Some cycles of the space of long knots and Vassiliev invariants

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(partially joint with Saki Kanou, arXiv.2203.15329)

Algebraic and Geometric Models for Spaces and Related Topics
September 2, 2022

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Overview (1) : Configuration space integral, trivalent graphs and Vassiliev invariants

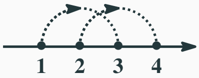
Aim : *non-trivalent* graph cocycles $\xrightarrow[\text{(CSI)}]{\text{configuration space integral}}$ *Vassiliev invariants*

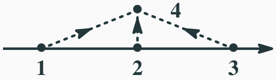
Background : Main objective is the topology of $\mathcal{K}_n := \{\text{long knots in } \mathbb{R}^n\}$. Today we focus on $H^0_{dR}(\mathcal{K}_3) = \{\mathbb{R}\text{-valued knot invariants}\}$.

- (R. Bott-C. Taubes, T. Kohno 1994, ...) For any Vassiliev invariant V,

$$\exists\,\Gamma : \text{a formal sum of } \textit{trivalent graphs} \xmapsto[CSI]{I} I(\Gamma) = V \ (\in \Omega^0_{dR}(\mathcal{K}_3) \quad \text{closed}).$$

Example. $I(X - Y) = v_2$: Casson's knot invariant, where

$X =$ 

$Y =$ 

- (A. Cattaneo, P. Cotta-Ramusino, R. Longoni 2002) $\exists\ \mathcal{G}^{k,*}_n$: *graph complexes* $(k \geq 2)$,

$$I : \mathcal{G}^{k,*}_n \to \Omega^{k(n-3)+*}_{dR}(\mathcal{K}_n) \quad \text{cochain maps } (n \geq 4),$$

$$I : H^{0}(\mathcal{G}^k_n) \hookrightarrow H^{k(n-3)}_{dR}(\mathcal{K}_n), \qquad (0 \leftrightarrow \text{trivalent graphs})$$

$$H^{0}(\mathcal{G}^{k}_n) \cong \{\text{Vassiliev invariants of order } k\} \text{ if } n \text{ is odd.}$$

Conjecture. $I : \bigoplus_k \mathcal{G}^{k,*}_n \to \Omega^*_{dR}(\mathcal{K}_n)$ is a quasi-isomorphism $(n \geq 4)$.

The trivalent part supports the conjecture, and can be seen as “high dim analogues of Vassiliev invariants”.

Overview (2-1) : Non-trivalent graphs

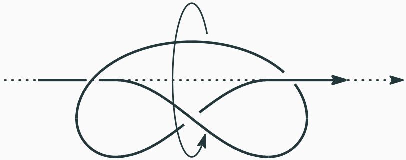
Little is known about non-trivalent graphs, but ∃ some results that suggest some relation to Vassiliev invariants.

Theorem (S. 2012)

$n \geq 3$ odd $\implies \exists\, \Gamma^1_3 \in \mathcal{G}^{3,1}_n$: a graph cocycle *of defect* 1, such that $I(\Gamma^1_3) \in H^{3(n-3)+1}_{dR}(\mathcal{K}_n)$ is not zero. In particular if $n = 3$,

$$\langle I(\Gamma^1_3), G_f \rangle = \int_{G_f} I(\Gamma^1_3) = v_2(f) \quad (f \in \mathcal{K}_3),$$

where G_f is *Gramain's* 1*-cycle* of $\mathcal{K}_3$.



Theorem (Longoni 2004, K. Pelatt-D. Sinha 2017)

If $n \geq 4$ is even, then $\exists\, G_L$: a graph cocycle of defect 1 such that $I(G_L) \in H^{3(n-3)+1}_{dR}(\mathrm{Emb}(S^1, \mathbb{R}^n))$ is not zero.

Overview (2-2) : Non-trivalent graphs

The cocycles below seem to be equal to $I(\Gamma_3^1)$, although they are defined in different ways.

Theorem (V. Turchin 2006)

$\exists\, v_3^1 \in$ (mod 2 Vassiliev spectral sequence for $\mathcal{K}_3$) with $\langle v_3^1, G_f \rangle = v_2(f) \mod 2$.

Theorem (A. Mortier 2015)

$\exists\, \alpha_3^1$: a 1-cocycle of $\mathcal{K}_3$ that satisfies $\langle \alpha_3^1, G_f \rangle = v_2(f)$ and

$$\langle \alpha_3^1, FH_f \rangle = 6v_3(f) - \mathrm{fr}(f)v_2(f). \qquad (*)$$

Here FH_f is *Fox-Hatcher 1-cycle* of $\widetilde{\mathcal{K}}_3$ (explained later).

We have a result analogous to $(*)$:

Theorem (S. Kanou-S. 2022)

$\langle I(\Gamma_3^1), FH_f \rangle = 6v_3(f) - \mathrm{fr}(f)v_2(f)$.

Overview (3) : defect 1 graphs and Vassiliev invariants

At present no way is known to produce non-trivalent graph cocycles, but the computations in [Kanou-S] in fact show the following.

Theorem (S. 2022)

For any graph cocycle $\Gamma^1_k \in \mathcal{G}^{k,1}_n$, we can evaluate $I(\Gamma^1_k) \in \Omega^{k(n-3)+1}_{dR}(\mathcal{K}_n)$ over some cycles (generalizing G_f and FH_f; explained later) in a combinatorial way.

In particular, if $n = 3$ then

- $\langle I(\Gamma^1_k), G\rangle$ is a Vassiliev invariant of order $\leq k-1$.
- $\langle I(\Gamma^1_k), FH\rangle$ is a Vassiliev invariant of order $\leq k$.

Remark

The above Theorem does not claim the non-triviality of $I(\Gamma^1_k) \in H^{k(n-3)+1}_{dR}(\mathcal{K}_n)$. It seems that we need rather complicated combinatorics to prove the non-triviality.

(Framed) long knots

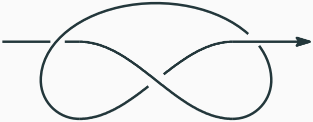
Definition

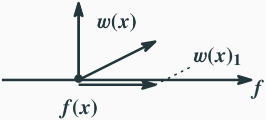
- A long knot is an embedding $f\colon \mathbb{R}^1 \hookrightarrow \mathbb{R}^n$ satisfying

$$|x| \geq 1 \implies f(x) = (x, 0, \ldots, 0).$$

- A framed long knot is a pair $\widetilde{f} = (f, w) \in \mathcal{K}_n \times \Omega_{I_n}\mathrm{SO}(n)$ such that the first column of $w(x)$ is $f'(x)/|f'(x)|$.
- $\mathcal{K}_n = \{\text{long knots in } \mathbb{R}^n\}$, $\quad \widetilde{\mathcal{K}}_n := \{\text{framed long knots in } \mathbb{R}^n\}$.

$H^0(\mathcal{K}_3; A) = \mathbf{Map}\,(\pi_0(\mathcal{K}_3), A) = \{A$-valued isotopy invariants of (long) knots$\}$.





Lemma (Sinha)

$$\begin{array}{ccc} \widetilde{\mathcal{K}}_n & \xrightarrow{\ \simeq\ } & \mathcal{K}_n \times \Omega_{I_{n-1}}\mathrm{SO}(n-1) \\ & {}_{p}\searrow \quad \mathcal{K}_n \quad \swarrow_{\mathrm{pr}_1} & \end{array}$$

where $p\colon \widetilde{\mathcal{K}}_n \to \mathcal{K}_n$ is the first projection. In particular $\widetilde{\mathcal{K}}_3 \simeq \mathcal{K}_3 \times \mathbb{Z}$.

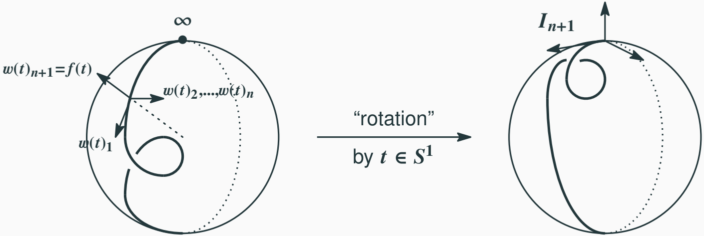
Fox-Hatcher 1-cycle of $\widetilde{\mathcal{K}}_n$

A framed long knot can also be seen as a pair (f, w), where

- $f : S^1 \hookrightarrow S^n$, with specified $f(0) = \infty$ and $f'(0)$,
- $w \in \Omega_{I_{n+1}} SO(n+1)$ with the first column $f'/|f'|$ and the last column f.

An "S^1-action" $FH : S^1 \times \widetilde{\mathcal{K}}_n \to \widetilde{\mathcal{K}}_n$ is defined as follows;

for $t \in S^1$ and $\widetilde{f} \in \widetilde{\mathcal{K}}_n$, "rotate" $\widetilde{f}$ by the multiplication of $w(t)^{-1} \in SO(n+1)$ and re-parametrize it to get $t \cdot \widetilde{f}$.



The orbit $FH_{\widetilde{f}} : S^1 \to \widetilde{\mathcal{K}}_n$ of $\widetilde{f} \in \widetilde{\mathcal{K}}_n$ is called the **Fox-Hatcher 1-cycle**.

Gramain and Fox-Hatcher cycles

Note that FH looks very similar to the S^1-action of $LX = \mathbf{Map}(S^1, X)$.

Fact (P. Salvatore-Turchin(-A. Kupers) 2022)

There is an action of the *framed little 2-disks operad* on $\widetilde{\mathcal{K}}_n$ and $H_*(\widetilde{\mathcal{K}}_n)$ is equipped with a *BV algebra* structure. The 1-cycles G_f and $FH_{\widetilde{f}}$ can be written respectively in terms of the *Gerstenhaber bracket* and the *BV operator*.

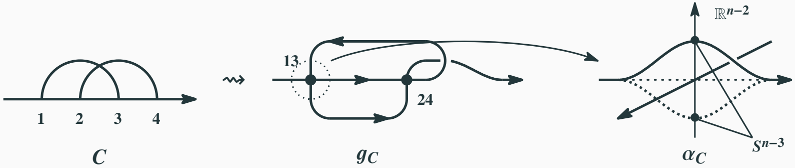
Remark

This is an affirmative answer to the question raised by R. Budney and Salvatore. The existence of the BV structure on $H_*(\widetilde{\mathcal{K}}_n)$ was found earlier [S].

Chord diagrams and cycles of $\mathcal{K}_n$

Given a *chord diagram* C with k chords, we have

- a "long immersion" $g_C : \mathbb{R}^1 \looparrowright \mathbb{R}^3$ ($\subset \mathbb{R}^n$) with k transverse double points that encode the chords of C, and
- $\alpha_C : (S^{n-3})^{\times k} \to \mathcal{K}_n$ that "resolves" the double points of g_C.



$n \geq 4 \implies [\alpha_C] \in H_{k(n-3)}(\mathcal{K}_n)$ depends only on C.

$n = 3 \implies$ for any choice of g_C, $\exists f \in \mathcal{K}_3$ such that

$$\alpha_C = \sum_{\epsilon_1,\dots,\epsilon_k \in \{+1,-1\}} \epsilon_1 \cdots \epsilon_k f_{\epsilon_1,\dots,\epsilon_k}.$$

Definition

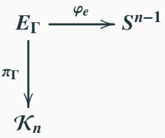
A knot invariant V is a *Vassiliev invariant of order $\leq k$* if, for any C with $k+1$ chords and any choice of g_C,

$$V(\alpha_C) \quad \left(= \sum_{\epsilon_1,\dots,\epsilon_{k+1} \in \{+1,-1\}} \epsilon_1 \cdots \epsilon_{k+1} V(f_{\epsilon_1,\dots,\epsilon_{k+1}}) \right) \quad = 0.$$

Configuration space integral (CSI)

$\mathrm{Conf}_k(X) := \{(x_1, \dots, x_k) \in X^{\times k} \mid x_i \neq x_j \text{ if } i \neq j\}$.

Configuration spaces associated with graphs:



- $E_\Gamma := \left\{ \begin{array}{l} (f, (y_1, y_2, \dots)) \\ \in \mathcal{K}_n \times \mathrm{Conf}_*(\mathbb{R}^n) \end{array} \middle| \begin{array}{l} y_i = f(x_i), \\ \exists\, x_1 < \cdots < x_s \end{array} \right\}$.
- For an edge e, define $\varphi_e : E_\Gamma \to S^{n-1}$ by

$$\varphi_e(f, y) := \begin{cases} (y_j - y_i)/|y_j - y_i| & e = \overrightarrow{ij}, \\ f'(x_i)/|f'(x_i)| & e \text{ is a loop at the vertex } i. \end{cases}$$

- $\omega_\Gamma := \bigwedge_{e:\text{ edges}} \varphi_e^* \mathrm{vol}_{S^{n-1}} \in \Omega_{dR}^{(n-1)\sharp\{\text{edges}\}}(E_\Gamma)$.
- The first projection $\pi_\Gamma : E_\Gamma \to \mathcal{K}_n$ is a fiber bundle with fiber $\subset \mathrm{Conf}_*(\mathbb{R}^n)$.
- $I(\Gamma) := \pi_{\Gamma *}\omega_\Gamma \in \Omega^*_{dR}(\mathcal{K}_n)$, where $\pi_{\Gamma *}$ is the integration along the fiber.

Lemma

$\mathcal{G}_n^{k,l} := \mathbb{R}\langle \Gamma \mid b_1(\Gamma) = k$ and $\sum_{v \in V(\Gamma)}(\text{valence}(v) - 3) = l \rangle$ (l : the *defect*). If $\Gamma \in \mathcal{G}_n^{k,l}$, then $\deg I(\Gamma) = k(n-3) + l$.

Coboundary operation on graphs

Proposition

1. (Axelrod-Singer) $\exists$ Fiberwise compactification of E_Γ so that φ_e extends smoothly on the compactification (thus $I(\Gamma)$ converges).
2. $dI(\Gamma) = I(\partial\Gamma)$ (essentially by Stokes' theorem), where $\partial\colon \mathcal{G}_n^{k,l} \to \mathcal{G}_n^{k,l+1}$ is given by

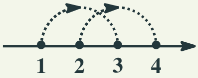
$$\partial\Gamma = \sum_{e\in\{\text{edges and arcs}\}} \pm\Gamma/e, \qquad \Gamma \xmapsto{\text{collapsing } e} \Gamma/e.$$

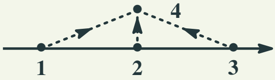
Warning. In fact we need a "correction term" when $n = 3$.

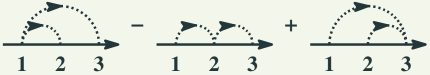
Thus $I\colon \mathcal{G}_n^{k,*} \to \Omega_{dR}^{k(n-3)+*}(\mathcal{K}_n)$ is a cochain map ($n \geq 4$).

Example

We have $\partial X = \partial Y$, where $X, Y \in \mathcal{G}_n^{2,0}$ and $\partial X = \partial Y \in \mathcal{G}_n^{2,1}$ are

$X =$ 

$Y =$ 

$\partial X =$  $= \partial Y$

Thus $v_2 := I(X - Y)$ is a $2(n-3)$-cocycle.

Defect 0 graphs $\rightsquigarrow$ non-zero cohomology classes

Question. How to show the non-triviality of $[I(\Gamma)] \in H^{k(n-3)}_{dR}(\mathcal{K}_n)$ for a cocycle $\Gamma \in \mathcal{G}^{k,0}_n$?

Answer. Evaluate $I(\Gamma)$ over a cycle α_C for some chord diagram C.

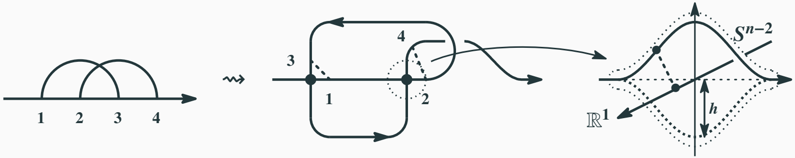
Suppose $\Gamma = aC + \sum_i a_i \Gamma_i$, where Γ_i are graphs of defect 0 other than C.

Proposition

$\langle I(\Gamma), \alpha_C \rangle = \pm a$.

Outline of proof. $\langle I(\Gamma), \alpha_C \rangle = a\langle I(C), \alpha_C \rangle + \sum_i a_i \langle I(\Gamma_i), \alpha_C \rangle$.

- $\langle I(C), \alpha_C \rangle = \int_{\alpha_C} \omega_C$ localizes to the following configurations, and each edge serves as $\mathrm{Link}(S^{n-2}, \mathbb{R}^1) = \pm 1$.



C $\langle I(C), \alpha_C \rangle$ α_C

- For $\forall$ configurations in E_{Γ_i}, $\exists$ a doublepoint of f_C to which no edge comes, and $\langle I(\Gamma_i), \alpha_C \rangle$ is defined continuously for $h \geq 0$. Thus $\langle I(\Gamma_i), \alpha_C \rangle \to 0$ ($h \to 0$) by dimensional reason. □

The case $n = 3$

$\Gamma \in \mathcal{G}_3^{k,0}$ a cocycle $\rightsquigarrow$ $V := I(\Gamma) \in H^0_{dR}(\mathcal{K}_3)$ is a knot invariant.

Choose crossings $c_1, \ldots, c_k$ of a diagram of $f \in \mathcal{K}_3$ that determine a chord diagram C.

$$\Longrightarrow \quad V(\alpha_C) = \langle I(\Gamma), \alpha_C \rangle = \sum_{\epsilon_1,\ldots,\epsilon_k \in \{+1,-1\}} \epsilon_1 \cdots \epsilon_k V(f_{\epsilon_1,\ldots,\epsilon_k}) \quad (= D^k V(f)).$$

Since $\langle I(\Gamma), \alpha_C \rangle = \pm a$ depends only on C, the invariant V is of order $\leq k$;

for C with $k+1$ chords,

$$D^{k+1}V(f) = D^k V(f_{\epsilon_{k+1}=+1}) - D^k V(f_{\epsilon_{k+1}=-1}) = a - a = 0.$$

Theorem (Bott-Taubes, Kohno, ...)

All the Vassiliev invariants for (long) knots can be obtained from graph cocycles of defect **0** (= trivalent) in this way.

Defect 1 graphs

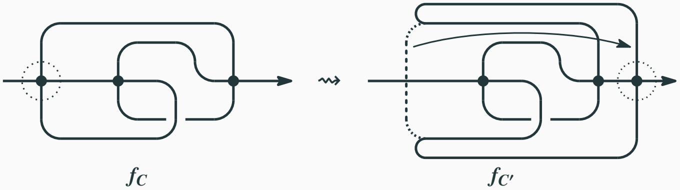
Question. How to show the non-triviality of $[I(\Gamma^1_k)] \in H^{k(n-3)+1}_{dR}(\mathcal{K}_n)$ for a cocycle $\Gamma \in \mathcal{G}^{k,1}_n$?

"Answer". Evaluate $I(\Gamma^1_k)$ over a cycle $FH(\alpha_C)$ (or $G(\alpha_C)$) for some chord diagram C;

$$FH(\alpha_C)\colon S^1 \times (S^{n-3})^{\times k} \xrightarrow{\mathrm{id}_{S^1} \times \alpha_C} S^1 \times \widetilde{\mathcal{K}}_n \xrightarrow{FH} \widetilde{\mathcal{K}}_n.$$

Diagramatic description of $FH(\alpha_C)$.

- S^{n-3}-factors serve as "resolutions" of doublepoints of f_C, and
- S^1-factor is decomposed into $2m$ *Fox-Hatcher moves*



where $m = \sharp\{$crossings in the diagram of $f_C\}$ ($m = 4$ for the above f_C).

The corresponding chord diagrams cyclically change as:

Thus $FH(\alpha_C)$ is homologous to $FH(\alpha_{C'})$ if $C \xrightarrow{\text{cyclic change}} C'$.

Defect 1 graphs

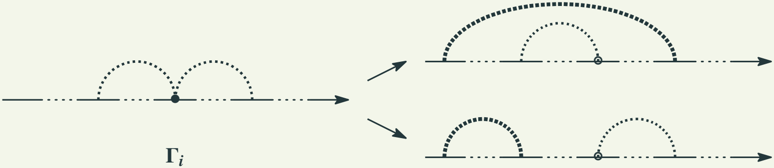
Let $\Gamma^1_k = \sum_i a_i \Gamma_i \in \mathcal{G}^{k,1}_n$ be a graph cocycle. Each Γ_i is one of the following two types:

(i) semi-chord diagrams

(ii) graphs with free vertices

Theorem (S. 2022)

$\langle I(\Gamma^1_k),\ p_* FH(\alpha_C)\rangle = \sum_{i\in\mathcal{I}} \pm a_i$, where $i \in \mathcal{I} \iff \Gamma_i$ is a semi-chord diagrams one of whose "resolutions"



is equal to a cyclic transformation of C.

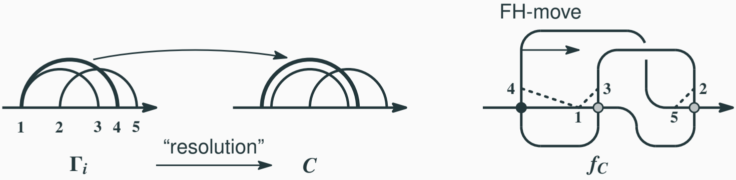
A rough sketch of the proof

Very similar to the proof of "$\langle I(\Gamma), \alpha_C\rangle = \pm a$" in defect 0 case.

- If one of the "resolution" of Γ_i is a cyclic transformation of C, then

$$\langle I(\Gamma_i), p_* FH(\alpha_C)\rangle = \int_{p_* FH(\alpha_C)} \omega_{\Gamma_i}$$

localizes to the following configurations.



- If the resolutions of Γ_j are not equal to C, then $\exists$ a doublepoint of f_C to which no edge comes, and $\langle I(\Gamma_j), FH(\alpha_C)\rangle$ is defined continuously for $h \geq 0$. Thus $\langle I(\Gamma_j), FH(\alpha_C)\rangle \to 0$ ($h \to 0$) by dimensional reason. □

The case $n = 3$

For a graph cocycle Γ^1_k of order k and defect 1, we have an invariant $V(\widetilde{f}) := \langle I(\Gamma^1_k),\ p_* FH_{\widetilde{f}} \rangle$ of framed long knots.

Choose crossings $c_1, \dots, c_k$ of a diagram of $\widetilde{f} \in \widetilde{\mathcal{K}}_3$ that determine a chord diagram C.

$$\Longrightarrow \quad \langle I(\Gamma^1_k),\ p_* FH(\alpha_C) \rangle = \sum_{\epsilon_1, \dots, \epsilon_k \in \{+1,-1\}} \epsilon_1 \cdots \epsilon_k V(\widetilde{f}_{\epsilon_1, \dots, \epsilon_k}) = D^k V(\widetilde{f}).$$

Now we can compute $D^k V(\widetilde{f})$ in a combinatorial way, and in particular $D^k V(\widetilde{f})$ depends only on C. Thus V is of order $\leq k$;

for C with $k + 1$ chords,

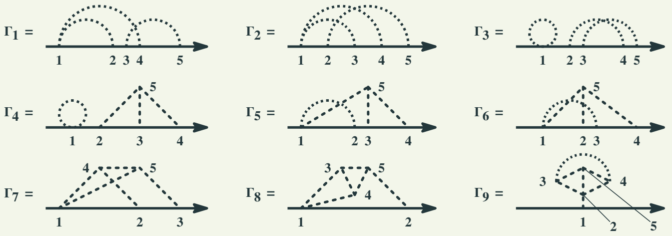
$$D^{k+1} V(\widetilde{f}) = D^k V(\widetilde{f}_{\epsilon_{k+1}=+1}) - D^k V(\widetilde{f}_{\epsilon_{k+1}=-1}) = 0.$$

Non-triviality of V is unknown!

A non-trivial example

Example (S. 2008, 2012)

$n \geq 3$ odd $\implies$ $\exists$ a graph cocycle $\Gamma^1_3 = \sum_{1\leq i\leq 9} a_i\Gamma_i$ of defect 1,



$(a_1, \ldots, a_9) = (-2, 1, 2, -2, 2, -1, 1, -1, 1)$.

Theorem (Kanou-S., 2022)

$v(\widetilde{f}) := \langle I(\Gamma),\ p_* FH_{\widetilde{f}} \rangle = 6v_3(\widetilde{f}) - \mathrm{fr}(\widetilde{f})v_2(f)$.

Since v is of order ≤ 3, this should be a linear combination of v_3, $\mathrm{fr} \cdot v_2$, fr^3, v_2, fr^2 and fr. The coefficients can be determined by explicit computations.

Questions

1. Is $I(\Gamma^1_k) \in H^{k(n-3)+1}_{dR}(\mathcal{K}_n)$ non-zero in general?
   — We might need rather complicated combinatorics.
2. How about cocycles of defect > 1?
   — We need more cycles. The action of (framed) little disks operad might produce them.
3. Are there any cycles other than α_C?
4. How about $\mathbf{Emb}^{(fr)}_{\mathrm{long}}(\mathbb{R}^m, \mathbb{R}^n)$?