow{f} Y) \to \mathbb{F}(X \xrightarrow{f} Y)$$

in [Part I: Bivariant Theories, §10.1 Fixed point theorems for coherent sheaves] of [FM]. (The definition of $K(X \xrightarrow{f} Y)$ is complicated, see §10.1 of [FM]).

**REMARK**: The above bivariant theory $\mathbb{F}$ is considered as the target of a Grohendieck transformation

$$tr : K(X \xrightarrow{f} Y) \to \mathbb{F}(X \xrightarrow{f} Y)$$

in [Part I: Bivariant Theories, §10.1 Fixed point theorems for coherent sheaves] of [FM]. (The definition of $K(X \xrightarrow{f} Y)$ is complicated, see §10.1 of [FM]). It remains to see whether one could use the universal bivariant theory $\mathbb{M}^{\mathcal{C}}_{\mathcal{S}}$ or some modified version for some fixed point problem.

**REMARK**: The above bivariant theory $\mathbb{F}$ is considered as the target of a Grohendieck transformation

$$tr : K(X \xrightarrow{f} Y) \to \mathbb{F}(X \xrightarrow{f} Y)$$

in [Part I: Bivariant Theories, §10.1 Fixed point theorems for coherent sheaves] of [FM]. (The definition of $K(X \xrightarrow{f} Y)$ is complicated, see §10.1 of [FM]). It remains to see whether one could use the universal bivariant theory $\mathbb{M}^{\mathcal{C}}_{\mathcal{S}}$ or some modified version for some fixed point problem.

So far we consider the category of finite sets, but we can extend the above arguments to the category $\mathcal{ENS}$ of sets, namely, infinite sets can be allowed. In this case, we need to define a confined map and a specialized map.

**REMARK**: The above bivariant theory $\mathbb{F}$ is considered as the target of a Grohendieck transformation

$$tr : K(X \xrightarrow{f} Y) \to \mathbb{F}(X \xrightarrow{f} Y)$$

in [Part I: Bivariant Theories, §10.1 Fixed point theorems for coherent sheaves] of [FM]. (The definition of $K(X \xrightarrow{f} Y)$ is complicated, see §10.1 of [FM]). It remains to see whether one could use the universal bivariant theory $\mathbb{M}^{\mathcal{C}}_{\mathcal{S}}$ or some modified version for some fixed point problem.

So far we consider the category of finite sets, but we can extend the above arguments to the category $\mathcal{ENS}$ of sets, namely, infinite sets can be allowed. In this case, we need to define a confined map and a specialized map.

**DEFINITION**: For the category $\mathcal{ENS}$ of sets,

1. a map $f : X \to Y$ is called **confined** if the inverse image of a finite subset of $Y$ is finite, equivalently, **if every fiber is a finite set**.

**REMARK**: The above bivariant theory $\mathbb{F}$ is considered as the target of a Grohendieck transformation

$$tr : K(X \xrightarrow{f} Y) \to \mathbb{F}(X \xrightarrow{f} Y)$$

in [Part I: Bivariant Theories, §10.1 Fixed point theorems for coherent sheaves] of [FM]. (The definition of $K(X \xrightarrow{f} Y)$ is complicated, see §10.1 of [FM]). It remains to see whether one could use the universal bivariant theory $\mathbb{M}^{\mathcal{C}}_{\mathcal{S}}$ or some modified version for some fixed point problem.

So far we consider the category of finite sets, but we can extend the above arguments to the category $\mathcal{ENS}$ of sets, namely, infinite sets can be allowed. In this case, we need to define a confined map and a specialized map.

**DEFINITION**: For the category $\mathcal{ENS}$ of sets,

1. a map $f : X \to Y$ is called **confined** if the inverse image of a finite subset of $Y$ is finite, equivalently, **if every fiber is a finite set**.
2. a map $f : X \to Y$ is called **specialized** if each fiber is finite and the number of elements of the fiber is the same at each element of $Y$. In other words, $X \cong Y \times F$ with **a finite set** $F$ and $f : X \to Y$ **is isomorphic to the projection** $pr_1 : Y \times F \to Y$.

## Theorem

*For* $\mathcal{ENS}$, $\mathcal{C}$ *of confined maps,* $\mathcal{S}$ *of specialized maps,*

## Theorem

*For $\mathcal{ENS}$, $\mathcal{C}$ of confined maps, $\mathcal{S}$ of specialized maps,*

1. *There exists a unique Grothendieck transformation*

$$\gamma_{\mathbb{F}} : \mathbb{M}^{\mathcal{C}}_{\mathcal{S}} \to \mathbb{F}$$

*such that for any* $f : X \to Y \in \mathcal{S}$, $\gamma_{\mathbb{F}}(\theta_{\mathbb{M}^{\mathcal{C}}_{\mathcal{S}}}(f)) = \theta_{\mathcal{S}}(f) = \chi(f)\theta_1(f)$.

## Theorem

*For $\mathcal{ENS}$, $\mathcal{C}$ of confined maps, $\mathcal{S}$ of specialized maps,*

1. *There exists a unique Grothendieck transformation*

$$\gamma_{\mathbb{F}} : \mathbb{M}^{\mathcal{C}}_{\mathcal{S}} \to \mathbb{F}$$

   *such that for any $f : X \to Y \in \mathcal{S}$, $\gamma_{\mathbb{F}}(\theta_{\mathbb{M}^{\mathcal{C}}_{\mathcal{S}}}(f)) = \theta_{\mathcal{S}}(f) = \chi(f)\theta_1(f)$.*

2. *We get the same "SGA 6", "BFM-RR" and "Verdier-RR" as above.*

## Theorem

*For $\mathcal{ENS}$, $\mathcal{C}$ of confined maps, $\mathcal{S}$ of specialized maps,*

1. *There exists a unique Grothendieck transformation*

$$\gamma_{\mathbb{F}} : \mathbb{M}^{\mathcal{C}}_{\mathcal{S}} \to \mathbb{F}$$

   *such that for any $f : X \to Y \in \mathcal{S}$, $\gamma_{\mathbb{F}}(\theta_{\mathbb{M}^{\mathcal{C}}_{\mathcal{S}}}(f)) = \theta_{\mathcal{S}}(f) = \chi(f)\theta_1(f)$.*

2. *We get the same "SGA 6", "BFM-RR" and "Verdier-RR" as above.*

## Theorem

*For $\mathcal{ENS}$, $\mathcal{C}$ of confined maps, $\mathcal{S}$ of specialized maps,*

1. *There exists a unique Grothendieck transformation*

$$\gamma_{\mathbb{F}} : \mathbb{M}^{\mathcal{C}}_{\mathcal{S}} \to \mathbb{F}$$

   *such that for any $f : X \to Y \in \mathcal{S}$, $\gamma_{\mathbb{F}}(\theta_{\mathbb{M}^{\mathcal{C}}_{\mathcal{S}}}(f)) = \theta_{\mathcal{S}}(f) = \chi(f)\theta_1(f)$.*

2. *We get the same "SGA 6", "BFM-RR" and "Verdier-RR" as above.*

For more examples of bivariant theories and Riemann-Roch formulas, see

- Fulton-MacPherson's book [FM] (Mem. AMS.)

## Theorem

*For $\mathcal{ENS}$, $\mathcal{C}$ of confined maps, $\mathcal{S}$ of specialized maps,*

1. *There exists a unique Grothendieck transformation*

$$\gamma_{\mathbb{F}} : \mathbb{M}^{\mathcal{C}}_{\mathcal{S}} \to \mathbb{F}$$

   *such that for any $f : X \to Y \in \mathcal{S}$, $\gamma_{\mathbb{F}}(\theta_{\mathbb{M}^{\mathcal{C}}_{\mathcal{S}}}(f)) = \theta_{\mathcal{S}}(f) = \chi(f)\theta_1(f)$.*

2. *We get the same "SGA 6", "BFM-RR" and "Verdier-RR" as above.*

For more examples of bivariant theories and Riemann-Roch formulas, see

- Fulton-MacPherson's book [FM] (Mem. AMS.)
- Fulton's "Intersection Theory" (Springer)

## Theorem

*For $\mathcal{ENS}$, $\mathcal{C}$ of confined maps, $\mathcal{S}$ of specialized maps,*

1. *There exists a unique Grothendieck transformation*

$$\gamma_{\mathbb{F}} : \mathbb{M}^{\mathcal{C}}_{\mathcal{S}} \to \mathbb{F}$$

   *such that for any $f : X \to Y \in \mathcal{S}$, $\gamma_{\mathbb{F}}(\theta_{\mathbb{M}^{\mathcal{C}}_{\mathcal{S}}}(f)) = \theta_{\mathcal{S}}(f) = \chi(f)\theta_1(f)$.*

2. *We get the same "SGA 6", "BFM-RR" and "Verdier-RR" as above.*

For more examples of bivariant theories and Riemann-Roch formulas, see

- Fulton-MacPherson's book [FM] (Mem. AMS.)
- Fulton's "Intersection Theory" (Springer)

## Theorem

*For $\mathcal{E}\mathcal{N}\mathcal{S}$, $\mathcal{C}$ of confined maps, $\mathcal{S}$ of specialized maps,*

1. *There exists a unique Grothendieck transformation*

$$\gamma_{\mathbb{F}} : \mathbb{M}^{\mathcal{C}}_{\mathcal{S}} \to \mathbb{F}$$

   *such that for any $f : X \to Y \in \mathcal{S}$, $\gamma_{\mathbb{F}}(\theta_{\mathbb{M}^{\mathcal{C}}_{\mathcal{S}}}(f)) = \theta_{\mathcal{S}}(f) = \chi(f)\theta_1(f)$.*

2. *We get the same "SGA 6", "BFM-RR" and "Verdier-RR" as above.*

For more examples of bivariant theories and Riemann-Roch formulas, see

- Fulton-MacPherson's book [FM] (Mem. AMS.)
- Fulton's "Intersection Theory" (Springer)

For some applications, see, e.g.,

- Behrend-Ginot-Noohi-Xu, "String Topology for Stacks", Astérisque 343 (2012)

## Theorem

*For $\mathcal{ENS}$, $\mathcal{C}$ of confined maps, $\mathcal{S}$ of specialized maps,*

1. *There exists a unique Grothendieck transformation*

$$\gamma_{\mathbb{F}} : \mathbb{M}^{\mathcal{C}}_{\mathcal{S}} \to \mathbb{F}$$

*such that for any* $f : X \to Y \in \mathcal{S}$, $\gamma_{\mathbb{F}}(\theta_{\mathbb{M}^{\mathcal{C}}_{\mathcal{S}}}(f)) = \theta_{\mathcal{S}}(f) = \chi(f)\theta_1(f)$.

2. *We get the same "SGA 6", "BFM-RR" and "Verdier-RR" as above.*

For more examples of bivariant theories and Riemann-Roch formulas, see

- Fulton-MacPherson's book [FM] (Mem. AMS.)
- Fulton's "Intersection Theory" (Springer)

For some applications, see, e.g.,

- Behrend-Ginot-Noohi-Xu, "String Topology for Stacks", Astérisque 343 (2012)
- Lipman-Tarrío-Lopéz; "Bivariance, Grothendieck duality and Hochschild homology I: construction of a bivariant theory", Asian J. Math. 15 (2011), 453-500

## Theorem

*For $\mathcal{ENS}$, $\mathcal{C}$ of confined maps, $\mathcal{S}$ of specialized maps,*

1. *There exists a unique Grothendieck transformation*

$$\gamma_{\mathbb{F}} : \mathbb{M}^{\mathcal{C}}_{\mathcal{S}} \to \mathbb{F}$$

   *such that for any $f : X \to Y \in \mathcal{S}$, $\gamma_{\mathbb{F}}(\theta_{\mathbb{M}^{\mathcal{C}}_{\mathcal{S}}}(f)) = \theta_{\mathcal{S}}(f) = \chi(f)\theta_1(f)$.*

2. *We get the same "SGA 6", "BFM-RR" and "Verdier-RR" as above.*

For more examples of bivariant theories and Riemann-Roch formulas, see

- Fulton-MacPherson's book [FM] (Mem. AMS.)
- Fulton's "Intersection Theory" (Springer)

For some applications, see, e.g.,

- Behrend-Ginot-Noohi-Xu, "String Topology for Stacks", Astérisque 343 (2012)
- Lipman-Tarrío-Lopéz; "Bivariance, Grothendieck duality and Hochschild homology I: construction of a bivariant theory", Asian J. Math. 15 (2011), 453-500
- Lipman-Tarrío-Lopéz; "Bivariance, Grothendieck duality and Hochschild homology II: the fundamental class of a flat scheme-map", Advances in Math. 257 (2014), 365-461.

## Theorem

*For $\mathcal{ENS}$, $\mathcal{C}$ of confined maps, $\mathcal{S}$ of specialized maps,*

1. *There exists a unique Grothendieck transformation*

$$\gamma_{\mathbb{F}} : \mathbb{M}^{\mathcal{C}}_{\mathcal{S}} \to \mathbb{F}$$

*such that for any* $f : X \to Y \in \mathcal{S}$, $\gamma_{\mathbb{F}}(\theta_{\mathbb{M}^{\mathcal{C}}_{\mathcal{S}}}(f)) = \theta_{\mathcal{S}}(f) = \chi(f)\theta_1(f)$.

2. *We get the same "SGA 6", "BFM-RR" and "Verdier-RR" as above.*

For more examples of bivariant theories and Riemann-Roch formulas, see

- Fulton-MacPherson's book [FM] (Mem. AMS.)
- Fulton's "Intersection Theory" (Springer)

For some applications, see, e.g.,

- Behrend-Ginot-Noohi-Xu, "String Topology for Stacks", Astérisque 343 (2012)
- Lipman-Tarrío-Lopéz; "Bivariance, Grothendieck duality and Hochschild homology I: construction of a bivariant theory", Asian J. Math. 15 (2011), 453-500
- Lipman-Tarrío-Lopéz; "Bivariance, Grothendieck duality and Hochschild homology II: the fundamental class of a flat scheme-map", Advances in Math. 257 (2014), 365-461.
- Déglise, "Bivariant Theories in Motivic Stable Homotopy", Documenta Math. 23 (2018), 997-1076.

Thank you very much for your attention!