Spaces of non-resultant systems of bounded multiplicity and homotopy stability

Kohhei Yamaguchi

Department of Mathematics, University of Electro-Communications
Chofu, Tokyo 182-8585 Japan

This talk is based on the joint work with
Andrzej Kozlowski (University of Warsaw)

September 16, 2023
Geometric or algebraic models of spaces and their related topics
(空間の幾何学的・代数的モデルとその周辺)
Shinshu University, Matsumoto, Japan

§1. Introduction

Motivation Let X be an m-dimensional toric variety over $\mathbb{C}$, and let Σ denote the fan in $\mathbb{R}^m$ which is associated to X. We write such a toric variety as $X = X_\Sigma$.

Let $\Sigma(1) = \{\rho_k : 1 \leq k \leq r\}$ denote the set of all one dimensional cones in Σ, and let $\mathbf{n}_k \in \mathbb{Z}^m$ be the primitive generator of ρ_k for each $1 \leq k \leq r$.

For each $D = (d_1, \cdots, d_r) \in \mathbb{N}^r$ and a field $\mathbb{F}$, one can define the certain space

$$\mathrm{Poly}_n^{D,\Sigma}(\mathbb{F}) \subset \mathbb{F}[z]^r$$

satisfying the following condition:

$$(1.1) \qquad \mathrm{Poly}_n^{D,\Sigma}(\mathbb{F}) = \mathrm{Hol}_D^*(S^2, X_\Sigma) \quad \text{if } (\mathbb{F}, n) = (\mathbb{C}, 1) \text{ and } \sum_{k=1}^r d_k \mathbf{n}_k = \mathbf{0}_m$$

as long as the primitive generators $\{\mathbf{n}_k\}_{k=1}^r$ satisfiy certain conditions.

The space $\mathrm{Poly}_n^{D,\Sigma}(\mathbb{F})$ is called the space of non-resultant systems of bounded multiplicity n (with coefficients $\mathbb{F}$) determined by a toric variety X_Σ.

We already investigated the homotopy type of the space $\mathrm{Poly}_n^{D,\Sigma}(\mathbb{F})$ for the case $\mathbb{F} = \mathbb{C}$ in the following:

> [KY12] A. Kozlowski and K. Yamaguchi, Spaces of non-resultant systems of bounded multiplicity determined by a toric variety, Topology Appl., (2023)

We would like to investigate the homotopy type of the space $\mathrm{Poly}_n^{D,\Sigma}(\mathbb{F})$ for the case $\mathbb{F} = \mathbb{R}$. Note that

$$(1.2)\qquad \begin{cases} \mathrm{Poly}_1^{D,\Sigma}(\mathbb{C}) = \mathrm{Hol}_D^*(S^2, X_\Sigma) & \text{if } n = 1 \text{ and } \displaystyle\sum_{k=1}^{r} d_k \mathbf{n}_k = \mathbf{0}, \\ \mathrm{Poly}_n^{D,\Sigma}(\mathbb{R}) = (\mathrm{Poly}_n^{D,\Sigma}(\mathbb{C}))^{\mathbb{Z}_2}, & \end{cases}$$

where $D = (d_1, \cdots, d_r) \in \mathbb{N}^r$ and the $\mathbb{Z}_2$-action on $\mathrm{Poly}_n^{D,\Sigma}(\mathbb{C})$ is induced from the complex conjugation on $\mathbb{C}$.

If $X_\Sigma = \mathbb{C}\mathrm{P}^{m-1}$ and $D = (d, \cdots, d) \in \mathbb{N}^m$, then we write

$$(1.3)\qquad \mathrm{Poly}_n^{D,\Sigma}(\mathbb{F}) = \mathrm{Poly}_n^{d,m}(\mathbb{F}).$$

In this talk, we would like to consider the space $\mathrm{Poly}_n^{d,m}(\mathbb{F})$ for $\mathbb{F} = \mathbb{C}$, or $\mathbb{F} = \mathbb{R}$.

1.1 Homology (Homotopy) stability

Definition Let $f : X \to Y$ be a based continuous map.

(i) A map f is called *a homology (resp. homotopy) equivalence through dimension D if*

$$f_* : H_k(X;\mathbb{Z}) \to H_k(Y;\mathbb{Z}) \quad (\text{resp. } f_* : \pi_k(X) \to \pi_k(Y)) \tag{1.4}$$

is an isomorphism for any $k \leq D$.

(ii) A map f is called *a homology (resp. homotopy) equivalence up to dimension D if*

$$f_* : H_k(X;\mathbb{Z}) \to H_k(Y;\mathbb{Z}) \quad (\text{resp. } f_* : \pi_k(X) \to \pi_k(Y)) \tag{1.5}$$

is an isomorphism for any $k < D$ and an epimorphism for $k = D$. □

Definition (i) Let $\mathrm{Map}(X,Y)$ denote the space consisting of all continuous maps $f : X \to Y$ with the compact open topology, and let

(1.6) $$\mathrm{Map}^*(X,Y) \subset \mathrm{Map}(X,Y)$$

be the subspace of all base point preserving maps $f : (X,*) \to (Y,*)$.

(ii) For a based homotopy class $D \in \pi_0(\mathrm{Map}^*(X,Y)) = [X,Y]$, we denote by

(1.7) $$\mathrm{Map}^*_D(X,Y) \subset \mathrm{Map}^*(X,Y)$$

the path component containing the homotopy class D.

(iii) When X and Y are complex manifolds, let

(1.8) $$\mathrm{Hol}^*_D(X,Y) = \{f \in \mathrm{Map}^*_D(X,Y) : f \text{ is holomorphic map}\} \subset \mathrm{Map}^*_D(X,Y).$$

Then we have the natural inclusion

(1.9) $$i_D : \mathrm{Hol}^*_D(X,Y) \xrightarrow{\subset} \mathrm{Map}^*_D(X,Y) \quad \Box$$

Homology (Homotopy) stability

Definition Let $\mathcal{F}=\{f_d : d\in\mathbb{N}\}$ denote the family of based continuous maps

$$(1.10)\qquad X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} X_3 \xrightarrow{f_3} X_4 \xrightarrow{f_4} X_5 \xrightarrow{f_5} X_6 \xrightarrow{f_6} X_6 \xrightarrow{f_7} \cdots$$

Then we say that the family $\mathcal{F}$ satisfies *the homology stability (resp. homotopy stability)* if each map f_d is a homology equivalence (resp. homotopy equivalence) through (resp. up to) dimension m_d such that $\lim_{d\to\infty} m_d=\infty.$

In this situation, let X_∞ denote the colimit (or homotopy colimit) given by

$$(1.11)\qquad X_\infty=\mathrm{colim}_d X_d \qquad \text{(taken from the family of maps } \mathcal{F}=\{f_d\}_{d=1}^{\infty})$$

If the above homology stability (or homotopy stability) holds, we see that the natural map

$$(1.12)\qquad \iota_d : X_d\to X_\infty$$

is a homology equivalence (resp. homotopy equivalence) though (up to) dimension m_d.

In this meaning we also say that the map

$$\iota_d : X_d \to X_\infty \qquad \text{(or space } X_d\text{)}$$

satisfies *the homology stability (resp. the homotopy stability).*

Moreover, if each space X_d is a finite dimensional space, we can say that the space X_d is *a finite dimensional homology (or homotopy) model* of the space X_∞.

Theorem 1.1 (G. Segal, (1979)) *If $m \geq 2$, the inclusion map*

$$i_d : \mathrm{Hol}_d^*(S^2, \mathbb{C}\mathrm{P}^{m-1}) \xrightarrow{\subset} \mathrm{Map}_d^*(S^2, \mathbb{C}\mathrm{P}^{m-1}) = \Omega_d^2 \mathbb{C}\mathrm{P}^{m-1} \simeq \Omega^2 S^{2m-1}$$

is a homotopy equivalence up to dimension $(2m-3)d$. □

Remark We can identify $\mathrm{Hol}_d^*(S^2, \mathbb{C}\mathrm{P}^{m-1})$ with the space Hol_d^* defined by

$$\mathrm{Hol}_d^* = \left\{ (f_1(z), \cdots, f_m(z)) \;\middle|\; \begin{array}{l} \text{Each } f_k(z) \in \mathbb{C}[z] \text{ is a monic polynomial} \\ \text{of degree } d, \text{ and all polynomials} \\ f_1(z), \cdots, f_m(z) \text{ have no common root} \end{array} \right\}.$$

Then the inclusion map $i_d : \mathrm{Hol}_d^* \xrightarrow{\subset} \Omega_d^2 \mathbb{C}\mathrm{P}^{m-1}$ can be given by

(1.13) $$i_d(f_1(z), \cdots, f_m(z))(\alpha) = \begin{cases} [f_1(\alpha) : \cdots : f_m(\alpha)] & \text{if } \alpha \in \mathbb{C} \\ [1 : 1 : \cdots : 1] & \text{if } \alpha = \infty \end{cases}$$

for $(f_1(z), \cdots, f_m(z)) \in \mathrm{Hol}_d^*$ and $\alpha \in \mathbb{C} \cup \infty = S^2 = \mathbb{C}\mathrm{P}^1$.

Proof of Theorem 1.1 Let $\mathrm{Hol}_d^* = \mathrm{Hol}_d^*(S^2, \mathbb{C}\mathrm{P}^{n-1})$.

There is a family of *stabilization maps*

$$(1.14) \qquad \cdots \to \mathrm{Hol}_d^* \xrightarrow{s_d} \mathrm{Hol}_{d+1}^* \xrightarrow{s_{d+1}} \mathrm{Hol}_{d+2}^* \xrightarrow{s_{d+2}} \mathrm{Hol}_{d+3}^* \xrightarrow{s_{d+3}} \mathrm{Hol}_{d+4}^* \xrightarrow{s_{d+4}} \cdots$$

such that each s_d is a homology equivalence up to dimension $(2n-3)d$.

Moreover, there is a homotopy equivalence (this map is called *a scanning map*)

$$(1.15) \qquad S : \mathrm{Hol}_\infty^* = \lim_{d\to\infty} \mathrm{Hol}_d^* \xrightarrow{\simeq} \Omega^2 W_n(\mathbb{C}\mathrm{P}^\infty),$$

where $W_n(X)$ denotes *the n-th fat wedge* of a base space X defined by

$$(1.16) \qquad W_n(X) = \{(x_1, \cdots, x_n) \in X^n : x_i = * \text{ for some } i\}.$$

Furthermore, one can show that there is a fibration sequence

$$\mathbb{C}\mathrm{P}^{n-1} \to W_n(\mathbb{C}\mathrm{P}^\infty) \to (\mathbb{C}\mathrm{P}^\infty)^{n-1} \quad (\therefore \quad \Omega_0^2 \mathbb{C}\mathrm{P}^{n-1} \simeq \Omega^2 W_n(\mathbb{C}\mathrm{P}^\infty))$$

and that $\lim_{d\to\infty} i_d = S$ (up to homotopy equivalence).

Thus, we see that the inclusion map

$$i_d : \mathrm{Hol}_d^* \to \Omega_d^2 \mathbb{C}\mathrm{P}^{n-1} \simeq \Omega^2 S^{2n-1}$$

is a homology equivalence up to dimension $(2n-3)d$.

If $n \geq 3$, Hol_d^* is simply connected and we see that i_d is a homotopy equivalence up to dimension $(2n-3)d$.

If $n = 2$, we can show that the space Hol_d^* is simple up to dimension d and the assertion follows. □

Remark (i) The idea of the above proof can be used for our case $\mathrm{Poly}_n^{d,m}(\mathbb{F})$ when $\mathbb{F} = \mathbb{C}$ or $\mathbb{R}$.

(ii) A space X is called *simple up to dimension d* if $\pi_1(X)$ acts on the homotopy group $\pi_k(X)$ trivially for any $k < d$. □

1.2 Spaces of non-resultant systems of bounded multiplicity

Definition (B.Farb-J.Wofson, (2016))

For $m, n, d \in \mathbb{N}$ with $(m,n) \neq (1,1)$ and a field $\mathbb{F}$ with algebraic closure $\overline{\mathbb{F}}$, let $\mathrm{Poly}_n^{d,m}(\mathbb{F})$ denote the space defined by

$$\mathrm{Poly}_n^{d,m}(\mathbb{F}) = \left\{ (f_1(z), \cdots, f_m(z)) \in \mathbb{F}[z]^m \;\middle|\; \begin{array}{l} \text{Each } f_k(z) \text{ is a monic} \\ \text{polynomial of degree } d, \\ f_1(z), \cdots, f_m(z) \text{ have} \\ \text{no common root in } \overline{\mathbb{F}} \text{ with} \\ \text{multiplicity } \geq n \end{array} \right\}.$$

Thus, there is an incleasing filtration:

$$\mathrm{Poly}_1^{d,m}(\mathbb{F}) \subset \mathrm{Poly}_2^{d,m}(\mathbb{F}) \subset \cdots \subset \mathrm{Poly}_d^{d,m}(\mathbb{F}) \subset \mathrm{Poly}_{d+1}^{d,m}(\mathbb{F}) = \mathrm{P}_d(\mathbb{F})^m,$$

where $\mathrm{P}_d(\mathbb{F}) = \{z^d + \sum_{k=1}^d a_k z^{d-k}; a_k \in \mathbb{F}\} \cong \mathbb{F}^d$.
The space $\mathrm{Poly}_{d+1}^{d,m}(\mathbb{F})$ is called *the space of non-resultant systems of bounded multiplicity with coefficients* $\mathbb{F}$. □

Example If $(n, \mathbb{F}) = (1, \mathbb{C})$, $\mathrm{Poly}_n^{d,m}(\mathbb{F}) = \mathrm{Hol}_d^*(S^2, \mathbb{C}\mathrm{P}^{m-1})$.

Example Let $\mathbb{F} = \mathbb{R}$, $(d, m) = (8, 2)$, and let $f(z)$ and $g(z)$ denote the monic polynomials of degree 8 given by $(f(z), g(z)) = ((z^2+1)^3 z^2, (z^2+1)^4)$.

(i) $\alpha = (f(z), g(z)) \notin \mathrm{Poly}_3^{8,2}(\mathbb{R})$ if $n = 3$.

(ii) $\alpha = (f(z), g(z)) \in \mathrm{Poly}_4^{8,2}(\mathbb{R})$ if $n = 4$.

Proof The assertions (i) and (ii) follows from the fact that $\{z = \pm\sqrt{-1}\}$ is a common root of $\{f(z), g(z)\}$ of multiplicity 3. □

Remark B. Farb and J. Wofson considered the space $\mathrm{Poly}_n^{d,m}(\mathbb{F})$ for investigating *the homological density of algebraic cycles* in a manifold. □

Definition (i) For a connected space X, let $F(X,d)$ denote

the ordered configuration space of distinct d points of X defined by

(1.17) $$F(X,d)=\{(x_1,\cdots,x_d)\in X^d : x_i\neq x_k \text{ if } i\neq k\}.$$

(ii) The symmetric group S_d of d letters acts on $F(X,d)$ by the permutation of coordinates freely. We denote by $C_d(X)$ the orbit space

(1.18) $$C_d(X)=F(X,d)/S_d.$$

The space $C_d(X)$ is called *the unordered configuration space* of distinct d points of X.

(iii) Let $D_d(X)$ denotes the equvariant half smash product given by

(1.19) $$D_d(X)=F(\mathbb{C},d)_+\wedge_{S_d} X^{\wedge d},$$

where $$\begin{cases} F(X,d)_+ & = F(X,d)\cup\{*\} \quad \text{(disjoint union)}, \\ X^{\wedge d} & = X\wedge X\wedge\cdots\wedge X \quad (d\text{-times}). \end{cases}$$

Theorem 1.2 (I.James, V.Snaith, F.Cohen-M.Mahowald-R.Milgram)

There are stable homotopy equivalences

(1.20)
$$\begin{cases} \Omega S^{N+1} & \simeq_s \bigvee_{k=1}^{\infty} S^{kN} = S^N \vee S^{2N} \vee S^{3N} \vee \cdots \\ \Omega^2 S^{2N+1} & \simeq_s \bigvee_{k=1}^{\infty} \Sigma^{2(N-1)k} D_k, \end{cases}$$

where Σ^k *denotes the* *k-fold reduced suspension*, *and* let

$$D_k = D_k(S^1) = F(\mathbb{C}, k)_+ \wedge_{S_k} (S^1)^{\wedge k}. \qquad \square$$

Stable version of Theorem 1.1

Theorem 1.3 (F.Cohen-R.Cohen-B.Mann-R.Milgram, (1991))

If $m \geq 2$, there is a stable homotopy equivalence

$$\mathrm{Hol}_d^*(S^2, \mathbb{C}\mathrm{P}^{m-1}) \simeq_s \bigvee_{k=1}^{d} \Sigma^{2(m-2)k} D_k. \tag{1.21}$$

Moreover, the following diagram is commutative up to stable homotopy equivalence

$$\begin{array}{ccc} \mathrm{Hol}_d^*(S^2, \mathbb{C}\mathrm{P}^{m-1}) & \xrightarrow[\subset]{i_d} & \Omega_d^2 \mathbb{C}\mathrm{P}^{m-1} \simeq \Omega^2 S^{2m-1} \\ \downarrow \simeq_s & & \downarrow \simeq_s \\ \bigvee_{k=1}^{d} \Sigma^{2(m-2)k} D_k & \xrightarrow[\subset]{} & \bigvee_{k=1}^{\infty} \Sigma^{2(m-2)k} D_k \end{array} \tag{1.22}$$

Here, the map i_d is a homotopy equivalence up to dimension $(2m-3)d$. □

Definition For each polynomial $f(z) \in \mathbb{C}[z]$, define the n-tuple

$$F_n(f) = F_n(f)(z) \in \mathbb{C}[z]^n \tag{1.23}$$

of polynomials by

$$F_n(f)(z) = \big(f(z), f(z) + f'(z), f(z) + f''(z), \cdots, f(z) + f^{(n-1)}(z)\big) \in \mathbb{C}[z]^n.$$

If $f(z) \in \mathbb{C}[z]$ is a monic polynomial of degree d, $F_n(f)$ is the n-tuples of monic polynomials of the same degree d. □

Remark Let $f(z) \in \mathbb{C}[z]$ of $\deg f \geq n$, and let $\alpha \in \mathbb{C}$. Then

$$\begin{aligned} & f(z) \text{ is can be divided by } (z-\alpha)^n \\ \Leftrightarrow\ & f(\alpha) = f'(\alpha) = f''(\alpha) = \cdots = f^{(n-1)}(\alpha) = 0 \\ \Leftrightarrow\ & F_n(f)(\alpha) = (0, 0, \cdots, 0) = \mathbf{0}_n \in \mathbb{C}^n \end{aligned}$$ □

Definition Define *the natural map*

$$(1.24) \qquad i^{d,m}_{n,\mathbb{C}} : \mathrm{Poly}^{d,m}_n(\mathbb{C}) \to \Omega^2_d \mathbb{C}\mathrm{P}^{mn-1} \simeq \Omega^2 S^{2mn-1} \quad \text{by}$$

$$i^{d,m}_{n,\mathbb{C}}(f_1,\cdots,f_m)(\alpha) = \begin{cases} [F_n(f_1)(\alpha) : F_n(f_2)(\alpha) : \cdots : f_n(f_m)(\alpha)] & \text{if } \alpha \in \mathbb{C} \\ [1:1:\cdots:1] & \text{if } \alpha = \infty \end{cases}$$

for $(f_1,\cdots,f_m) \in \mathrm{Poly}^{d,m}_n(\mathbb{C})$, where we identify $S^2 = \mathbb{C} \cup \infty$. □

1.3 The space $\mathrm{Poly}_n^{d,m}(\mathbb{C})$

Theorem 1.4; G. Segal (The case $mn = 2 \Leftrightarrow (m,n) = (1,2)$ or $(2,1)$)

(i) (**G. Segal, (1976);** $(m,n) = (1,2)$)

The natural map

(1.25) $$i_{2,\mathbb{C}}^{d,1} : C_d(\mathbb{C}) \cong \mathrm{Poly}_2^{d,1}(\mathbb{C}) \to \Omega_d^2 S^2$$

is a homology equivalence up to dimension $\lfloor d/2 \rfloor$, *where* $\lfloor x \rfloor$ *denotes the integer part of a real number* x.

(ii) (**G. Segal, (1979);** $(m,n) = (2,1)$) **(The special case of Theorem 1.1)**

The natural map

(1.26) $$i_{1,\mathbb{C}}^{d,2} : \mathrm{Poly}_2^{d,1}(\mathbb{C}) = \mathrm{Hol}_d^*(S^2, \mathbb{C}\mathrm{P}^1) \to \Omega_d^2 \mathbb{C}\mathrm{P}^1 \simeq \Omega^2 S^3$$

is a homotopy equivalence up to dimension d. □

Remark (i) The natural map $i^{d,1}_{2,\mathbb{C}}: C_d(\mathbb{C}) \cong \mathrm{Poly}^{d,1}_2(\mathbb{C}) \to \Omega^2_d S^2$ can be identified with the map $i^{d,1}_{2,\mathbb{C}}(f) : (\mathbb{C}\cup\infty,\infty) \to (\mathbb{C}\cup\infty, 1)$ given by

$$i^{d,1}_{2,\mathbb{C}}(f)(\alpha) = \begin{cases} \dfrac{f(\alpha)+f'(\alpha)}{f(\alpha)} = 1 + \displaystyle\sum_{k=1}^{d} \dfrac{1}{\alpha - a_k} & \text{if } \alpha \notin \{a_1, \cdots, a_d\} \\ \infty & \text{if } \alpha = a_k \end{cases}$$

for $\alpha \in \mathbb{C}\cup\infty = S^2$ and $f = f(z) = \prod_{k=1}^{d}(z-a_k) \in \mathrm{Poly}^{d,1}_2(\mathbb{C})$.

Proof The above identification follows from the following homeomorphism:

$$\begin{array}{ccc} \mathbb{C}\mathrm{P}^1 & \xrightarrow[\cong]{} & S^2 = \mathbb{C}\cup\infty \\ i^{d,1}_{2,\mathbb{C}}(f)(\alpha) = [f(\alpha) : f(\alpha)+f'(\alpha)] & \longrightarrow & \frac{f(\alpha)+f'(\alpha)}{f(\alpha)} \end{array} \qquad \square$$

(ii) Note that there is a homotopy equivalence $C_d(\mathbb{C}) \simeq K(\beta_d, 1)$, where β_d denotes the Artin's braid group of d-strings.

Theorem 1.5 (V. Vassiliev, (1992))

There is a stable homotopy equivalence

$$C_d(\mathbb{C}) \cong \mathrm{Poly}_2^{d,1}(\mathbb{C}) \simeq_s \mathrm{Hol}^*_{\lfloor d/2 \rfloor}(S^2, \mathbb{C}\mathrm{P}^1), \tag{1.27}$$

where $\lfloor x \rfloor$ *denotes the integer part of a real number* x. □

Theorem 1.6 (Guest-Kozlowski-Yamaguchi, (1998))

If $n \geq 3$, *there is a homotopy equivalence*

$$\mathrm{Poly}_n^{d,1}(\mathbb{C}) = \mathrm{SP}_n^d(\mathbb{C}) \simeq \mathrm{Hol}^*_{\lfloor d/n \rfloor}(S^2, \mathbb{C}\mathrm{P}^{n-1}). \tag{1.28}$$

Here, let $\mathrm{SP}_n^d(\mathbb{C})$ *denote the space of all monic polynomials* $f(z) \in \mathbb{C}[z]$ *of degree* d *without roots of multiplicity* $\geq n$. □

Remark There is a homeomorphism $\mathrm{SP}_2^d(\mathbb{C}) \cong C_d(\mathbb{C})$.

Theorem 1.7 (Kozlowski-Yamaguchi (2017); The case $mn \geq 3$)

(i) *If $mn \geq 3$, the natural map*

(1.29) $$i^{d,m}_{n,\mathbb{C}} : \mathrm{Poly}^{d,m}_n(\mathbb{C}) \to \Omega^2_d \mathbb{C}\mathrm{P}^{mn-1} \simeq \Omega^2 S^{2mn-1}$$

is a homotopy equivalence through dimension $D_{\mathbb{C}}(d; m, n)$, where

(1.30) $$D_{\mathbb{C}}(d; m, n) = (2mn-3)(\lfloor d/n \rfloor + 1) - 1.$$

(ii) *If $mn \geq 3$, there is a homotopy equivalence*

(1.31) $$\mathrm{Poly}^{d,m}_n(\mathbb{C}) \simeq \mathrm{Hol}^*_{\lfloor \frac{d}{n} \rfloor}(S^2, \mathbb{C}\mathrm{P}^{mn-1}) = \mathrm{Poly}^{\lfloor \frac{d}{n} \rfloor, mn}_1(\mathbb{C})$$

(iii) *If $mn \geq 3$, there is a stable homotopy equivalence*

(1.32) $$\mathrm{Poly}^{d,m}_n(\mathbb{C}) \simeq_s \bigvee_{k=1}^{\lfloor \frac{d}{n} \rfloor} \Sigma^{2(mn-2)k} D_k, \text{ where } D_k = D_k(S^1).$$ □

§2. The space $\mathrm{Poly}_n^{d,m}(\mathbb{R})$

From now on, we shall consider the homotopy type of the space $\mathrm{Poly}_n^{d,m}(\mathbb{F})$ for the case $\mathbb{F} = \mathbb{R}$.

References

[KY13] A. Kozlowski and K. Yamaguchi, Spaces of non-resultant systems of bounded multiplicity with real coefficients, (arXiv:2212.05494), preprint.

[KY14] A. Kozlowski and K. Yamaguchi, Homotopy stability of spaces of non-resultant systems of bounded multiplicity with real coefficients, (arXiv:2305.00307), preprint.

2.1 The $\mathbb{Z}_2$-actions induced from the complex conjugation

Let $\mathbb{Z}_2 = \{\pm 1\}$ denote the multiplicative cyclic group of order 2.

Then the complex conjugation on $\mathbb{C}$ naturally extends to the $\mathbb{Z}_2$-actions on

$$S^2 = \mathbb{C} \cup \infty,\ \mathbb{C}\mathrm{P}^N,\ \Omega^2_d\mathbb{C}\mathrm{P}^N,\ \mathrm{Poly}^{d,m}_n(\mathbb{C}), \quad \text{etc.}$$

For example, the space $\Omega^2_d\mathbb{C}\mathrm{P}^N$ has the natural $\mathbb{Z}_2$-action given by $\overline{f}(\alpha) = f(\overline{\alpha})$.

Definition Let G be a group and let X be a G-space.

For each subgroup $H \subset G$, let X^H denote *the H-fixed point set* of X given by

(2.1) $$X^H = \{x \in X : h \cdot x = x \text{ for any } h \in H\}.$$

Example

$$(S^2)^{\mathbb{Z}_2} = S^1 = \mathbb{R} \cup \infty,\ (\mathbb{C}\mathrm{P}^N)^{\mathbb{Z}_2} = \mathbb{R}\mathrm{P}^N,\ (\mathrm{Poly}^{d,m}_n(\mathbb{C}))^{\mathbb{Z}_2} = \mathrm{Poly}^{d,m}_n(\mathbb{R}).$$

Lemma 2.1 *There is a homotopy equivalence*

$$(\Omega^2_d\mathbb{C}\mathrm{P}^N)^{\mathbb{Z}_2} \simeq \Omega^2 S^{2N+1} \times \Omega S^N. \tag{2.2}$$ □

Definition Let G be group, and X and Y be G-spaces.

(i) Let $f : X \to Y$ be a based G-map. Then the map f is called *a G-equivariant homotopy (resp. homology) equivalence through dimension D* if

$$f^H = f|X^H : X^H \to Y^H \tag{2.3}$$

is a homotopy (resp. homology) equivalence through dimension D for any subgroup $H \subset G$.

(ii) Similarly, the map f is *a G-equivariant homotopy (resp. homology) equivalence up to dimension D* if

$$f^H : X^H \to Y^H \tag{2.4}$$

is a homotopy (resp. homology) equivalence up to dimension D for any subgroup $H \subset G$. □

2.2 The case $mn = 2$ for $\mathrm{Poly}_n^{d,m}(\mathbb{R})$

Note that

(2.5) $$mn = 2 \ \Leftrightarrow \ (m,n) = (2,1) \text{ or } (m,n) = (1,2).$$

First, consider the case $(m,n) = (2,1)$.

Definition (the case $(m,n) = (2,1)$)

Let $S^2 = \mathbb{C} \cup \infty$ and let $(\Omega_d^2 \mathbb{C}\mathrm{P}^1)_j^{\mathbb{Z}_2} \subset (\Omega_d^2 \mathbb{C}\mathrm{P}^1)^{\mathbb{Z}_2}$ denote the subspace given by

(2.6) $$(\Omega_d^2 \mathbb{C}\mathrm{P}^1)_j^{\mathbb{Z}_2} = \{ f \in (\Omega_d^2 \mathbb{C}\mathrm{P}^1)^{\mathbb{Z}_2} : \deg(f|S^1) = j \}.$$

Theorem 2.2 (G. Segal (1979); the case $(m,n)=(2,1)$)

(i) *The space* $\mathrm{Poly}_1^{d,2}(\mathbb{R})$ *consists of* $(d+1)$ *connected components*

(2.7) $$\{\mathrm{Poly}_{1,j}^{d,2}(\mathbb{R}) : j = d-2k, 0 \leq k \leq d\}.$$

(ii) *If* $j = d-2k$ *and* $0 \leq k \leq d$*, the natural inclusion map*

(2.8) $$i_{1,j}^{d,2} : \mathrm{Poly}_{1,j}^{d,2}(\mathbb{R}) \xrightarrow{\subset} (\Omega_d^2\mathbb{C}\mathrm{P}^1)_j^{\mathbb{Z}_2} \simeq \Omega_d^2\mathbb{C}\mathrm{P}^1 \simeq \Omega^2 S^3$$

is a homotopy equivalence up to dimension $\frac{1}{2}(d-|j|)$*.*

Here, the map $i_{1,j}^{d,2}$ *is defined by*

(2.9) $$i_{1,j}^{d,2}(f(z),g(z))(\alpha) = \begin{cases} [f(\alpha):g(\alpha)] & \text{if } \alpha \in \mathbb{C} \\ [1:1] & \text{if } \alpha = \infty \end{cases}$$

for $(f(z),g(z)) \in \mathrm{Poly}_{1,j}^{d,2}(\mathbb{R})$ *and* $\alpha \in S^2 = \mathbb{C} \cup \infty$*.* □

Definition (the case $(m,n)=(1,2)$) Let $f(z)\in \mathrm{Poly}^{d,1}_2(\mathbb{R})$.

If $\alpha\in\mathbb{C}\setminus\mathbb{R}$ is a root of $f(z)$, then $\overline{\alpha}$ is also a root of $f(z)$. Thus, we can write

$$(2.10)\qquad f(z)=\Big(\prod_{i=1}^{d-2j}(z-x_i)\Big)\Big(\prod_{k=1}^{j}(z-\alpha_k)(z-\overline{\alpha}_k)\Big)$$

for some $(\{x_i\}_{i=1}^{d-2j},\{\alpha_k\}_{k=1}^{j})\in \mathbb{C}_{d-2j}(\mathbb{R})\times C_j(\mathrm{H}_+)$, where

$$\mathrm{H}_+=\{\alpha\in\mathbb{C}:\mathrm{Im}\ \alpha>0\}.$$

(i) Define the subspace $\mathrm{Poly}^{d,1}_{2,j}(\mathbb{R})\subset \mathrm{Poly}^{d,1}_2(\mathbb{R})$ by

$$\begin{aligned}\mathrm{Poly}^{d,1}_{2,j}(\mathbb{R})&=\left\{ f(z)\in \mathrm{Poly}^{d,1}_2(\mathbb{R})\ \middle|\ \begin{array}{l} f(z)\text{ is represented as the form of}\\ (2.10)\text{ for some } (\{x_i\}_{i=1}^{d-2j},\{\alpha_k\}_{k=1}^{j})\\ \text{in } C_{d-2j}(\mathbb{R})\times C_j(\mathrm{H}_+)\end{array}\right\}\\ &\cong C_{d-2j}(\mathbb{R})\times C_j(\mathrm{H}_+)\\ &\cong \mathbb{R}^{d-2j}\times C_j(\mathbb{C})\simeq C_j(\mathbb{C})\end{aligned}$$

(ii) Define the natural map

(2.11) $$i^{d,1}_{2,j} : \mathrm{Poly}^{d,1}_{2,j}(\mathbb{R}) \to \Omega^2_j \mathbb{C}\mathrm{P}^1 \simeq \Omega^2_j S^2 \simeq \Omega^2 S^3$$ by

(2.12) $$i^{d,1}_{2,j}(f(z))(\alpha) = \begin{cases} [f(\alpha) : f(\alpha) + f'(\alpha)] & \text{if } \alpha \in \mathbb{C} \\ [1 : 1] & \text{if } \alpha = \infty \end{cases}$$

for $f(z) \in \mathrm{Poly}^{d,1}_{2,j}(\mathbb{R})$ and $\alpha \in S^2 = \mathbb{C} \cup \infty$. □

Proposition I (the case $(m,n)=(1,2)$)

(i) *The space* $\mathrm{Poly}_2^{d,1}(\mathbb{R})$ *consists of* $(\lfloor d/2 \rfloor + 1)$ *connected components*

(2.13) $$\{\mathrm{Poly}_{2,j}^{d,1}(\mathbb{R}) : 0 \leq j \leq \lfloor d/2 \rfloor\}.$$

(ii) *There is a natural map*

(2.14) $$i_{2,j}^{d,1} : \mathrm{Poly}_{2,j}^{d,1}(\mathbb{R}) \to \Omega_j^2 \mathbb{C}\mathrm{P}^1 \simeq \Omega_j^2 S^2 \simeq \Omega^2 S^3$$

which is a homology equivalence up to dimension $\lfloor j/2 \rfloor$ *if* $3 \leq j \leq \lfloor d/2 \rfloor$*, and it is a homotopy equivalence through dimension* 1 *if* $j=2$.

(iii) *If* $0 \leq j \leq \lfloor d/2 \rfloor$*, there is a homotopy equivalence*

(2.15) $$\mathrm{Poly}_{2,j}^{d,1}(\mathbb{R}) \simeq K(\beta_j, 1),$$

where β_j *denotes the Artin braid group on* j*-strings.*

In particular, the space $\mathrm{Poly}_{2,j}^{d,1}(\mathbb{R})$ *is contractible if* $j \in \{0,1\}$ *and it is homotopy equivalent to the space* S^1 *if* $j=2$. □

2.3 The case $mn \geq 3$ for $\mathrm{Poly}_n^{d,m}(\mathbb{R})$

Definition Let $mn \geq 3$ and consider the restriction of *the natural map*

$$i_{n,\mathbb{C}}^{d,m}|\mathrm{Poly}_n^{d,m}(\mathbb{R}) : \mathrm{Poly}_n^{d,m}(\mathbb{R}) \to \Omega_d^2\mathbb{C}\mathrm{P}^{mn-1} \quad \text{given by}$$

$$i_{n,\mathbb{C}}^{d,m}(f_1,\cdots,f_m)(\alpha) = \begin{cases} [F_n(f_1)(\alpha) : F_n(f_2)(\alpha) : \cdots : f_n(f_m)(\alpha)] & \text{if } \alpha \in \mathbb{C} \\ [1:1:\cdots:1] & \text{if } \alpha = \infty \end{cases}$$

for $(f_1,\cdots,f_m) \in \mathrm{Poly}_n^{d,m}(\mathbb{R})$, where we identify $S^2 = \mathbb{C} \cup \infty$.

Since $i_{n,\mathbb{C}}^{d,m}(\mathrm{Poly}_n^{d,m}(\mathbb{R})) \subset (\Omega_d^2\mathbb{C}\mathrm{P}^{mn-1})^{\mathbb{Z}_2}$, the restriction map

$$i_{n,\mathbb{R}}^{d,m} = i_{n,\mathbb{C}}^{d,m}|\mathrm{Poly}_n^{d,m}(\mathbb{R})$$

gives *the natural map*

$$\text{(2.16)} \qquad \boxed{i_{n,\mathbb{R}}^{d,m} : \mathrm{Poly}_n^{d,m}(\mathbb{R}) \to (\Omega_d^2\mathbb{C}\mathrm{P}^{mn-1})^{\mathbb{Z}_2} \simeq \Omega^2 S^{2mn-1} \times \Omega S^{mn-1}}$$

Definition Let $mn \geq 3$, and $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

Let

(2.17)
$$s^{d,m}_{n,\mathbb{K}} : \mathrm{Poly}^{d,m}_n(\mathbb{K}) \to \mathrm{Poly}^{d+1,m}_n(\mathbb{K})$$

denote *the stabilization map* given by adding roots from the infinity.

One can define the map $s^{d,m}_{n,\mathbb{K}}$ satisfying the following condition:

(2.18)
$$s^{d,m}_{n,\mathbb{R}} = (s^{d,m}_{n,\mathbb{C}})^{\mathbb{Z}_2} = s^{d,m}_{n,\mathbb{C}} | \mathrm{Poly}^{d,m}_n(\mathbb{R}).$$

Thus, we have the following two stabilization maps:

(2.19)
$$\begin{cases} s^{d,m}_{n,\mathbb{R}} : \mathrm{Poly}^{d,m}_n(\mathbb{R}) \to \mathrm{Poly}^{d+1,m}_n(\mathbb{R}) \\ s^{d,m}_{n,\mathbb{C}} : \mathrm{Poly}^{d,m}_n(\mathbb{C}) \to \mathrm{Poly}^{d+1,m}_n(\mathbb{C}) \end{cases}$$

such that

$$s^{d,m}_{n,\mathbb{R}} = (s^{d,m}_{n,\mathbb{C}})^{\mathbb{Z}_2} = s^{d,m}_{n,\mathbb{C}} | \mathrm{Poly}^{d,m}_n(\mathbb{R}).$$

The main results

Theorem I (The case $mn \geq 3$) (i) *The natural map*

(2.20) $$i_{n,\mathbb{R}}^{d,m} : \mathrm{Poly}_n^{d,m}(\mathbb{R}) \to (\Omega_d^2 \mathbb{C}\mathrm{P}^{mn-1})^{\mathbb{Z}_2} \simeq \Omega^2 S^{2mn-1} \times \Omega S^{mn-1}$$

is a homotopy equivalence through dimension $D(d;m,n)$ if $mn \geq 4$, and it is a homology equivalence through dimension $D(d;m,n) = \lfloor d/n \rfloor$ if $mn = 3$. Here, the positive integer $D(d;m,n)$ is given by

(2.21) $$D(d;m,n) = (mn-2)(\lfloor d/n \rfloor + 1) - 1.$$

(ii) *If $mn \geq 3$, there is a stable homotopy equivalence*

$$\mathrm{Poly}_n^{d,m}(\mathbb{R}) \simeq_s \Big(\bigvee_{i=1}^{\lfloor d/n \rfloor} S^{(mn-2)i} \Big) \vee \Big(\bigvee_{i \geq 0, j \geq 1, i+2j \leq \lfloor d/n \rfloor} \Sigma^{(mn-2)(i+2j)} D_j \Big),$$

where D_j denotes the space $D_j = D_j(S^1) = F(\mathbb{C}, j)_+ \wedge_{S_j} (S^1)^{\wedge j}$. □

By (ii) of Theorem I, we also easily obtain the following result:

Example I *If $mn \geq 3$ and $k \geq 2$, there is a stable homotopy equivalence*

$$\mathrm{Poly}_n^{kn,m}(\mathbb{R}) \simeq_s \mathrm{Poly}_n^{(k-1)n,m}(\mathbb{R}) \vee \Big(\bigvee_{j=0}^{\lfloor \frac{k}{2} \rfloor} \Sigma^{k(mn-2)} D_j\Big),$$

where we set $(D_0 := S^0)$. □

Definition

(i) The space X is called *simple up to dimension D* if the group $\pi_1(X)$ acts trivially on the group $\pi_i(X)$ for any $i < D$.

(ii) In particular, the space X is called *simple* if the group $\pi_1(X)$ acts trivially on the group $\pi_i(X)$ for any $i \geq 1$. □

Theorem II (The case $(m,n)=(3,1)$) *Let $(m,n)=(3,1)$. Then the space $\mathrm{Poly}_1^{d,3}(\mathbb{R})$ is simple if $d \equiv 1 \pmod 2$ and it is simple up to dimension d if $d \equiv 0 \pmod 2$.*

Theorem III (The case $(m,n)=(3,1)$) *The natural map*

$$i^{d,3}_{1,\mathbb{R}} : \mathrm{Poly}_1^{d,3}(\mathbb{R}) \to (\Omega^2_d \mathbb{C}\mathrm{P}^2)^{\mathbb{Z}_2} \simeq \Omega^2 S^5 \times \Omega S^2 \simeq \Omega^2 S^5 \times \Omega S^3 \times S^1 \tag{2.22}$$

is a homotopy equivalence through dimension d if $d \equiv 1 \pmod 2$, and it is a homotopy equivalence up to dimension d if $d \equiv 0 \pmod 2$. □

Remark $mn = 3 \Leftrightarrow (m,n) = (3,1)$ or $(1,3)$

$$D(d;m,n) = \begin{cases} d & \text{if } (m,n) = (3,1) \\ \lfloor d/3 \rfloor & \text{if } (m,n) = (1,3) \end{cases}$$

By using (i) of Theorem I of the case $(m,n) = (1,3)$, we have:

Corollary I* (The case $(m,n) = (1,3)$) *If $(m,n) = (1,3)$, the natural map*

$$i^{d,1}_{3,\mathbb{R}} : \mathrm{Poly}^{d,1}_3(\mathbb{R}) \to (\Omega^2_d \mathbb{C}\mathrm{P}^2)^{\mathbb{Z}_2} \simeq \Omega^2 S^5 \times \Omega S^2 \simeq \Omega^2 S^5 \times \Omega S^3 \times S^1$$

is a homology equivalence through dimension $\lfloor d/3 \rfloor$. □

By using (ii) of Theorem I, we easily obtain the following:

Corollary I** *If $mn \geq 3$, there is a stable homotopy equivalence*

(2.23) $$\mathrm{Poly}_n^{d,m}(\mathbb{R}) \simeq_s \mathrm{Poly}_1^{\lfloor d/n \rfloor, mn}(\mathbb{R}). \qquad \square$$

Remark Note that there is a homotopy equivalence

(2.24) $$\mathrm{Poly}_n^{d,m}(\mathbb{C}) \simeq \mathrm{Poly}_1^{\lfloor d/n \rfloor, mn}(\mathbb{C}) \quad \text{if } mn \geq 3.$$

Example I** (i) *If $mn = 3$, there is a stable homotopy equivalence*

(2.25) $$\mathrm{Poly}_n^{d,m}(\mathbb{R}) \simeq_s \mathrm{Poly}_n^{d_1,m}(\mathbb{R}) \quad \textit{if and only if } \lfloor d/n \rfloor = \lfloor d_1/n \rfloor.$$

(ii) *If $mn \geq 4$, there is a homotopy equivalence*

(2.26) $$\mathrm{Poly}_n^{d,m}(\mathbb{R}) \simeq \mathrm{Poly}_n^{d_1,m}(\mathbb{R}) \quad \textit{if and only if } \lfloor d/n \rfloor = \lfloor d_1/n \rfloor. \qquad \square$$

Corollary IV

(i) *If* $mn \geq 4$, *the natural map*

$$i_{n,\mathbb{C}}^{d,m} : \mathrm{Poly}_n^{d,m}(\mathbb{C}) \to \Omega_d^2 \mathbb{C}\mathrm{P}^{mn-1} \simeq \Omega^2 S^{2mn-1}$$

is a $\mathbb{Z}_2$*-equivariant homotopy equivalence through dimension* $D(d;m,n)$.

(ii) *Let* $(m,n)=(3,1)$. *Then the natural map*

$$i_{1,\mathbb{C}}^{d,3} : \mathrm{Poly}_1^{d,3}(\mathbb{C}) \to \Omega_d^2 \mathbb{C}\mathrm{P}^2 \simeq \Omega^2 S^5$$

is a $\mathbb{Z}_2$*-equivariant homotopy equivalence through dimension* d *if* $d \equiv 1 \pmod 2$, *and it is a* $\mathbb{Z}_2$*-equivariant homotopy equivalence up to dimension* d *if* $d \equiv 0 \pmod 2$.

(iii) *If* $(m,n)=(1,3)$, *the natural map*

$$i_{n,\mathbb{C}}^{d,m} : \mathrm{Poly}_3^{d,1}(\mathbb{C}) \to \Omega_d^2 \mathbb{C}\mathrm{P}^2 \simeq \Omega^2 S^5$$

is a $\mathbb{Z}_2$*-equivariant homology equivalence through dimension* $\lfloor d/3 \rfloor$. □

The key points of the proof of Theorem I are the following two points:

1. Scanning maps (defined by G. Segal).
2. Vassiliev spectral sequence.

For example, by using the Vassiliev spectral sequence one can prove the following Theorem V.

Theorem V (The homology stability)

Let $mn \geq 3$. *Then the stabilization map*

$$s^{d,m}_{n,\mathbb{R}} : \mathrm{Poly}^{d,m}_n(\mathbb{R}) \to \mathrm{Poly}^{d+1,m}_n(\mathbb{R})$$

is a homology equivalence if $\lfloor d/n \rfloor = \lfloor (d+1)/n \rfloor$, *and it is a homology equivalence through dimension* $D(d;m,n)$ *otherwise, where*

$$D(d;m,n) = (mn-2)(\lfloor d/n \rfloor + 1) - 1.$$

Sketch proof of Theorem V Consider the Vassiliev spectral sequence

$$\text{(2.27)} \qquad \left\{E^{t;d}_{k,s},\ d^t : E^{t;d}_{k,s} \to E^{t;d}_{k+t,s+t-1}\right\} \Rightarrow \tilde{H}_{s-k}(\mathrm{Poly}^{d,m}_n(\mathbb{R});\mathbb{Z}).$$

One can prove that there is a natural isomorphism

$$\text{(2.28)} \qquad E^{1;d}_{k,s} \cong \Big(\bigoplus_{j=1}^{k} \tilde{H}_{s-(mn-1)k}(\Sigma^{(mn-2)j} D_j;\mathbb{Z})\Big) \oplus \tilde{H}_{s-(mn-1)k}(S^0;\mathbb{Z})$$

if $1 \leq k \leq \lfloor d/n \rfloor$.

The stabilization map $s^{d,m}_{n,\mathbb{R}} : \mathrm{Poly}^{d,m}_n(\mathbb{R}) \to \mathrm{Poly}^{d+1,m}_n(\mathbb{R})$ naturally induces the homomorphism of spectral sequences

$$\{\theta^t_{k,s} : E^{1;d}_{k,s} \to E^{1;d+1}_{k,s}\}.$$

By using the comparison theorem of spectral sequences, we can obtain the assertion. □

Example ($(m,n)=(4,3)$) For example, for $(m,n)=(4,3)$, we have:

$$
\begin{aligned}
\mathrm{Poly}_3^{3,4}(\mathbb{R}) &\simeq_s S^{10},\\
\mathrm{Poly}_3^{6,4}(\mathbb{R}) &\simeq_s \mathrm{Poly}_3^{3,4}(\mathbb{R}) \vee S^{20} \vee \Sigma^{20} D_1,\\
\mathrm{Poly}_3^{9,4}(\mathbb{R}) &\simeq_s \mathrm{Poly}_3^{6,4}(\mathbb{R}) \vee S^{30} \vee \Sigma^{30} D_1,\\
\mathrm{Poly}_3^{12,4}(\mathbb{R}) &\simeq_s \mathrm{Poly}_3^{9,4}(\mathbb{R}) \vee S^{40} \vee \Sigma^{40} D_1 \vee \Sigma^{40} D_2,\\
\mathrm{Poly}_3^{15,4}(\mathbb{R}) &\simeq_s \mathrm{Poly}_3^{12,4}(\mathbb{R}) \vee S^{50} \vee \Sigma^{50} D_1 \vee \Sigma^{50} D_2,\\
\mathrm{Poly}_3^{18,4}(\mathbb{R}) &\simeq_s \mathrm{Poly}_3^{15,4}(\mathbb{R}) \vee S^{60} \vee \Sigma^{60} D_1 \vee \Sigma^{60} D_2 \vee \Sigma^{60} D_3,\\
\mathrm{Poly}_3^{21,4}(\mathbb{R}) &\simeq_s \mathrm{Poly}_3^{18,4}(\mathbb{R}) \vee S^{70} \vee \Sigma^{70} D_1 \vee \Sigma^{70} D_2 \vee \Sigma^{70} D_3,\\
\mathrm{Poly}_3^{24,4}(\mathbb{R}) &\simeq_s \mathrm{Poly}_3^{21,4}(\mathbb{R}) \vee S^{80} \vee \Sigma^{80} D_1 \vee \Sigma^{80} D_2 \vee \Sigma^{80} D_3 \vee \Sigma^{80} D_4,\\
\mathrm{Poly}_3^{27,4}(\mathbb{R}) &\simeq_s \mathrm{Poly}_3^{24,4}(\mathbb{R}) \vee S^{90} \vee \Sigma^{90} D_1 \vee \Sigma^{90} D_2 \vee \Sigma^{90} D_3 \vee \Sigma^{90} D_4,\\
\mathrm{Poly}_3^{30,4}(\mathbb{R}) &\simeq_s \mathrm{Poly}_3^{27,4}(\mathbb{R}) \vee S^{100} \vee \Big(\bigvee_{k=1}^{5} \Sigma^{100} D_k\Big),\\
\mathrm{Poly}_3^{33,4}(\mathbb{R}) &\simeq_s \mathrm{Poly}_3^{30,4}(\mathbb{R}) \vee S^{110} \vee \Big(\bigvee_{k=1}^{5} \Sigma^{110} D_k\Big), \cdots\cdots \quad \text{etc}
\end{aligned}
$$

For example, by Corollary V*, the stabilization map

$$s^{d,4}_{3,\mathbb{R}} : \mathrm{Poly}^{d,4}_3(\mathbb{R}) \to \mathrm{Poly}^{d+1,4}_3(\mathbb{R})$$

is a homotopy equivalence if $\lfloor \frac{d}{3} \rfloor = \lfloor \frac{d+1}{3} \rfloor$ and it is is a homotopy equivalence through dimension $D(d;4,3) = 10(\lfloor \frac{d}{3} \rfloor + 1) - 1$ if $\lfloor \frac{d}{3} \rfloor < \lfloor \frac{d+1}{3} \rfloor$.
For example, since $6 = \lfloor 18/3 \rfloor = \lfloor 19/3 \rfloor = \lfloor 20/3 \rfloor < \lfloor 21/3 \rfloor = 7$,

$$\begin{array}{ccccc}
\mathrm{Poly}^{18,4}_3(\mathbb{R}) & \xrightarrow[\simeq]{s^{18,4}_{3,\mathbb{R}}} & \mathrm{Poly}^{19,4}_3(\mathbb{R}) & \xrightarrow[\simeq]{s^{19,4}_{3,\mathbb{R}}} & \mathrm{Poly}^{20,4}_3(\mathbb{R}) \\
\simeq_s \uparrow & & & & \downarrow s^{20,4}_{3,\mathbb{R}} \\
\mathrm{Poly}^{15,4}_3(\mathbb{R}) \vee X_0 & & & & \mathrm{Poly}^{21,4}_3(\mathbb{R}) \\
 & & & & \downarrow \simeq_s \\
 & & & & \mathrm{Poly}^{18,4}_3(\mathbb{R}) \vee Y_0
\end{array}$$

where we write $\begin{cases} X_0 & = & S^{60} \vee \Sigma^{60} D_1 \vee \Sigma^{60} D_2 \vee \Sigma^{60} D_3, \\ Y_0 & = & S^{70} \vee \Sigma^{70} D_1 \vee \Sigma^{70} D_2 \vee \Sigma^{70} D_3. \end{cases}$ □