Cartan calculi on the free loop spaces

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空間の代数的・幾何的モデルとその周辺

This talk is based on joint work with Kuribayashi Katsuhiko, Shun Wakatsuki and Toshihiro Yamaguchi (arXiv:2207.05941)

Plan

1. Background and Main result (rough)
2. Definition of homotopy Cartan calculi
3. Homotopy Cartan calculi on the free loop spaces
4. Computational examples and Motivated result

Notations

- M : simply-connected closed manifold
- $LM := \mathrm{Map}(S^1, M)$: the free loop space
- $\mathrm{aut}_1(M) := \{f \in \mathrm{Map}(M, M) \mid f \simeq id_M\}$
  : the monoid of self-homotopy equivalences of M
- $H^*(-)$: the singular cohomology over a field $\mathbb{K}$

Classical example of Cartan calculus

- $(\Omega^*(M), d)$: de Rham complex
- $X \in \mathrm{Der}(C^\infty(M))$: vector field
- $L_X : \Omega^n(M) \to \Omega^n(M)$: Lie derivative
- $\iota_X : \Omega^n(M) \to \Omega^{n-1}(M)$: interior product (contraction)

The Lie derivative and the interior product induce maps

$$L,\ \iota : \mathrm{Der}(C^\infty(M)) \longrightarrow \mathrm{Der}(\Omega^*(M))$$

which satisfy $L_X = [d, \iota_X]$ called **Cartan (magic) formula**.

Problem : Does there exist a "second stage" of Cartan calculus:

$$L,\ e : \mathrm{Der}(\Omega^*(M)) \longrightarrow \quad ??$$

Cartan formula in Hochschild theory

- A : algebra
- $C_*(A) = \{C_n(A), d\}$: Hochshild chain complex
  - $C_n(A) = A^{\otimes(n+1)}$
  - $d : C_n(A) \to C_{n-1}(A)$ the differential is defined by the multiplication

For $\theta \in \mathrm{Der}(A)$, we define a map $L_\theta : C_*(A) \to C_*(A)$ by

$$L_\theta(a_0, \ldots, a_n) = \sum_i (a_0, \ldots, a_{i-1}, \theta(a_i), a_{i+1}, \ldots, a_n).$$

Proposition (Loday)

There are well-defined homomorphism of Lie algebras:

$$HH^1(A) \cong \mathrm{Der}(A)/\{\textit{inner derivation}\} \longrightarrow \mathrm{End}(HH_*(A)),\ [\theta] \mapsto L_\theta.$$

Main results (rough)

(1) We have

- an algebraic construction of a Cartan calculus

$$L, e : \mathrm{Der}(\wedge V) \longrightarrow \mathrm{End}(C_*(\wedge V)),$$

where $\wedge V \xrightarrow{\simeq} \Omega^*(M)$ is a minimal Sullivan model for M over $\mathbb{R}$.

- a geometric construction of a Cartan calculus

$$L, e : \pi_*(\mathrm{aut}_1(M)) \otimes \mathbb{R} \longrightarrow \mathrm{End}(H^*(LM; \mathbb{R})).$$

(2) There exists a commutative diagram for $* \geq 2$;

$$\begin{array}{ccc} H_*(\mathrm{Der}(\wedge V)) & \xrightarrow[e]{L} & \mathrm{End}(HH_*(\wedge V)) \\ \text{Sullivan's iso} \uparrow \cong & & \uparrow \cong \\ \pi_*(\mathrm{aut}_1(M)) \otimes \mathbb{R} & \xrightarrow[\pm e]{L} & \mathrm{End}(H^*(LM; \mathbb{R})). \end{array}$$

Homotopy Cartan calculus (1)

- $(\mathfrak{g}, \delta, [\ ,\ ])$: differential graded Lie algebra
- (M, d, B) : mixed complex
  - $d, B : M \to M$: linear maps of degree 1, -1, respectively.
  - $d^2 = 0$, $B^2 = 0$, $[d, B] = dB + Bd = 0$.

Definition (Fiorenza–Kowalzig)

A **homotopy pre-Cartan calculus** *of* $\mathfrak{g}$ *on* M *consists of linear maps*

$e : \mathfrak{g} \to \mathrm{End}(M)$ *of degree* 1,

$L : \mathfrak{g} \to \mathrm{End}(M)$ *of degree* 0,

$S : \mathfrak{g} \to \mathrm{End}(M)$ *of degree* -1

such that

$$L_\theta = [B, e_\theta] + [d, S_\theta] + S_{\delta\theta}, \qquad [d, e_\theta] + e_{\delta\theta} = 0, \qquad [B, S_\theta] = 0$$

for any $\theta \in \mathfrak{g}$.

Homotopy Cartan calculus (2)

Definition (Fiorenza–Kowalzig)

A **homotopy Cartan calculus** *on* M *is a homotopy pre-Cartan calculus* $(\mathfrak{g}, e, L, S)$ *endowed with a linear map* $T : \mathfrak{g} \otimes \mathfrak{g} \to \mathrm{End}(M)$ *of degree* 0 *such that*

$$[e_\theta, L_\rho] - e_{[\theta,\rho]} = [d, T_{\theta,\rho}] - T_{\delta\theta,\rho} - (-1)^{\deg\theta} T_{\theta,\delta\rho},$$
$$[S_\theta, L_\rho] - S_{[\theta,\rho]} = [B, T_{\theta,\rho}].$$

for any $\theta, \rho \in \mathfrak{g}$*. Here,* T *is called Gelfan'd-Daletskiĭ-Tsygan homotopy***.**

Remark

If $T = 0$*, then* e *is regarded as a morphisms of* $(\mathfrak{g}, d, [\ ,\ ])$*-modules.*

Proposition (Fiorenza-Kowalzig)

Let $(\mathfrak{g}, e, L, S, T)$ *be a homotopy Cartan calculus on* M*. Then* $L : \mathfrak{g} \to \mathrm{End}(M)$ *is a morphism of dg Lie algebras.*

Cartan calculus on the Hochschild chain complex (1)

- A : augmented differential graded algebra
- $C_*(A) = (A \otimes T(s\bar{A}), d)$: Hochschild chain complex, where
  - $\bar{A}$: augmentation ideal
  - $(s\bar{A})^n = \bar{A}^{n+1}$: suspension
- $B : C_*(A) \longrightarrow C_*(A)$: Connes' B-operator

  $B|_{A\otimes T^n(s\bar{A})} = s \circ (1 + t + \cdots + t^n)$, where
  - $s(a_0[a_1|\cdots|a_n]) = 1[a_0|\cdots|a_n]$,
  - $t_n(a_0[a_1|\cdots|a_n]) = \pm a_n[a_0|\cdots|a_{n-1}]$.

Proposition (Burghelea – Vigué-Poirrier)

$(C_*(A), d, B)$: *mixed complex*

Cartan calculus on the Hochschild chain complex (2)

For any $\theta \in \mathrm{Der}(A)$, we define L_θ, e_θ, $S_\theta : C_*(A) \to C_*(A)$ by

- $L_\theta = \sum_i L_{\theta,i}$

  $$L_{\theta,i}(a_0[a_1|\cdots|a_n]) = \begin{cases} \theta(a_0)[a_1|a_2|\cdots|a_n] & (i=0), \\ \pm a_0[a_1|\cdots|\theta(a_i)|\cdots|a_n] & (1 \le i \le n). \end{cases}$$

- $e_\theta|_A = 0, \qquad e_\theta(a_0[a_1|\cdots|a_n]) = \pm a_0\theta(a_1)[a_2|\cdots|a_n]$

- $$S_\theta|_{A\otimes T^n(s\bar{A})} = \sum_{j=1}^{n}\left(\sum_{k=0}^{n-j} s \circ t_n^k\right) \circ L_{\theta,j}$$

Proposition (KNWY)

$(\mathrm{Der}(A), e, L, S, T=0)$: *homotopy Cartan calculus on* $(C_*(A), d, B)$.

Geometric description of e and L

$f : S^n \to \mathrm{aut}_1(M) \in \pi_n(\mathrm{aut}_1(M))$

$\mathrm{ad}_f : S^n \times M \longrightarrow M$, $(u,x) \mapsto f(u)(x)$: adjoint map of f

$L(\mathrm{ad}_f) : LS^n \times LM \cong L(S^n \times M) \longrightarrow LM$

We define linear maps $L, e : \pi_*(\mathrm{aut}_1(M)) \to \mathrm{End}(H^*(LM))$ by

$$L_f : H^*(LM) \xrightarrow{L(\mathrm{ad}_f)^*} H^*(LS^n \times LM) \xrightarrow{\int_{[S^n]}} H^*(LM),$$

$$e_f : H^*(LM) \xrightarrow{L(\mathrm{ad}_f)^*} H^*(LS^n \times LM) \xrightarrow{\int_{\overline{[S^n]}}} H^*(LM),$$

where

- $\int_\omega$: the integration along a cohomology class $\omega \in H^*(LS^n)$,
- $[S^n] \in H^n(S^n) \hookrightarrow H^n(LS^n)$: the fundamental class,
- $\overline{[S^n]} \in H^{n-1}(LS^n)$: the image of $[S^n]$ induced by the composite

$$H^*(LM) \xrightarrow{(S^1\text{-action})^*} H^*(S^1 \times LM) \xrightarrow{\int_{[S^1]}} H^*(LM).$$

Sullivan model for LM

$\wedge V = (\wedge V, d)$: minimal Sullivan model for M, that is,

- $V = \{V^n\}_{n \geq 2}$: graded $\mathbb{Q}$-vector space
- $(\wedge V, d) = \big(TV / \langle x \otimes y - (-1)^{|x||y|} y \otimes x \rangle, d\big)$

  : free comm diff graded alg with d a decomposable differential
- $H^*(\wedge V, d) \cong H^*(M; \mathbb{Q})$ as alg

We define a mixed complex $(\wedge V \otimes \wedge sV, d, B)$, where

- $(sV)^n = V^{n+1}$
- $B : \wedge V \otimes \wedge sV \to \wedge V \otimes \wedge sV$, $B(v) = sv$, $B(sv) = 0$
- $d : \wedge V \otimes \wedge sV \to \wedge V \otimes \wedge sV$, $d(sv) = -Bd(v)$

Theorem (Sullivan – Vigué-Poirrier)

$H^*(\wedge V \otimes \wedge sV, d) \cong H^*(LM; \mathbb{Q})$ *as algebras.*

Cartan calculus on the Sullivan model for LM

Notation : $\mathcal{L} := (\wedge V \otimes \wedge sV, d)$.

For $\theta \in \mathrm{Der}(\wedge V)$, we define derivations L_θ, $e_\theta : \mathcal{L} \longrightarrow \mathcal{L}$ by

$$L_\theta(v) = \theta(v), \qquad L_\theta(\bar{v}) = (-1)^{|\theta|} s\theta(v),$$

$$e_\theta(v) = 0, \qquad e_\theta(\bar{v}) = (-1)^{|\theta|}\theta(v).$$

These induce maps

$$L, e : \mathrm{Der}(\wedge V) \longrightarrow \mathrm{Der}(\mathcal{L}).$$

Proposition (KNWY)

$(\mathrm{Der}(\wedge V), e, L, S = 0, T = 0)$: *homotopy Cartan calculus on* $(\mathcal{L}, d, B)$.

The isomorphisms in Main result

1. Sullivan's isomorphism

$f : S^n \to \mathrm{aut}_1(M) \in \pi_n(\mathrm{aut}_1(M)) \otimes \mathbb{Q}$

$\mathrm{ad}_f : S^n \times M \longrightarrow M$: adjoint map of f

$\mathcal{M}_{\mathrm{ad}_f} : \wedge V \longrightarrow \mathcal{M}_{S^n} \otimes \wedge V \xrightarrow{\simeq} H^*(S^n; \mathbb{Q}) \otimes \wedge V$: Sullivan rep for f

We write $\mathcal{M}_{\mathrm{ad}_f}(\alpha) = 1 \otimes \alpha + [S^n] \otimes \theta(\alpha)$. Then, Sullivan's isomorphism

$$\Phi : \pi_*(\mathrm{aut}_1(M)) \otimes \mathbb{Q} \xrightarrow{\cong} H_*(\mathrm{Der}(\wedge V))$$

is defined by $\Phi(f) = \theta$.

2. Burghelea and Vigué-Poirier's quasi-isomorphism

$$\Theta : C_*(\wedge V) \longrightarrow \mathcal{L}, \quad \Theta(a_0[a_1|\cdots|a_n]) = \frac{1}{n!} a_0 s a_1 \cdots s a_n.$$

Theorem (KNWY)

$$
\begin{CD}
H_*(\mathrm{Der}(\wedge V)) @>L>e> \mathrm{End}(HH_*(\wedge V)) \\
@A{\textit{Sullivan's iso}}A{\cong}A @AA{\cong}A \\
\pi_*(\mathrm{aut}_1(M)) \otimes \mathbb{R} @>L>(-1)^* e> \mathrm{End}(H^*(LM;\mathbb{R}))
\end{CD}
$$

(Sketch) Theorem is shown by the following commutative diagram;

$$
\begin{CD}
H_*(\mathrm{Der}(\wedge V)) @>L>e> \mathrm{End}(HH_*(\wedge V)) \\
@| @V{\cong}V{\Theta}V \\
H_*(\mathrm{Der}(\wedge V)) @>L>e> \mathrm{End}(H^*(\mathcal{L})) \\
@A{\text{Sullivan's iso}}A{\cong}A @A{\cong}AA \\
\pi_*(\mathrm{aut}_1(M)) \otimes \mathbb{R} @>L>(-1)^* e> \mathrm{End}(H^*(LM;\mathbb{R}))
\end{CD}
$$

Computational examples (1)

Example

$M = \mathbb{C}P^2$: complex projective space

$\wedge V = (\wedge(x, y), d)$, $|x| = 2$, $|y| = 5$, $dx = 0$, $dy = x^3$

: a minimal Sullivan model for $\mathbb{C}P^2$

Then

- $H^*(L\mathbb{C}P^2; \mathbb{Q})$ is spanned by

$$\{1,\ \alpha_n,\ x\alpha_n,\ \beta_n,\ x\beta_n\}_{n\geq 0}, \quad |\alpha_n| = 4n+1,\ |\beta_n| = 4n+2.$$

- $H_*(\mathrm{Der}(\wedge V))$ is spanned by $\{(y, 1), (y, x) : \wedge V \to \wedge V\}$.

For $\theta = (y, 1)$, we have

1. $L_\theta(\alpha_n) = 0, \qquad L_\theta(\beta_n) = -3n\alpha_{n-1},$
2. $e_\theta(\alpha_n) = nx\alpha_{n-1}, \qquad e_\theta(\beta_n) = -n\beta_{n-1}.$

Computational examples (2)

Example

For $\theta = (y, x)$, we have

1. $L_\theta(\alpha_n) = 0, \qquad L_\theta(\beta_n) = -2nx\alpha_{n-1},$
2. $e_\theta(\alpha_n) = nx\alpha_{n-1}, \qquad e_\theta(\beta_n) = -nx\beta_{n-1}.$

For determining a basis of $H^*(\mathcal{L})$, we rely on the software "kohomology".

Theorem (KNWY)

For a simply-connected closed manifold M,

$$e : \pi_*(\mathrm{aut}_1(M)) \otimes \mathbb{Q} \longrightarrow \mathrm{Der}(H^*(LM; \mathbb{Q}))$$

is injective.

Motivating result

Recall the reduced operator (Batalin-Vilkovisky (BV)-operator)

$$\tilde{\Delta} : \tilde{H}^*(LM;\mathbb{Q}) \xrightarrow{(S^1\text{-action})^*} \tilde{H}^*(S^1 \times LM;\mathbb{Q}) \xrightarrow{\int_{[S^1]}} \tilde{H}^{*-1}(LM;\mathbb{Q})$$

Definition (KNWY21)

M is **BV-exact** *if* $\mathrm{Im}\tilde{\Delta} = \mathrm{Ker}\tilde{\Delta}$.

[KNWY21] A reduction of the string bracket to the loop product, arXiv:2109.10536.

Theorem (KNWY21)

A simpy-connected space M admitting positive weights is BV exact. In particular, a formal space is BV exact.

In the proof, we use the (homotopy) Cartan calculus

$$L,\ e : \mathrm{Der}(\wedge V) \longrightarrow \mathrm{Der}(\mathcal{L}).$$

Equivariant version of the Lie derivative L

Given $f : S^n \to \mathrm{aut}_1(M) \in \pi_n(\mathrm{aut}_1(M))$. We consider the composite

$$u_f : S^n \xrightarrow{f} \mathrm{aut}_1(M) \xrightarrow{L(\ )} \mathrm{aut}_1(LM)$$

Observe that $\mathrm{ad}(u_f) : S^n \times LM \to LM$ is an S^1-equivariant map, where S^1-action on S^n is defined to be trivial. Thus, we have a map

$$v_f := \overline{\mathrm{ad}(u_f) \times_{S^1} 1} : (S^n \times LM) \times_{S^1} ES^1 \longrightarrow LM \times_{S^1} ES^1$$

and a derivation

$$\bar{L}_f : H^*_{S^1}(LM) \xrightarrow{v_f^*} H^*_{S^1}(S^n \times LM) \xrightarrow{\int_{[S^n]}} H^*_{S^1}(LM).$$

Theorem (KNWY)

The map $\bar{L} : \pi_*(\mathrm{aut}_1(M)) \to \mathrm{Der}(H^*_{S^1}(LM))$ *is a morphism of Lie alg.*

Reference

[FK] D. Fiorenza, N. Kowalzig. Higher brackets on cyclic and negative cyclic (co)homology. Int. Math. Res. Not., 2020(23): 9148–9209.

[KNWY21] K. Kuribayashi, T. Naito, S. Wakatsuki, T. Yamaguchi. A reduction of the string bracket to the loop product, arXiv:2109.10536.

[KNWY] K. Kuribayashi, T. Naito, S. Wakatsuki, T. Yamaguchi. Cartan calculi on the free loop spaces, arXiv:2207.05941.

Thank you for listening!