Tangent spaces of diffeological spaces and their variants

Taho Masaki

The University of Tokyo, D1

August 26, 2024

Diffeological space

- Diffeological spaces are one of the generalizations of smooth manifolds.
  - manifolds with corners
  - orbifolds
  - infinite dimensional manifolds
- The category of diffeological spaces is complete, cocomplete, and cartesian closed. [BH]

Diffeological space

X: a set
$\mathrm{Param}_n(X) := \{p\colon U_p \to X \mid U_p \subset \mathbb{R}^n \text{ is an open set, } n \in \mathbb{N}\ \}$
$\mathrm{Param}(X) := \bigcup_{n\in\mathbb{N}} \mathrm{Param}_n(X)$

diffeology

$\mathcal{D} \subset \mathrm{Param}(X)$ is called a *diffeology* on X if

- **Covering** Every constant parametrization $U \to X$ is in $\mathcal{D}$.
- **Locality** For all $(p\colon U_p \to X) \in \mathrm{Param}(X)$, if there is an open covering $\{U_\alpha\}$ of U_p such that $p|_{U_\alpha} \in \mathcal{D}$ for all α, then p itself is in $\mathcal{D}$.
- **Smooth compatibility** For all $(p\colon U_p \to X) \in \mathcal{D}$, every open set V in $\mathbb{R}^m$ and $f\colon V \to U_p$ that is smooth as a map between Euclidean spaces, $p \circ f\colon V \to X$ is also in $\mathcal{D}$.

$(X, \mathcal{D}_X)$, $(Y, \mathcal{D}_Y)$: diffeological spaces
$f\colon X \to Y$ is said to be *smooth* $:\Longleftrightarrow$ For all $p \in \mathcal{D}_X$, $f \circ p \in \mathcal{D}_Y$

Examples

- M: a C^∞-manifold
  $\mathcal{D}_M := \{p\colon U_p \to M \mid p \text{ is a } C^\infty\text{-map}\}$
  : the standard diffeology of M
  $f\colon (M, \mathcal{D}_M) \to (N, \mathcal{D}_N)$ is smooth $\Longleftrightarrow$ f is a C^∞-map.
- X: a set
  $\mathcal{D}_{min} := \{p\colon U_p \to X \mid p \text{ is locally constant}\}$
  : the minimum diffeology of X
  $\mathcal{D}_{max} := \mathrm{Param}(X)$: the maximum diffeology of X

X: a set, $\mathcal{F} \subset \mathrm{Param}(X)$
$\langle \mathcal{F} \rangle$: the smallest diffeology of X which contains $\mathcal{F}$

Quotient diffeology/subdiffeology

$(X.\mathcal{D}_X)$: a diffeological space, $\sim$: an equivalent relation on X
$\pi\colon X \to X/\sim$: the quotient map
$\langle \{\pi \circ q \mid q \in \mathcal{D}_X\} \rangle$: the *quotient diffeology* of $X/\sim$

$(X, \mathcal{D}_X)$: a diffeological space, $A \subset X (i\colon A \hookrightarrow X)$.
$\mathcal{D}_A := \{p \in \mathrm{Param}(A) \mid i \circ p \in \mathcal{D}_X\}$: the *subdiffeology* of A.

Examples of diffeological spaces

- $+_{quot} := (\mathbb{R}_1 \coprod \mathbb{R}_2)/(0_1 \sim 0_2)$
- For $n \in \mathbb{N}$, we equip the quotient diffeology with $\mathbb{R}^n/O(n)$
  Then $\mathbb{R}^n/O(n)$ is not diffeomorphic to $\mathbb{R}^m/O(m)$ if $m \neq n$.
- For $\alpha \in \mathbb{R}\backslash\mathbb{Q}$, $T_\alpha := \mathbb{R}/(\mathbb{Z} + \alpha\mathbb{Z})$ is called an irrational torus.
  T_α is diffeomorphic to T_β
  $\iff \exists a, b, c, d \in \mathbb{Z}$ such that $\alpha = \dfrac{a + b\beta}{c + d\beta}$.
- $[0, \infty) \subset \mathbb{R}$ with subdiffeology is NOT diffeomorphic to $\mathbb{R}^n/O(n)$ for any $n \in \mathbb{N}$.
- $+_{sub} := \{(x, y) \in \mathbb{R}^2 \mid xy = 0\} \subset \mathbb{R}^2$ with subdiffeology is NOT diffeomorphic to $+_{quot}$.
- $\langle\{p\colon \mathbb{R} \to \mathbb{R}^2 \mid p \text{ is a } C^\infty\text{-map}\}\rangle$: the wire diffeology on $\mathbb{R}^2$.

D-topology

$(X, \mathcal{D}_X)$: a diffeological space
$\mathcal{O} := \{U \subset X \mid \forall p \in \mathcal{D}_X, p^{-1}(U) \text{ is open}\}$: D-topology

example

manifolds M: a manifold
The D-topology of M as a standard diffeological space is the topology of M as a manifold.

quotient X: a diffeological space,
$\sim$: an equivalence relation on X
The D-topology of $X/\sim$ is the quotient topology of the D-topology of X.

Internal tangent spaces [CW]

Category	Objects	Morphisms
Eucl_*	based Euclidean open sets	based C^∞-maps
Dflg_*	based diffeological spaces	based smooth maps

$i_* \colon \mathrm{Eucl}_* \to \mathrm{Dflg}_*$: the inclusion functor
$T' \colon \mathrm{Eucl}_* \to \mathrm{Vect}$: the standard tangent functor

internal tangent functor

$T := \mathrm{Lan}_{i_*} T' \colon \mathrm{Dflg}_* \to \mathrm{Vect}$: the internal tangent functor
For a diffeological space X and a point $x \in X$, we write

$$T_x(X) := T(X, x) = \left(\bigoplus_{p \colon (U_p, 0) \to (X, x)} T_0'(U_p) \right) / R,$$

$$\left(R = \left\langle v - w \,\middle|\, \begin{array}{l} v \in T_0'(U_p),\ w \in T_0'(U_q), \\ T'(f)v = w, \text{ where } p = q \circ f. \end{array} \right\rangle \right)$$

X: a diffeological space, $x \in X$
$G(X,x) := \{f\colon B_f \to \mathbb{R} \mid x \in B_f \subset X$ is D-open, f is smooth$\}/\sim$,
where $(f\colon B_f \to \mathbb{R}) \sim (g\colon B_g \to \mathbb{R})$
$:\Longleftrightarrow x \in \exists B \subset (B_f \cap B_g)$: D-open set s.t. $f|_B = g|_B$

$G(X,x)$ is a diffeological $\mathbb{R}$-algebra with standard operations.

external tangent space [CW]

X: a diffeological space, $x \in X$

$$\hat{T}_x(X) = \left\{ D\colon G(X,x) \to \mathbb{R} \;\middle|\; \begin{array}{l} D \text{ is smooth and linear.} \\ D \text{ satisfies the Leibniz rule.} \end{array} \right\}$$

: the external tangent space

Examples

Spaces	Internal	External
n-dim. manifold	$\mathbb{R}^n$	$\mathbb{R}^n$
$+_{quot}$	$\mathbb{R}^2$	$\mathbb{R}^2$
$+_{sub}(\subset \mathbb{R}^2)$	$\mathbb{R}^2$	$\mathbb{R}^2$
$[0, \infty)(\subset \mathbb{R})$	0	$\mathbb{R}$
$\mathbb{R}^n / O(n)$	0	$\mathbb{R}$
$T_\alpha = \mathbb{R}/(\mathbb{Z} + \alpha\mathbb{Z})$	$\mathbb{R}$	0
$(X, \mathcal{D}_{min})$	0	0
$(X, \mathcal{D}_{max})$	0	0
$\mathbb{R}^2_{wire}$	uncountable-dim.	$\mathbb{R}^2$
$\mathbb{R}^2/\{(x, 0) \in \mathbb{R}^2\}$	uncountable-dim.	uncountable-dim.

Question Are $\hat{T}_x(X)$ and $\mathrm{Ran}_{i_*} T'(X, x)$ isomorphic?

Answer

- They are NOT isomorphic in general. However in many case, they are isomorphic.
- By slightly modifying the definitions, we get an isomorphism.

X: a diffeological space, $x \in X$

$$\hat{T}_x(X) = \left\{ D\colon G(X,x) \to \mathbb{R} \;\middle|\; \begin{array}{l} D \text{ is smooth and linear.} \\ D \text{ satisfies the Leibniz rule.} \end{array} \right\}$$

: the external tangent space

$$\hat{T}^R_x(X) = \left\{ D\colon G(X,x) \to \mathbb{R} \;\middle|\; \begin{array}{l} D \text{ is linear.} \\ D \text{ satisfies the Leibniz rule.} \end{array} \right\}$$

: the right tangent space

$$\hat{\mathbb{T}}_x(X) = \left\{ D\colon C^\infty(X,\mathbb{R}) \to \mathbb{R} \;\middle|\; \begin{array}{l} D \text{ is smooth and linear.} \\ D \text{ satisfies the Leibniz rule.} \end{array} \right\}$$

: the global external tangent space

$$\hat{\mathbb{T}}^R_x(X) = \left\{ D\colon C^\infty(X,\mathbb{R}) \to \mathbb{R} \;\middle|\; \begin{array}{l} D \text{ is linear.} \\ D \text{ satisfies the Leibniz rule.} \end{array} \right\}$$

: the global right tangent space

$$\hat{\mathbb{T}}_x^R(X) = \left\{ D\colon C^\infty(X,\mathbb{R}) \to \mathbb{R} \;\middle|\; \begin{array}{l} D \text{ is linear.} \\ D \text{ satisfies the Leibniz rule.} \end{array} \right\}$$

: the global right tangent space

Proposition[T, 2024]

X: a diffeological space, $x \in X$

$$\mathrm{Ran}_{i_*}\, T'(X,x) \cong \hat{\mathbb{T}}_x^R(X)$$

Modification

Category	Objects	Morphisms
Eucl_*^{loc}	based Euclidean open sets	germs of C^∞-maps
Dflg_*^{loc}	based diffeological spaces	germs of smooth maps

$i_*^{loc} : \mathrm{Eucl}_*^{loc} \to \mathrm{Dflg}_*^{loc}$: the inclusion functor
$T^{\mathrm{Eucl}_*^{loc}} : \mathrm{Eucl}_*^{loc} \to \mathrm{Vect}$: the standard tangent functor

Main Theorem[T, 2024]

X: a diffeological space, $x \in X$

$$\mathrm{Ran}_{i_*^{loc}} T^{\mathrm{Eucl}_*^{loc}}(X, x) \cong \hat{T}_x^R(X)$$

sketch of proof

$\mathrm{Ran}_{i_*^{loc}} T^{\mathrm{Eucl}_*^{loc}}(X,x)$ is written as an element of $\{v_g\}_{g\in \mathrm{Obj}((X,x)\downarrow i_*^{loc})} \in \prod_{g\in \mathrm{Obj}((X,x)\downarrow i_*^{loc})} T_{g(x)}^{\mathrm{Eucl}_*^{loc}} V_g$ which satisfies the following condition:

$$\forall g,h \in \mathrm{Obj}((X,x)\downarrow i_*^{loc}), \forall s\colon g \to h \text{ in } (X,x)\downarrow i_*^{loc},$$
$$T^{\mathrm{Eucl}_*^{loc}}(s)v_g = v_h$$

So we define the map
$\beta\colon \mathrm{Ran}_{i_*^{loc}} T^{\mathrm{Eucl}_*^{loc}}(X,x) \to \hat{T}_x^R(X); \{v_f\}_f \mapsto [f \mapsto v_f[\mathrm{id}_{\mathbb{R}}]]$.

$$(\text{If } f \in G(X,x), \text{ then } v_f \in T_{f(x)}\mathbb{R} \cong \mathbb{R}.)$$

The inverse is defined by the universality of the right Kan extension.

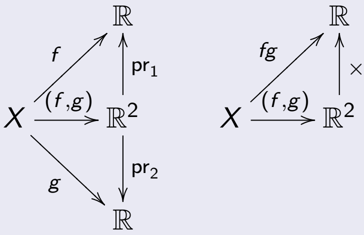
Well-definedness of β

$v_{fg}([\mathrm{id}_{\mathbb{R}}]) = f(x)v_g([\mathrm{id}_{\mathbb{R}}]) + g(x)v_f([\mathrm{id}_{\mathbb{R}}]$ (Leibniz rule)

We define $(f,g)\colon X \to \mathbb{R}^2; y \mapsto (f(y), g(y))$.

Then we have

$T(\mathrm{pr}_1)(v_{(f,g)}) = v_f$, $T(\mathrm{pr}_2)(v_{(f,g)}) = v_g$, and $T(\times)v_{(f,g)} = v_{fg}$



$v_{fg}([\mathrm{id}_{\mathbb{R}}]) = f(x)v_g([\mathrm{id}_{\mathbb{R}}]) + g(x)v_f([\mathrm{id}_{\mathbb{R}}])$ (Leibniz rule)

$$
\begin{aligned}
v_{fg}([\mathrm{id}_{\mathbb{R}}]) &= T(\times)v_{(f,g)}([\mathrm{id}_{\mathbb{R}} \colon \mathbb{R} \to \mathbb{R}]) \\
&= v_{(f,g)}([\times \colon \mathbb{R}^2 \to \mathbb{R}]) = v_{(f,g)}([\mathrm{pr}_1 \colon \mathbb{R}^2 \to \mathbb{R}] * [\mathrm{pr}_2 \colon \mathbb{R}^2 \to \mathbb{R}]) \\
&= \mathrm{pr}_1((f(x), g(x)))v_{(f,g)}([\mathrm{pr}_2 \colon \mathbb{R}^2 \to \mathbb{R}]) \\
&\qquad + \mathrm{pr}_2((f(x), g(x)))v_{(f,g)}([\mathrm{pr}_1 \colon \mathbb{R}^2 \to \mathbb{R}]) \\
&= f(x)v_g([\mathrm{id}_{\mathbb{R}}]) + g(x)v_f([\mathrm{id}_{\mathbb{R}}])
\end{aligned}
$$

$(T(\mathrm{pr}_1)(v_{(f,g)}) = v_f,\ T(\mathrm{pr}_2)(v_{(f,g)}) = v_g$, and $T(\times)v_{(f,g)} = v_{fg})$

Thus $[f \mapsto v_f[\mathrm{id}_{\mathbb{R}}]]$ satisfies the Leibniz rule.
Similarly, we can prove that $[f \mapsto v_f[\mathrm{id}_{\mathbb{R}}]]$ is linear.

Smoothly regular

Theorem[T, 2024]

X: a diffeological space, $x \in X$
If X is smoothly regular at x, then

$$\hat{T}^R_x(X) \cong \hat{\mathbb{T}}^R_x(X)$$

Also,

$$\hat{T}_x(X) \cong \hat{\mathbb{T}}_x(X)$$

Here, X is smoothly regular at x
$:\Longleftrightarrow x \in \forall U \subset X$: D-open,

$$\exists f : X \to \mathbb{R}: \text{ smooth map s.t. } f(y) = \begin{cases} 1 & (y = x) \\ 0 & (y \notin U). \end{cases}$$

I-topology

X: a diffeological space

$$\mathcal{O}_I := \left\{ U \subset X \,\middle|\, \begin{array}{l} \forall u \in U, \exists V \subset \mathbb{R}\text{: open and} \\ \exists f : X \to \mathbb{R}\text{: smooth map s.t. } u \in f^{-1}(V) \subset U \end{array} \right\}$$

: I-topology (the smallest topology such that all elements of $C^\infty(X, \mathbb{R})$ are continuous)

Proposition

X: a diffeological space X is smoothly regular at x
$\iff x \in \forall U \subset X$: D-open, $x \in \exists V \subset U$: I-open set of X

Examples

- Manifolds with the standard diffeology are smoothly regular.
- $+_{quot}$, $[0, \infty)$, $T_\alpha = \mathbb{R}/(\mathbb{Z} + \alpha\mathbb{Z})$, $(X, \mathcal{D}_{min})$, $(X, \mathcal{D}_{max})$, and $\mathbb{R}^2_{wire}$ are all smoothly regular.
- $\mathbb{R}/(0,1)$ is NOT smoothly regular.
  Note that $\{[1/2]\} \subset X$ is a D-open set (the inverse image $(0,1) \subset \mathbb{R}$ is an open set). However, every smooth map $f\colon X \to \mathbb{R}$ satisfies $f([1/2]) = f([1])$ (by the continuity of f). Therefore, $\{[1/2]\} \subset X$ is not an I-open set of X.
- The orbit space A of the action $\mathbb{R} \curvearrowright \mathbb{R}^2; t \cdot (x, y) = (x + ty, y)$ is NOT smoothly regular.
- The orbit space B of the action $\mathbb{R} \curvearrowright \mathbb{R}^2; t \cdot (x, y) = (2^t x, 2^{-t} y)$ is NOT smoothly regular.

$X := \text{十}_{quot}/\{(x,0) \mid x \neq 0\}$ is NOT smoothly regular.
Moreover, $\hat{T}^R_{[(1,0)]}(X) \ncong \hat{\mathbb{T}}^R_{[(1,0)]}(X)$.

<u>proof</u>
Note that $\{[(1,0)]\} \subset X$ is a D-open set.
Therefore, $\hat{T}^R_{[(1,0)]}(X) \cong \hat{T}^R_{[(1,0)]}(\{[(1,0)]\}) \cong 0$.

On the other hands,

$$
\begin{aligned}
&C^\infty(X, \mathbb{R}) \\
&\cong \{f \colon \text{十}_{quot} \to \mathbb{R} \mid \forall x \in \mathbb{R}\backslash\{0\}, f(x,0) = f(1,0)\} \\
&= \{f \colon \text{十}_{quot} \to \mathbb{R} \mid \forall x \in \mathbb{R}, f(x,0) = f(1,0)\} \\
&\cong C^\infty(\mathbb{R}, \mathbb{R})
\end{aligned}
$$

Therefore, $\hat{\mathbb{T}}^R_{[(1,0)]}(X) \cong \hat{\mathbb{T}}^R_0(\mathbb{R}) \cong \mathbb{R}$.

References

John C. Baez and Alexander E. Hoffnung.
Convenient categories of smooth spaces.
Transactions of the American Mathematical Society, 363(11):5789–5825, 2011.

J. Daniel Christensen and Enxin Wu.
Tangent spaces and tangent bundles for diffeological spaces.
Cah. Topol. Géom. Différ. Catég., 57(1):3–50, 2016.

Patrick Iglesias-Zemmour.
Diffeology, volume 185 of *Mathematical Surveys and Monographs*.
American Mathematical Society, Providence, RI, 2013.

References

J.-M. Souriau.
Groupes différentiels.
In *Differential geometrical methods in mathematical physics (Proc. Conf., Aix-en-Provence/Salamanca, 1979)*, volume 836 of *Lecture Notes in Math.*, pages 91–128. Springer, Berlin, 1980.

Masaki Taho.
Tangent spaces of diffeological spaces and their variants, 2024, arXiv:2406.04703

Martin Vincent.
Diffeological differential geometry.
Master Thesis, University of Copenhagen, 2008.

Thank you for listening!