Clifford extensions

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Clifford algebras play important roles in various fields and the construction of Clifford algebras contains that of complex numbers, quaternions, and so on (see e.g. [4]). In this talk, we generalize the construction of Clifford algebras and introduce the notion of Clifford extensions. Clifford extensions are constructed as Frobenius extensions and we have already known that Frobenius extensions of Auslander-Gorenstein rings are also Auslander-Gorenstein rings. It should be noted that little is known about constructions of Auslander-Gorenstein rings although Auslander-Gorenstein rings appear in various fields of current research in mathematics including noncommutative algebraic geometry, Lie algebras, and so on (see e.g. [1], [2], [3] and [5]).

We use the notation A/R to denote that a ring A contains a ring R as a subring. Let $n \geq 2$ be an integer. We fix a set of integers $I = \{0, 1, \ldots, n-1\}$ and a ring R. First, we construct a split Frobenius extension Λ/R of second kind using a certain pair (σ, c) of $\sigma \in \mathrm{Aut}(R)$ and $c \in R$. Namely, we define an appropriate multiplication on a free right R-module Λ with a basis $\{v_i\}_{i \in I}$. We show that this construction can be iterated arbitrary times. Then we deal with the case where $n = 2$ and study the iterated Frobenius extensions. For $m \geq 1$ we construct ring extensions Λ_m/R using the following data: a sequence of elements $c_1, c_2, \cdots$ in $\mathrm{Z}(R)$ and signs $\varepsilon(i,j)$ for $1 \leq i, j \leq m$. Namely, we define an appropriate multiplication on a free right R-module Λ_m with a basis $\{v_x\}_{x \in I^m}$. We show that Λ_m is obtained by iterating the construction above m times, that Λ_m/R is a split Frobenius extension of first kind, and that if $c_i \in \mathrm{rad}(R)$ for $1 \leq i \leq m$ then $R/\mathrm{rad}(R) \xrightarrow{\sim} \Lambda_m/\mathrm{rad}$. We call Λ_m Clifford extensions of R because they have the following properties similar to Clifford algebras. For each $x = (x_1, \ldots, x_m) \in I^m$ we set $S(x) = \{i \mid x_i = 1\}$. Also we set $v_x = t_i$ for $x \in I^m$ with $S(x) = \{i\}$. Then the following hold: (C1) $t_i^2 = v_0 c_i$ for all $1 \leq i \leq m$; (C2) $t_i t_j + t_j t_i = 0$ unless $i = j$; (C3) $v_x = t_{i_1} \cdots t_{i_r}$ if $S(x) = \{i_1, \ldots, i_r\}$ with $i_1 < \cdots < i_r$.

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2010 Mathematics Subject Classification: Primary: 16S35, 16S50; Secondary: 16E10.
Keywords: Auslander-Gorenstein ring, Clifford extension, Frobenius extension.
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