ON HOCHSCHILD COHOMOLOGY OF A SELF-INJECTIVE SPECIAL BISERIAL ALGEBRA OBTAINED BY A CIRCULAR QUIVER WITH DOUBLE ARROWS

AYAKO ITABA

ABSTRACT. This reprot is a survey of our result in [I]. We calculate the dimensions of the Hochschild cohomology groups of a self-injective special biserial algebra Λ_s obtained by a circular quiver with double arrows. Moreover, we give a presentation of the Hochschild cohomology ring modulo nilpotence of Λ_s by generators and relations. This result shows that the Hochschild cohomology ring modulo nilpotence of Λ_s is finitely generated as an algebra.

1. Introduction

Let K be an algebraically closed field. Let Q be a finite connected quiver and KQ a path algebra and I an admissible ideal of KQ. Then $A = KQ/I$ is a finite-dimensional K-algebra. Also, we denote the origin of a by $o(a)$ and the terminus of a by $t(a)$ for $a \in Q_1$.

For a finite-dimensional albegra A over K, the Hochschild cohomology groups $\mathrm{HH}^n(A)$ of A is defined by

$$\mathrm{HH}^n(A) := \mathrm{Ext}^n_{A^e}(A, A) \ (n \geq 0),$$

where $A^e{:=}A^{\mathrm{op}} \otimes_K A$ is the enveloping algebra of A. Moreover, the Hochschild cohomology rings $\mathrm{HH}^*(A)$ of A is graded algebra defined by

$$\mathrm{HH}^*(A) := \mathrm{Ext}^*_{A^e}(A, A) = \bigoplus_{i \geq 0} \mathrm{Ext}^i_{A^e}(A, A)$$

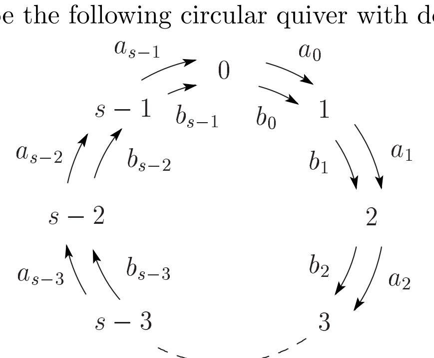
with the Yoneda product.

The low-dimensional Hochschild cohomology groups are described as follows:

- $\mathrm{HH}^0(A) = Z(A)$: the center of A.
- $\mathrm{HH}^1(A)$ is the space of derivations modulo the inner derivation. A deivations is a k-linear map $f : A \to A$ such that $f(ab) = af(b) + f(a)b$ for all $a, b \in A$. A derivation $f : A \to A$ is an inner derivation if there is some $x \in A$ such that $f(a) = ax - xa$ for all $a \in A$.

One important property of Hochschild cohomology is its invariance under derived equivalence. In gereral, it is duffucult to calucurate the Hochschild cohomology of a finite-dimensional albegra A.

For a positive integer s, let Γ_s be the following circular quiver with double arrows:



$e_i :=$ the trivial path at the vertex i, where the subscript i of e_i is regarded as modulo s. We set the elements $x = \sum_{i=0}^{s-1} a_i$ and $y = \sum_{i=0}^{s-1} b_i$ in the path algebra $K\Gamma_s$. $e_i x^n = x^n e_{i+n} = e_i x^n e_{i+n}$ and $e_i y^n = y^n e_{i+n} = e_i y^n e_{i+n}$ hold for $0 \leq i \leq s-1$ and $n \geq 0$. We denote by I the ideal generated by x^2, $xy + yx$

Date: June 30, 2014.
2010 *Mathematics Subject Classification.* 16D20, 16E40, 16G20.
Key words and phrases. Koszul algebra, special biserial algebra, self-injective algebra, Hochschild cohomology.

and y^2, that is, $I = \langle e_i x^2, e_i(xy + yx), e_i y^2 \mid 0 \le i \le s-1\rangle$. The bound quiver algebra $\Lambda_s := K\Gamma_s/I$ over K. Our purpose is to study the Hochschild cohomology of Λ_s. This algebra Λ_s is a Koszul self-injective special biserial algebra (see Proposition 2.2), but is not a weakly symmetric algebra for $s \ge 3$.

Remark 1.1. For $s = 1, 2, 4$, the Hochschild cohomology of Λ_s is reserched in [XH], [ST] and [F], respectively.

Definition 1.2. *We say that a graded projective resolution*

$$\cdots \to P^2 \xrightarrow{d^2} P^1 \xrightarrow{d^1} P^0 \xrightarrow{d^0} M \to 0$$

is linear and M is a linear module if for $n \ge 0$, the graded module P^n is generated in degree n. A graded algebra Λ is a Koszul algebra if Λ_0 is a linear module; that is, Λ_0 has a linear projective resolution

$$(\mathbb{L}, e) : \cdots \to L^2 \xrightarrow{e^2} L^1 \xrightarrow{e^1} L^0 \xrightarrow{e^0} \Lambda_0 \to 0.$$

as a right Λ-module.

Definition 1.3. *$A = KQ/I$ is said to be a special biserial algebra, if A satisfies the following* (SP1) *and* (SP2):

(SP1): *For each vertex $v \in Q_0$,*
- *#{the arrows starting at v}≤ 2; and*
- *#{the arrows ending with v}≤ 2.*

(SP2): *For each arrow $\alpha \in Q_1$,*
- *#{the arrows β with $\beta\alpha \notin I$}≤ 1; and*
- *#{the arrows γ with $\alpha\gamma \notin I$}≤ 1.*

In [GHMS], Green, Hartmann, Marcos and Solberg constructed a minimal projective bimodule resolution for any Koszul algebra by using some sets $\mathcal{G}^n$ $(n \ge 0)$ introduced in [GSZ]. These sets $\mathcal{G}^n$ $(n \ge 0)$ are also used in the papers [FO], [ST], [ScSn] in constructing a minimal projective bimodule resolution of several weakly symmetric algebras. By this same method, we give the minimal projective bimodule resolution of Λ_s for $s \ge 1$ and compute the Hochschild cohomology group $\mathrm{HH}^n(\Lambda_s)$ of Λ_s $(n \ge 0)$ in the case where $s \ge 3$.

In [SnSo], Snashall and Solberg have defined the support varieties of finitely generated modules over a finite-dimensional algebra by using the Hochschild cohomology ring modulo nilpotence. In [EHTSS], for any finite-dimensional algebra, Erdmann, Holloway, Taillefer, Snashall and Solberg have introduced certain finiteness conditions, denoted by (Fg), and showed that if a finite-dimensuonal algebra satisfies (Fg), then the supprot varieties have a lot analogous properties of support varieties for finite group algebras. These works inspire us to study the Hochschild cohomology rings modulo nilpotence of finite-dimensional algebras. We determine generators and relations of the Hochschild cohomology ring modulo nilpotence $\mathrm{HH}^*(\Lambda_s)/\mathcal{N}_{\Lambda_s}$ for all $s \ge 3$.

2. A projective bimodule resolution $(Q^\bullet, \partial^\bullet)$ of Λ_s

Let $\mathcal{G}^0$ be the set of all vertices of Q, $\mathcal{G}^1$ the set of all arrows of Q, and $\mathcal{G}^2$ a minimal set of uniform generators of I. In [GSZ], Green-Solberg-Zacharia showed that, for each $n \ge 3$, there are sets $\mathcal{G}^n$ of uniform elements in KQ such that we have a minimal projective resoluition $(P^\bullet, d^\bullet)$ of the right A-module $A/\mathrm{rad}\, A$ satisfying the following conditions:

(a) For $n \ge 0$, $P^n = \bigoplus_{x \in \mathcal{G}^n} t(x)A$.

(b) For $x \in \mathcal{G}^n$, there are unique elements $r_y, s_z \in KQ$, where $y \in \mathcal{G}^{n-1}$ and $z \in \mathcal{G}^{n-2}$, such that $x = \sum_{y \in \mathcal{G}^{n-1}} y r_y = \sum_{z \in \mathcal{G}^{n-2}} z s_z$.

(c) For $n \ge 1$, the differential $d^n : P^n \to P^{n-1}$ is defined by $d^n(t(x)\lambda) = \sum_{y \in \mathcal{G}^{n-1}} r_y t(x)\lambda$ for $x \in \mathcal{G}^n$ and $\lambda \in A$, where r_y denotes the element in the expression (b).

A minimal projective bimodule resolution of any Koszul algebra is given in [GHMS] by using the sets $\mathcal{G}^n$ $(n \ge 0)$ in [GSZ]. We construct sets $\mathcal{G}^n$ $(n \ge 0)$ for the right Λ_s-modules $\Lambda_s/\mathrm{rad}\, \Lambda_s$ by following [GHMS]. We give a projective bimodule resolution $(Q^\bullet, \partial^\bullet)$ of Λ_s.

In order to construct sets $\mathcal{G}^n$ for $\Lambda_s/\mathrm{rad}\, \Lambda_s$, we define the following elements in $K\Gamma_s$:

Definition 2.1. *For $0 \leq i \leq s-1$, we put $g^0_{i,0} := e_i$, and, for $n \geq 1$, we inductively define the elements $g^n_{i,j} \in K\Gamma_s$ as follows*:

- $g^n_{i,0} := g^{n-1}_{i,0} y$ *for* $0 \leq i \leq s-1$,
- $g^n_{i,j} := g^{n-1}_{i,j-1} x + g^{n-1}_{i,j} y$ *for* $0 \leq i \leq s-1$ *and* $1 \leq j \leq n-1$,
- $g^n_{i,n} := g^{n-1}_{i,n-1} x$ *for* $0 \leq i \leq s-1$.
- *We regard the subscript i of $g^n_{i,j}$ as modulo s.*

By using Definition 2.1, we put the set

$$\mathcal{G}^n = \{g^n_{i,j} \mid 0 \leq i \leq s-1;\ 1 \leq j \leq n-1\}$$

for all $n \geq 0$. It is easy to check that these sets satisfying the conditions (a), (b) and (c) in the beginning of this section.

Now, it can be seen that Λ_s is a self-injective Koszul algebra. We have the following proposition.

Proposition 2.2. *The algebra Λ_s is a self-injective Koszul algebra.*

In order to obtain a minimal Λ^e_s-projective resolution $(Q^\bullet, \partial^\bullet)$ of Λ_s, we need the following lemma.

Lemma 2.3. *For $n \geq 1$, we have the following equations hold*:

- $g^n_{i,0} = y g^{n-1}_{i+1,0}$ *for* $0 \leq i \leq s-1$,
- $g^n_{i,j} = y g^{n-1}_{i+1,j} + x g^{n-1}_{i+1,j-1}$ *for* $0 \leq i \leq s-1$ *and* $1 \leq j \leq n-1$,
- $g^n_{i,n} = x g^{n-1}_{i+1,n-1}$ *for* $0 \leq i \leq s-1$.

By using the sets $\mathcal{G}^n$ $(n \geq 0)$, we give a minimal projective resolution $(Q^\bullet, \partial^\bullet)$ of Λ_s as a right Λ^e_s-module.

First, we start with the definition of the projective module Q^n for $n \geq 0$. For $n \geq 0$, we denote the elements $o(g^n_{i,j}) \otimes t(g^n_{i,j})$ by $b^n_{i,j}$ in $\Lambda_s o(g^n_{i,j}) \otimes t(g^n_{i,j})\Lambda_s$ for $0 \leq i \leq s-1$ and $0 \leq j \leq n$, where the subscript i of $b^n_{i,j}$ is regarded as modulo s.

Definition 2.4. *We define the projective Λ^e_s-module Q^n by the following* :

$$Q^n := \bigoplus_{g \in \mathcal{G}^n} \Lambda_s o(g) \otimes t(g) \Lambda_s = \bigoplus_{i=0}^{s-1} \bigoplus_{j=0}^{n} \Lambda_s b^n_{i,j} \Lambda_s.$$

Next, by Definition 2.1 and Lemma 2.3, we also define the map ∂^n in the following definition.

Definition 2.5. *We define $\partial^0 : Q^0 \to \Lambda_s$ to be the multiplication map, and, for $n \geq 1$, $\partial^n : Q^n \to Q^{n-1}$ to be the Λ^e_s-homomorphism determined by*

- $b^n_{i,0} \longmapsto b^{n-1}_{i,0} y + (-1)^n y b^{n-1}_{i+1,0}$ *for* $0 \leq i \leq s-1$,
- $b^n_{i,j} \longmapsto (b^{n-1}_{i,j-1} x + b^{n-1}_{i,j} y) + (-1)^n (y b^{n-1}_{i+1,j} + x b^{n-1}_{i+1,j-1})$ *for* $0 \leq i \leq s-1;\ 1 \leq j \leq n-1$,
- $b^n_{i,n} \longmapsto b^{n-1}_{i,n-1} x + (-1)^n x b^{n-1}_{i+1,n-1}$ *for* $0 \leq i \leq s-1$.

By the direct computations, we see that the composite $\partial^n \partial^{n+1}$ is zero for all $n \geq 0$. Therefore, $(Q^\bullet, \partial^\bullet)$ is a complex of Λ^e_s-modules.

Now since Λ_s is Koszul by Proposition 2.2, the following theorem is immediatly from [GHMS].

Theorem 2.6. *$(Q^\bullet, \partial)$ is a minimal projective Λ^e_s-resolution of Λ_s.*

3. Hochschild cohomology groups $\mathrm{HH}^n(\Lambda_s)$

In this section, we calculate the Hochschild cohomology group $\mathrm{HH}^n(\Lambda_s)$ for $n \geq 0$. By applying the functor $\mathrm{Hom}_{\Lambda^e_s}(-, \Lambda_s)$ to the resolution $(Q^\bullet, \partial^\bullet)$, we have the complex

$$0 \longrightarrow \widehat{Q}^0 \xrightarrow{\widehat{\partial}^1} \widehat{Q}^1 \xrightarrow{\widehat{\partial}^2} \widehat{Q}^2 \xrightarrow{\widehat{\partial}^3} \cdots \xrightarrow{\widehat{\partial}^{n-1}} \widehat{Q}^{n-1} \xrightarrow{\widehat{\partial}^n} \widehat{Q}^n \xrightarrow{\widehat{\partial}^{n+1}} \widehat{Q}^{n+1} \xrightarrow{\widehat{\partial}^{n+2}} \cdots,$$

where $\widehat{Q}^n := \mathrm{Hom}_{\Lambda^e_s}(Q^n, \Lambda_s)$ and $\widehat{\partial}^n := \mathrm{Hom}_{\Lambda^e_s}(\partial^n, \Lambda_s)$. We recall that, for $n \geq 0$, the n-th Hochschild cohomology group $\mathrm{HH}^n(\Lambda_s)$ is defined to be the K-space $\mathrm{HH}^n(\Lambda_s) := \mathrm{Ext}^n_{\Lambda^e_s}(\Lambda_s, \Lambda_s) = \mathrm{Ker}\, \widehat{\partial}^{n+1} / \mathrm{Im}\, \widehat{\partial}^n$.

3.1. **The dimension of** $\operatorname{Im}\widehat{\partial}^{n+1}$**.** By direct computation, we get the dimension of $\operatorname{Im}\widehat{\partial}^{n+1}$ for $n \geq 0$:

Corollary 3.1. *Let $n = ms + r$ for integers $m \geq$ and $0 \leq r \leq s-1$. Then the dimension of $\operatorname{Im}\widehat{\partial}^{n+1}$ is as follows*:

$$\dim_K \operatorname{Im}\widehat{\partial}^{ms+r+1} = \begin{cases} s(ms+1) & \text{if } s \text{ odd, } m \text{ odd, } \operatorname{char} K \neq 2 \text{ and } r = 0, \\ (s-1)(ms+1) & \text{if } s \text{ even and } r = 0, \text{ if } m \text{ even and } r = 0, \text{ or} \\ & \quad \text{if } \operatorname{char} K = 2 \text{ and } r = 0, \\ s(ms+3) & \text{if } s \text{ odd, } m \text{ odd, } \operatorname{char} K \neq 2 \text{ and } r = 1, \\ (s-1)(ms+3) & \text{if } s \text{ even and } r = 1, \text{ if } m \text{ even and } r = 1, \text{ or} \\ & \quad \text{if } \operatorname{char} K = 2 \text{ and } r = 1, \\ 0 & \text{otherwise.} \end{cases}$$

3.2. **The dimension of** $\operatorname{Ker}\widehat{\partial}^{n+1}$**.** As an immediate consequence, we get the dimension of $\operatorname{Ker}\widehat{\partial}^{n+1}$ for $n \geq 1$:

Corollary 3.2. *Let $n = ms + r$ for integers $m \geq 0$ and $0 \leq r \leq s-1$. Then the dimension of $\operatorname{Ker}\widehat{\partial}^{n+1}$ is as follows*:

$$\dim_K \operatorname{Ker}\widehat{\partial}^{ms+r+1} = \begin{cases} 0 & \text{if } s \text{ odd, } m \text{ odd, } \operatorname{char} K \neq 2 \text{ and } r = 0, \\ ms+1 & \text{if } s \text{ even and } r = 0, \text{ if } m \text{ even and } r = 0, \text{ or} \\ & \quad \text{if } \operatorname{char} K = 2 \text{ and } r = 0, \\ s(ms+1) & \text{if } s \text{ odd, } m \text{ odd, } \operatorname{char} K \neq 2 \text{ and } r = 1, \\ (s+1)(ms+1)+2 & \text{if } s \text{ even and } r = 1, \text{ if } m \text{ even and } r = 1, \text{ or} \\ & \quad \text{if } \operatorname{char} K = 2 \text{ and } r = 1, \\ s(ms+3) & \text{if } r = 2, \\ 0 & \text{otherwise.} \end{cases}$$

3.3. **The dimension formula for the Hochschild cohomology groups** $\mathrm{HH}^n(\Lambda_s)$**.** We have the following theorem.

Theorem 3.3. *Let $n = ms + r$ for integers $m \geq 0$ and $0 \leq r \leq s-1$. Then, for $s \geq 3$, we have the dimension formula for $\mathrm{HH}^n(\Lambda_s)$*:

$$\dim_K \mathrm{HH}^{ms+r}(\Lambda_s) = \begin{cases} ms+1 & \text{if } s \text{ even and } r = 0, \text{ if } m \text{ even and } r = 0, \text{ or} \\ & \quad \text{if } \operatorname{char} K = 2 \text{ and } r = 0, \\ 2ms+4 & \text{if } s \text{ even and } r = 1, \text{ if } m \text{ even and } r = 1, \text{ or} \\ & \quad \text{if } \operatorname{char} K = 2 \text{ and } r = 1, \\ ms+3 & \text{if } s \text{ even and } r = 2, \text{ if } m \text{ even and } r = 2, \text{ or} \\ & \quad \text{if } \operatorname{char} K = 2 \text{ and } r = 2, \\ 0 & \text{otherwise.} \end{cases}$$

4. The Hochschild cohomology ring modulo nilpotence $\mathrm{HH}^*(\Lambda_s)/\mathcal{N}_s$

Recall that the Hochschild cohomology ring of the algebra Λ_s is defined to be the graded ring

$$\mathrm{HH}^*(\Lambda_s) := \operatorname{Ext}^*_{\Lambda_s^e}(\Lambda_s, \Lambda_s) = \bigoplus_{t \geq 0} \operatorname{Ext}^t_{\Lambda_s^e}(\Lambda_s, \Lambda_s)$$

with the Yoneda product. Denote $\mathcal{N}_{\Lambda_s}$ by the ideal generated by all homogeneous nilpotent elements in $\mathrm{HH}^*(\Lambda_s)$. Then the quotient algebra $\mathrm{HH}^*(\Lambda_s)/\mathcal{N}_{\Lambda_s}$ is called the Hochschild cohomology ring modulo nilpotence of Λ_s. Note that $\mathrm{HH}^*(\Lambda_s)/\mathcal{N}_{\Lambda_s}$ is a commutative graded algebra (see [SnSo]). Our purpose of this section is to find generators and relations of $\mathrm{HH}^*(\Lambda_s)/\mathcal{N}_{\Lambda_s}$ for $s \geq 3$. For simplicity, we denote the graded subalgebras $\bigoplus_{t \geq 0} \mathrm{HH}^{st}(\Lambda_s)$ of $\mathrm{HH}^*(\Lambda_s)$ by $\mathrm{HH}^{s*}(\Lambda_s)$ and $\bigoplus_{t \geq 0} \mathrm{HH}^{2st}(\Lambda_s)$ by $\mathrm{HH}^{2s*}(\Lambda_s)$. Also, we denote the Yoneda product in $\mathrm{HH}^*(\Lambda_s)$ by $\times$.

Theorem 4.1. *For $s \geq 3$, there are the following isomorphisms of commutative graded algebras*:

(i) *If s is odd and* $\operatorname{char} K \neq 2$, *then*

$$\begin{aligned}\mathrm{HH}^*(\Lambda_s)/\mathcal{N}_{\Lambda_s} &\cong \mathrm{HH}^{2s*}(\Lambda_s)\\ &\cong K[z_0,\dots,z_{2s}]/\langle z_k z_l - z_q z_r \mid k+l=q+r,\ 0\le k,l,q,r\le 2s\rangle,\end{aligned}$$

where $z_0,\dots,z_{2s}$ are in degree $2s$.

(ii) *If s is even or* $\operatorname{char} K = 2$, *then*

$$\begin{aligned}\mathrm{HH}^*(\Lambda_s)/\mathcal{N}_{\Lambda_s} &\cong \mathrm{HH}^{s*}(\Lambda_s)\\ &\cong K[z_0,\dots,z_s]/\langle z_k z_l - z_q z_r \mid k+l=q+r,\ 0\le k,l,q,r\le s\rangle,\end{aligned}$$

where $z_0,\dots,z_s$ are in degree s.

Therefore, $\mathrm{HH}^*(\Lambda_s)/\mathcal{N}_{\Lambda_s}$ *is finitely generated as an algebra.*

We conclude this report with the following remarks.

Remark 4.2. Let $E(\Lambda_s) = \bigoplus_{i\ge 0}\operatorname{Ext}^i(\Lambda_s/\operatorname{rad}\Lambda_s, \Lambda_s/\operatorname{rad}\Lambda_s)$ be the Ext algebra of Λ_s, and let $Z_{gr}(E(\Lambda_s))$ be the graded center of $E(\Lambda_s)$ (see [BGSS], for example). Denote by $\mathcal{N}'_{\Lambda_s}$ the ideal of $Z_{gr}(E(\Lambda_s))$ generated by all homogeneous nilpotent elements. Since Λ_s is a Koszul algebra by Proposition 2.2, it follows by [BGSS] that $Z_{gr}(E(\Lambda_s))/\mathcal{N}'_{\Lambda_s} \cong \mathrm{HH}^*(\Lambda_s)/\mathcal{N}_{\Lambda_s}$ as graded rings. Therefore, we have the same presentation of $Z_{gr}(E(\Lambda_s))/\mathcal{N}'_{\Lambda_s}$ by generators and relations as that in Theorem 4.1.

Remark 4.3. In [F], Furuya has discussed the Hochschild cohomology of some self-injective special biserial algebra A_T for $T \ge 0$, and in particular he has given a presentation of $\mathrm{HH}^*(A_T)/\mathcal{N}_{A_T}$ by generators and relations in the case $T = 0$. We easily see that the algebra Λ_4 is isomorphic to A_0. By setting $s = 4$ in Theorem 4.1, our presentation actually coincides with that in [F, Theorem 4.1].

References

[BGSS] R.-O. Buchweitz, E. L. Green, N. Snashall and Ø. Solberg, Multiplicative structures for Koszul algebras, Q. J. Math. 59 (2008), no. 4, 441-454.

[EHTSS] K. Erdmann, M. Holloway, R. Taillefer, N. Snashall and Ø. Solberg, Support varieties for selfinjective algebras, K-Theory 33 (2004), no. 1, 67-87.

[F] T. Furuya, Hochschild cohomology for a class of some self-injective special biserial algebras of rank four, arXiv: 1403.6375.

[FO] T. Furuya and D. Obara, Hochschild cohomology of a class of weakly symmetric algebras with radical cube zero, SUT J. Math. 48 (2012), no. 2, 117-143.

[GHMS] E.L. Green, G. Hartmann, E. N. Marcos and Ø. Solberg, Resolutions over Koszul algebras, Arch. Math. (Basel) 85 (2005), 118-127.

[GSZ] E.L. Green, Ø. Solberg and D. Zacharia, Minimal projective resolutions, Trans. Amer. Math. Soc. 353 (2001), 2915-2939.

[I] A. Itaba, On Hochschild cohomology of a self-injective special biserial algebra obtained by a circular quiver with double arrows, arXiv: 1404.2032.

[S] N. Snashall, Support varieties and the Hochschild cohomology ring modulo nilpotence, Proceedings of the 41st Symposium on Ring Theory and Representation Theory, 68-82, Symp. Ring Theory Represent. Theory Organ. Comm., Tsukuba, 2009.

[ScSn] S. Schroll and N. Snashall, Hochschild cohomology and support varieties for tame Hecke algebras, Q. J. Math. 62 (2011), no. 4, 1017-1029.

[SnSo] N. Snashall and Ø. Solberg, Support varieties and Hochschild cohomology rings, Proc. London Math. 81 (2004), 705-732.

[ST] N. Snashall and R. Taillefer, The Hochschild cohomology ring of a class of special biserial algebras, J. Algebra Appl. 9 (2010), no. 1, 73-122.

[XH] Y. Xu and Y. Han, Hochschild (co)homology of exterior algebras, Comm. Algebra 35 (2007), no. 1, 115-131.

Ayako Itaba, Department of Mathematics, Tokyo University of Science, Kagurazaka 1-3, Shinjuku, Tokyo 162-0827, Japan

E-mail address: `j1110701@ed.tus.ac.jp`