低次元球面上の有限群の滑らかな odd-Euler-characteristic action について

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空間の代数的・幾何的モデルとその周辺

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The flow of this talk

G : a finite group. M : a smooth manifold with smooth G-action. (Write $G \curvearrowright M$.)

Definitions

1. A **smooth action** of G on M is a group homomorphism

$$\Psi_M : G \to \mathrm{Diff}(M).$$

2. For a subgroup H of G, let M^H denote the ***H*-fixed-point set** of M, i.e.

$$M^H = \{x \in M \mid \Psi_M(h)(x) = x \text{ for all } h \in H\}.$$

3. $G \curvearrowright M$ is called an **odd-Euler-characteristic action** (resp. a **one-fixed-point action**) if $\chi(M^G) \equiv 1 \bmod 2$ (resp. $|M^G| = 1$).

G: a finite group. S^n : the standard n-sphere.

A_5: the alternating group on five letters. C_n : a cyclic group of order n.

S_5: the symmetric group on five letters.

Conjecture (M. Morimoto 2019 ?)

$\exists$ an effective one-fixed-point G-action on $S^6 \iff G \cong A_5$, $A_5 \times C_2$ or S_5.

Note that if $[G : A_5] = 2$ then G is isomorphic to either $A_5 \times C_2$ or S_5.

M.Morimoto recently has proved the following:

Theorem (M. Morimoto 2022 (1987))

If $G \cong A_5$, $A_5 \times C_2$ or S_5 then there are one-fixed-point G-actions on S^6.

Main result (1)

Σ : a $\mathbb{Z}$-homology 6-sphere (as a closed smooth manifold) with effective G-action.

We call $G \curvearrowright \Sigma$ is **orientation preserving** if the diffeomorphism

$$\Psi_\Sigma(g) : \Sigma \to \Sigma$$

preserves an orientation of Σ for each $g \in G$.

Theorem (T.)

Suppose $\exists$ an **orientation preserving** odd-Euler-characteristic G-action on Σ.

Then $G \cong A_5$ and $|\Sigma^G| = 1$.

Corollary

Suppose $\exists$ a **not orientation preserving** odd-Euler-characteristic G-action on Σ.

Then $G \cong A_5 \times C_2$ or S_5 and $|\Sigma^G| = 1$.

A proof of the corollary

Σ : a $\mathbb{Z}$-homology 6-sphere with effective G-action.

Hypothesis

1. If $\exists$ $G \underset{\mathrm{ori-pre}}{\curvearrowright} \Sigma$ with $\chi(\Sigma^G) \equiv 1 \bmod 2$ then $G \cong A_5$ and $|\Sigma^G| = 1$.
2. If $|G| = p^r$ and X is a finite G-complex then $\chi(X) \equiv \chi(X^G) \bmod p$.

Suppose $\exists$ a **not orientation preserving** odd-Euler-characteristic G-action on Σ.

Then

$$L = \{\, g \in G \mid \Psi_\Sigma(g) : \Sigma \to \Sigma \text{ preserves an orientation of } \Sigma \,\}$$

is a subgroup of G with $[G : L] = 2$. Thus, $G/L \curvearrowright \Sigma^L$ and $\Sigma^G = (\Sigma^L)^{G/L}$.

Since $\chi(\Sigma^G) \equiv \chi(\Sigma^L) \equiv 1 \bmod 2$, it holds that $L \underset{\mathrm{ori-pre}}{\curvearrowright} \Sigma$ and $\chi(\Sigma^L) \equiv 1 \bmod 2$.

By our main theorem, we have $L \cong A_5$ and $|\Sigma^L| = 1$.

Since $[G : A_5] = 2$ and $\Sigma^G \subset \Sigma^L$, $G \cong A_5 \times C_2$ or S_5 and $|\Sigma^G| = 1$. $\square$

Main result (2)

Theorem

Let $n \leq 5$. If S is a **homotopy** n-sphere then $\chi(S^G) \equiv 0 \bmod 2$.

Ξ : a $\mathbb{Z}_2$-homology 5-sphere with effective G-action.

Theorem (T.)

There are no odd-Euler-characteristic G-actions on Ξ, i.e. $\chi(\Xi^G) \equiv 0 \bmod 2$.

Remark

1. $\exists$ an A_5-action on a **homology** 3-sphere Σ with $|\Sigma^{A_5}| = 1$.
2. $\exists$ an A_5-action on a $\mathbb{Z}_p$-homology 5-sphere Ξ with $|\Xi^{A_5}| = 3$. (p: odd prime)

The flow of this talk

1. Motivation and Main theorem
2. Histories of exotic actions on spheres
    2.1 A conjecture posed by Montgomery–Samelson
    2.2 On finite groups having one-fixed-point actions on spheres
    2.3 On dimensions of spheres admitting one-fixed-point actions
3. An idea of our proof of the main theorem

Motivation for one-fixed-point actions on spheres

Conjecture (D. Montgomery–H. Samelson 1946)

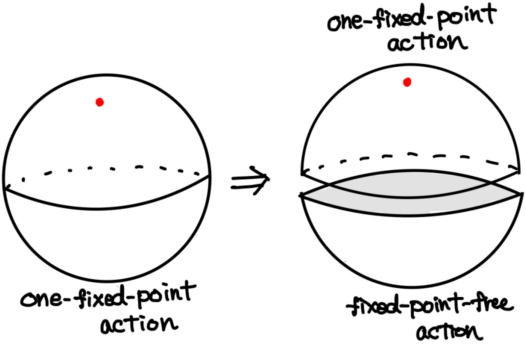
If a compact Lie group G acts smoothly on the n-sphere S^n in such a way as to have one stationary point, it is likely that there must be a second stationary point.

Theorem (E. Stein 1977 [First example])

Let $G = SL(2,5)$ or $SL(2,5) \times C_r$ with $(r,30) = 1$. Then G can act on S^7 with one fixed point.

$SL(2,5)$: the special linear group of order 120 or the nontrivial double covering of A_5.

A G-action on a manifold M is called a **fixed-point-free action** if $M^G = \emptyset$.



p, q : prime numbers.

Let $\mathcal{G}_p^q$ denote the family of finite groups G having a normal sequence

$$P \trianglelefteq H \trianglelefteq G$$

such that P is a p-group, H/P is a cyclic group and G/H is a q-group.

Remark

Let $D = D^n$ and $S = S^n$ (n: arbitrary). Suppose $G \in \mathcal{G}_p^q$.

1. $\chi(D^G) \equiv 1 \bmod q$. In particular, $\chi(D^G) \neq 0$.
2. $\chi(S^G) \equiv 0 \text{ or } 2 \bmod q$. In particular, $\chi(S^G) \neq 1$.

We call a finite group G an **Oliver group** if $G \notin \mathcal{G} = \bigcup_{p,q} \mathcal{G}_p^q$.

Theorem (R. Oliver 1975)

A fintie group G can act on some disk D with $D^G = \emptyset \iff G$ is an Oliver group.

G : a group.

Theorem (T. Petrie 1982)

Suppose that G satisfies at least one of the following.

1. G is an abelian Oliver group of odd order.
2. $G = SL(2, \mathbb{F})$ or $PSL(2, \mathbb{F})$ with $|\mathbb{F}| =$ odd. (Except for $SL(2, 3)$.)
3. $G = S^3$ or $SO(3)$.

Then G can act on some sphere S with exactly one G-fixed point.

Theorem (E. Laitinen–M. Morimoto 1998)

The follwong three conditions are equivalent.

- G is an Oliver group.
- G can act on some disk D with $D^G = \emptyset$.
- G can act on some sphere S with exactly one G-fixed point.

Question

What is the least dimension of a sphere which has a one-fixed-point G-action?

Moreover, which finite group can act on the sphere in such a way?

For a nonnegative integer m, $G \curvearrowright M$ is called an $\boldsymbol{m}$**-pseudofree action** if $\dim M^H \leq m$ for any nontrivial subgroup H of G.

Theorem (E. Laitinen–P. Traczyk 1986)

If S^6 has a 2-pseudofree one-fixed-point G-action then G is isomorphic to A_5.

Theorem (M. Morimoto 1987)

There are 2-pseudofree one-fixed-point A_5-actions on S^6.

More generally, A. Bak–M. Morimoto has proved that, for each $n \geq 6$, S^n possesses one-fixed-point A_5-actions. (Joint with A. Bak in the case that $n = 7,\ 8$)

One-fixed-point actions on S^4

Theorem (M. Furuta 1989)

Let S be a homotopy 4-sphere with orientation presreving G-action. Then $|S^G| \neq 1$.

Theorem (S. Demichelis 1989)

The G-fixed-point set of an orientation preserving G-action on any homology 4-sphere is empty set or a sphere S, where $0 \leq \dim S \leq 2$.

Theorem (M. Morimoto 1988)

Homotopy 4-spheres have no one-fixed-point actions of compact Lie group.

Comment (C. Giffen 1966)

The fixed point set of a finite cyclic group action on S^4 can be a knotted 2-sphere.

Theorem (N. P. Buchdahl–S. Kwasik–R. Schultz 1990 (M. Furuta ?))

Let M be an orientable, closed, connected 3-manifold with G-action.

If $|M^G| = 1$ then $E \neq \pi_1(M) \subset SU(2)$.

Remark

The poincare 3-sphere $M = S^3/SL(2,5)$ has a one-fixed-point A_5-action.

Theorem (N. P. Buchdahl–S. Kwasik–R. Schultz 1990)

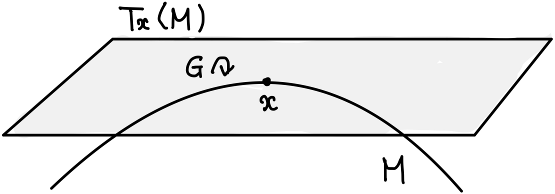
If Ξ is a $\mathbb{Z}$-homology 5-sphere with G-action then $|\Xi^G| \neq 1$.

The flow of this talk

1. Motivation and Main theorem
2. Histories of exotic actions on spheres
3. An idea of our proof of the main theorem
   - 3.1 Tangential representation
   - 3.2 Ideas of the proofs of the nonexistence of one-fixed-point actions on S^4 and S^5
   - 3.3 A relation between the Fitting subgroup $F(G)$ and the parity of $\chi((S^6)^G)$
   - 3.4 Outline of our proof of the main theorem

Tangential G-module and Tangential representation of G

M : an n-dimensional manifold with G-action.



For $x \in M^G$, the tangential space $T_x(M)$ inherits linearly the G-action on M.

We call a real representation $\rho_x : G \to O(n)$ a **tangential representation** at $x \in M^G$ associated with $T_x(M)$.

Tangential representation of G at x

Suppose M is orientable and the G-action on M is effective and orientation preserving.

Then we can get a faithful real representation $\rho_x : G \to SO(n)$.

Thus we may assume that a finite group G acting on M is a finite subgroup of $SO(n)$.

Ideas by S. Demichelis and N. P. Buchdahl–S. Kwasik–R. Schultz

Finite subgroups of $SO(n)$

1. Any finite subgroup of $SO(2)$ is a cyclic group.
2. If $G \subset SO(3)$ then $G \cong C_n,\ D_{2n},\ A_4,\ S_4$ or A_5.
3. The finite groups of $SO(4)$ are classified (up to conjugations).
4. The finite groups of $SO(5)$ are classified (up to conjugations).

The case S^4 by S. Demichelis

If a nontrivial normal p-subgroup of G then $(S^4)^G$ is empty set or a sphere.

The case S^5 by N. P. Buchdahl–S. Kwasik–R. Schultz

If each minimal normal subgroup H of G is not nonabelian simple then $|(S^5)^G| \neq 1$.

Let $F(G)$ denote the **Fitting subgroup** of G,i.e.

the unique maximal nilpotent normal subgroup of G.

Key Lemma (1)

Σ : a $\mathbb{Z}$-homology 6-sphere with **orientation preserving** effective G-action.

If $F(G)$ is nontrivial then $\chi(\Sigma^G)$ is even.

Key Lemma (2)

Ξ : a $\mathbb{Z}_2$-homology 5-sphere with **orientation preserving** effective G-action.

If $F(G)$ is nontrivial then $\chi(\Xi^G)$ is even.

The idea of a proof of Key lemma

$\rho : G \to SO(n)$: a faithful real representation of degree n.

If L is normal in G then ρ is decomposed to the two subrepresentations

$$\rho^L : G \to O(m) \text{ and } \rho_L : G \to O(l),$$

where $l + m = n$.

In the case when $L = F(G)$

Let $H = \ker \rho^{F(G)}$ and $K = \ker \rho_{F(G)}$.

1. $F(G) \subset H \subset SO(l)$.
2. $H \cap K = E$. ① $F(G) \cap F(K) = E \Rightarrow F(K) = E$. ($\because$ $F(G)$ is maximal)
3. $K \subset SO(m)$. ② $K \subset SO(m)$ with $F(K) = E$.
4. $G/K \subset O(l)$. ③ If $K = E$ then $G \subset O(l)$.

Σ : a $\mathbb{Z}$-homology 6-sphere with effective **orientation preserving** G-action.

Hypothesis

1. If $F(G)$ is noncyclic then $\chi(\Sigma^G)$ is even.
2. If $G \subset O(4)$ and $F(G)$ is nontrivial then $\chi(\Sigma^G)$ is even.

Suppose $F(G)$ is nontrivial and cyclic. Then $\deg \rho^{F(G)}$ is equal to either 0, 2 or 4.

① $\deg \rho^{F(G)} = 2$ Then $K = \ker \rho_{F(G)} \subset SO(2)$ with $F(K) = E$ (thus $K = E$).

Therefore, $G \cong G/K \subset O(4)$, and $\chi(\Sigma^G)$ is even.

② $\deg \rho^{F(G)} = 4$ Then $K \subset SO(4)$ with $F(K) = E$ (thus $K = E$ or A_5).

Therefore, $G \cong G/K \subset O(2)$, i.e. $G \cong C_n$ or D_{2n}, or $A_5 \trianglelefteq G$ (later).

Since C_n and D_{2n} belong to $\mathcal{G}_2^2$, $\chi(\Sigma^G) \equiv 0$ or $2 \bmod 2$ (hence $\chi(\Sigma^G) \equiv 0 \bmod 2$).

A proof of our main theorem

Σ^6 : a $\mathbb{Z}$-homology 6-sphere with effective **orientation preserving** G-action.

Proposition

If $\chi(\Sigma^G) \equiv 1 \bmod 2$ then there is a normal subgroup H of G isomorphic to

$$A_5,\ A_6,\ A_7,\ PSL(2,7),\ PSU(4,2) \text{ or } A_5 \times A_5.$$

Proof) This follows from key lemma: if $\chi(\Sigma^G) \equiv 1 \bmod 2$ then $F(G)$ is trivial.

A property on minimal normal subgroups

Each minimal normal subgroup of G is a **characteristically simple group**, i.e.

$$C_p \times \cdots \times C_p \ \text{ or } \ Q \times \cdots \times Q \ \ (Q \ : \text{ nonabelian})$$

Proposition (R. Brauer, H. F. Blichfeldt and J. H. Lindsey)

$H = Q \times \cdots \times Q \subset SO(6)$ is isomorphic to either

$$A_5,\ A_6,\ A_7,\ PSL(2,7),\ PSU(4,2) \text{ or } A_5 \times A_5.$$

Σ : a $\mathbb{Z}$-homology 6-sphere with **orientation preserving** effective G-action.

Theorem (T.)

If $\chi(\Sigma^G) \equiv 1 \mod 2$ then $G \cong A_5$ and $|\Sigma^G| = 1$.

The main theorem follow from the following results.

Proposition

If $\Sigma^G \neq \emptyset$ and G contains a normal subgroup of H isomorphic to

$$A_6,\ A_7,\ PSL(2,7),\ PSU(4,2) \text{ or } A_5 \times A_5$$

then $\Sigma^H \cong S^k$, where $0 \leq k \leq 1$. Moreover, $\Sigma^G = (\Sigma^H)^{G/H} \cong S^l$, where $0 \leq l \leq 1$.

Lemma

If A_5 is normal in G and $\chi(\Sigma^G) \equiv 1 \mod 2$ then $G = A_5$ and $|\Sigma^G| = 1$.

G: a finite group S^n: the n-sphere

Summery

1. Let $n \leq 5$. $|(S^n)^G| \neq 1 \Rightarrow \chi((S^n)^G) \not\equiv 1 \bmod 2$.
2. M. Morimoto's conjecture is true, i.e.

   $\exists\, G \curvearrowright S^6$ with $|(S^6)^G| = 1 \iff G \cong A_5,\ A_5 \times C_2$ or S_5.

   Moreover,

   $\exists\, G \curvearrowright S^6$ with $\chi((S^6)^G) \equiv 1 \bmod 2 \iff G \cong A_5,\ A_5 \times C_2$ or S_5.

Thank you for your attention !