# SPECIALIZATION ORDERS ON ATOM SPECTRA OF GROTHENDIECK CATEGORIES

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The aim of this talk is to provide systematic methods to construct Grothendieck categories with remarkable structures and to establish a theory of the specialization orders on the spectra of Grothendieck categories.

In commutative ring theory, Hochster characterized topological spaces appearing as the prime spectra of commutative rings ([Hoc69, Theorem 6 and Proposition 10]). Speed [Spe72] pointed out that Hochster's result gives the following characterization of partially ordered sets appearing as the prime spectra of commutative rings.

**Theorem 1** (Hochster [Hoc69, Proposition 10] and Speed [Spe72, Corollary 1])**.** *Let $P$ be a partially ordered set. Then $P$ is isomorphic to the prime spectrum of some commutative ring with the inclusion relation if and only if $P$ is an inverse limit of finite partially ordered sets in the category of partially ordered sets.*

In [Kan12a] and [Kan12b], we investigated Grothendieck categories by using the associated topological spaces called the *atom spectra* of them. For a Grothendieck category $\mathcal{A}$, the atom spectrum $\operatorname{ASpec}\mathcal{A}$ has a partial order. For a commutative ring $R$, the partially ordered set $\operatorname{ASpec}(\operatorname{Mod}R)$ is isomorphic to the prime spectrum $\operatorname{Spec}R$ with the inclusion relation. Hence we can consider that the atom spectrum of a Grothendieck category is a (noncommutative) generalization of the prime spectrum of a commutative ring, and it is natural to ask *which partially ordered sets appear as the atom spectra of Grothendieck categories.*

In order to answer this question, we introduce a construction of Grothendieck categories by using colored quivers. A sextuple $(Q_0, Q_1, C, s, t, u)$ is called a *colored quiver* if $(Q_0, Q_1, s, t)$ is a quiver (not necessarily finite), $C$ is a set (of colors), and $u\colon Q_1 \to C$ is a map. From a colored quiver which satisfies some condition of local finiteness, we construct a Grothendieck category associated to the colored quiver. By using this construction, we can show the following result, which is a complete answer to the above question.

**Theorem 2** ([Kan13])**.** *For any partially ordered set $P$, there exists a Grothendieck category $\mathcal{A}$ such that the atom spectrum $\operatorname{ASpec}\mathcal{A}$ is isomorphic to $P$ as a partially ordered set.*

## References

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