SPECIALIZATION ORDERS ON ATOM SPECTRA OF GROTHENDIECK CATEGORIES

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Abstract. This report is a survey of our result in [Kan13]. We introduce systematic methods to construct Grothendieck categories from colored quivers and develop a theory of the specialization orders on the atom spectra of Grothendieck categories. We showed that any partially ordered set is realized as the atom spectrum of some Grothendieck category, which is an analog of Hochster's result in commutative ring theory. In this report, we explain techniques in the proof by using examples.

1. Introduction

This report is a survey of our result in [Kan13].

There are important Grothendieck categories appearing in representation theory of rings and algebraic geometry: the category $\operatorname{Mod}\Lambda$ of (right) modules over a ring Λ, the category $\operatorname{QCoh}X$ of quasi-coherent sheaves on a scheme X ([Con00, Lem 2.1.7]), and the category of quasi-coherent sheaves on a noncommutative projective scheme introduced by Verevkin [Ver92] and Artin and Zhang [AZ94]. Furthermore, by using the Gabriel-Popescu embedding ([PG64, Proposition]), it is shown that any Grothendieck category can be obtained as the quotient category of the category of modules over some ring by some localizing subcategory.

In commutative ring theory, Hochster characterized the topological spaces appearing as the prime spectra of commutative rings with Zariski topologies ([Hoc69, Theorem 6 and Proposition 10]). Speed [Spe72] pointed out that Hochster's result gives the following characterization of the partially ordered sets appearing as the prime spectra of commutative rings.

Theorem 1.1 (Hochster [Hoc69, Proposition 10] and Speed [Spe72, Corollary 1]). *Let P be a partially ordered set. Then P is isomorphic to the prime spectrum of some commutative ring with the inclusion relation if and only if P is an inverse limit of finite partially ordered sets in the category of partially ordered sets.*

We showed a theorem of the same type for Grothendieck categories. In [Kan12a] and [Kan12b], we investigated Grothendieck categories by using the *atom spectrum* $\operatorname{ASpec}\mathcal{A}$ of a Grothendieck category $\mathcal{A}$. It is the set of equivalence classes of monoform objects, which generalizes the prime spectrum of a commutative ring.

In fact, our main result claims that any partially ordered set is realized as the atom spectrum of some Grothendieck categories.

Theorem 1.2. *Any partially ordered set is isomorphic to the atom spectrum of some Grothendieck category.*

In this report, we explain key ideas to show this theorem by using examples. For more details, we refer the reader to [Kan13].

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2. Atom spectrum

In this section, we recall the definition of atom spectrum and fundamental properties. Throughout this report, let $\mathcal{A}$ be a Grothendieck category. It is defined as follows.

Definition 2.1. An abelian category $\mathcal{A}$ is called a *Grothendieck category* if it satisfies the following conditions.

(1) $\mathcal{A}$ admits arbitrary direct sums (and hence arbitrary direct limits), and for every direct system of short exact sequences in $\mathcal{A}$, its direct limit is also a short exact sequence.
(2) $\mathcal{A}$ has a generator G, that is, every object in $\mathcal{A}$ is isomorphic to a quotient object of the direct sum of some copies of G.

Definition 2.2. A nonzero object H in $\mathcal{A}$ is called *monoform* if for any nonzero subobject L of H, there does not exist a nonzero subobject of H which is isomorphic to a subobject of H/L.

Monoform objects have the following properties.

Proposition 2.3. *Let H be a monoform object in $\mathcal{A}$. Then the following assertions hold.*

(1) *Any nonzero subobject of H is also monoform.*
(2) *H is uniform, that is, for any nonzero subobjects L_1 and L_2 of H, we have $L_1 \cap L_2 \neq 0$.*

Definition 2.4. For monoform objects H and H' in $\mathcal{A}$, we say that H is *atom-equivalent* to H' if there exists a nonzero subobject of H which is isomorphic to a subobject of H'.

Remark 2.5. The atom equivalence is an equivalence relation between monoform objects in $\mathcal{A}$ since any monoform object is uniform.

Now we define the notion of atoms, which was originally introduced by Storrer [Sto72] in the case of module categories.

Definition 2.6. Denote by $\operatorname{ASpec}\mathcal{A}$ the quotient set of the set of monoform objects in $\mathcal{A}$ by the atom equivalence. We call it the *atom spectrum* of $\mathcal{A}$. Elements of $\operatorname{ASpec}\mathcal{A}$ are called *atoms* in $\mathcal{A}$. The equivalence class of a monoform object H in $\mathcal{A}$ is denoted by $\overline{H}$.

The following proposition shows that the atom spectrum of a Grothendieck category is a generalization of the prime spectrum of a commutative ring.

Proposition 2.7. *Let R be a commutative ring. Then the map $\operatorname{Spec} R \to \operatorname{ASpec}(\operatorname{Mod} R)$ given by $\mathfrak{p} \mapsto \overline{(R/\mathfrak{p})}$ is a bijection.*

The notions of associated primes and support are also generalized as follows.

Definition 2.8. Let M be an object in $\mathcal{A}$.

(1) Define the *atom support* of M by
$$\operatorname{ASupp} M = \{\overline{H} \in \operatorname{ASpec}\mathcal{A} \mid H \text{ is a subquotient of } M\}.$$
(2) Define the set of *associated atoms* of M by
$$\operatorname{AAss} M = \{\overline{H} \in \operatorname{ASpec}\mathcal{A} \mid H \text{ is a subobject of } M\}.$$

The following proposition is a generalization of a proposition which is well known in the commutative ring theory.

Proposition 2.9. *Let $0 \to L \to M \to N \to 0$ be an exact sequence in $\mathcal{A}$. Then the following assertions hold.*

(1) $\operatorname{ASupp} M = \operatorname{ASupp} L \cup \operatorname{ASupp} N$.
(2) $\operatorname{AAss} L \subset \operatorname{AAss} M \subset \operatorname{AAss} L \cup \operatorname{AAss} N$.

A partial order on the atom spectrum is defined by using atom support.

Definition 2.10. Let α and β be atoms in $\mathcal{A}$. We write $\alpha \leq \beta$ if for any object M in $\mathcal{A}$ satisfying $\alpha \in \operatorname{ASupp} M$ also satisfies $\beta \in \operatorname{ASupp} M$.

Proposition 2.11. *The relation* $\leq$ *on* $\operatorname{ASpec}\mathcal{A}$ *is a partial order.*

In the case where $\mathcal{A}$ is the category of modules over a commutative ring R, the notion of associated atoms, atom support, and the partial order on the atom spectrum coincide with associated primes, support, and the inclusion relation between prime ideals, respectively, through the bijection in Proposition 2.7.

3. Construction of Grothendieck categories

In order to construct Grothendieck categories, we use colored quivers.

Definition 3.1. A *colored quiver* is a sextuple $\Gamma = (Q_0, Q_1, C, s, t, u)$ satisfying the following conditions.

(1) Q_0, Q_1, and C are sets, and $s\colon Q_1 \to Q_0$, $t\colon Q_1 \to Q_0$, and $u\colon Q_1 \to C$ are maps.
(2) For each $v \in Q_0$ and $c \in C$, the number of arrows r satisfying $s(r) = v$ and $u(r) = c$ is finite.

We regard the colored quiver Γ as the quiver (Q_0, Q_1, s, t) with the *color* $u(r)$ on each arrow $r \in Q_1$.

From now on, we fix a field K. From a colored quiver, we construct a Grothendieck category as follows.

Definition 3.2. Let $\Gamma = (Q_0, Q_1, C, s, t, u)$ be a colored quiver. Denote a free K-algebra on C by $S_C = K\langle s_c \mid c \in C\rangle$. Define a K-vector space M_Γ by $M_\Gamma = \bigoplus_{v \in Q_0} F_v$, where $F_v = x_v K$ is a one-dimensional K-vector space generated by an element x_v. Regard M_Γ as a right S_C-module by defining the action of $s_c \in S_C$ as follows: for each vertex v in Q,

$$x_v \cdot s_c = \sum_r x_{t(r)},$$

where r runs over all the arrows $r \in Q_1$ with $s(r) = v$ and $u(r) = c$. Denote by $\mathcal{A}_\Gamma$ the smallest full subcategory of $\operatorname{Mod} S_C$ which contains M_Γ and is closed under submodules, quotient modules, and direct sums.

The category $\mathcal{A}_\Gamma$ defined above is a Grothendieck category. The following proposition is useful to describe the atom spectrum of $\mathcal{A}_\Gamma$.

Proposition 3.3. *Let* $\Gamma = (Q_0, Q_1, C, s, t, u)$ *be a colored quiver. Then* $\operatorname{ASpec}\mathcal{A}_\Gamma$ *is isomorphic to the subset* $\operatorname{ASupp} M_\Gamma$ *of* $\operatorname{ASpec}(\operatorname{Mod} S_C)$ *as a partially ordered set.*

Example 3.4. Define a colored quiver $\Gamma = (Q_0, Q_1, C, s, t, u)$ by $Q_0 = \{v, w\}$, $Q_1 = \{r\}$, $C = \{c\}$, $s(r) = v$, $t(r) = w$, and $u(r) = c$. This is illustrated as

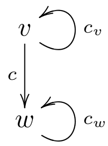
$$\begin{array}{c} v \\ {\scriptstyle c}\big\downarrow \\ w \end{array} .$$

Then we have $S_C = K\langle s_c\rangle = K[s_c]$, $M_\Gamma = x_v K \oplus x_w K$ as a K-vector space, and $x_v s_c = x_w$, $x_w s_c = 0$. The subspace $L = x_w K$ of M_Γ is a simple S_C-submodule, and L is isomorphic to M_Γ / L as an S_C-module. Hence we have

$$\operatorname{ASpec}\mathcal{A}_\Gamma = \operatorname{ASupp} M_\Gamma = \operatorname{ASupp} L \cup \operatorname{ASupp}\frac{M_\Gamma}{L} = \{\overline{L}\}.$$

The next example explains the way to distinguish simple modules corresponding different vertices.

Example 3.5. Let $\Gamma = (Q_0, Q_1, C, s, t, u)$ be the colored quiver



and let $N = x_v K$ and $L = x_w K$. Then we have an exact sequence

$$0 \to L \to M_\Gamma \to N \to 0$$

of K-vector spaces and this can be regarded as an exact sequence in $\operatorname{Mod} S_C$. Hence we have

$$\operatorname{ASpec} \mathcal{A}_\Gamma = \operatorname{ASupp} M_\Gamma = \operatorname{ASupp} L \cup \operatorname{ASupp} N = \{\overline{L}, \overline{N}\},$$

where $\overline{L} \neq \overline{N}$.

In order to realize a partially ordered set with nontrivial partial order, we use an infinite colored quiver.

Example 3.6. Let $\Gamma = (Q_0, Q_1, C, s, t, u)$ be the colored quiver

$$v_0 \xrightarrow{c_0} v_1 \xrightarrow{c_1} \cdots .$$

Let L be the simple S_C-module defined by $L = K$ as a K-vector space and $Ls_{c_i} = 0$ for each $i \in \mathbb{Z}_{\geq 0}$. Then we have $\operatorname{ASpec} \mathcal{A}_\Gamma = \{\overline{M_\Gamma}, \overline{L}\}$, where $\overline{M_\Gamma} < \overline{L}$.

Definition 3.7. For a colored quiver $\Gamma = (Q_0, Q_1, C, s, t, u)$, define the colored quiver $\widetilde{\Gamma} = (\widetilde{Q}_0, \widetilde{Q}_1, \widetilde{C}, \widetilde{s}, \widetilde{t}, \widetilde{u})$ as follows.

(1) $\widetilde{Q}_0 = \mathbb{Z}_{\geq 0} \times Q_0$.
(2) $\widetilde{Q}_1 = (\mathbb{Z}_{\geq 0} \times Q_1) \amalg \{r^i_{v,v'} \mid i \in \mathbb{Z}_{\geq 0},\ v, v' \in Q_0\}$.
(3) $\widetilde{C} = C \amalg \{c^i_{v,v'} \mid i \in \mathbb{Z}_{\geq 0},\ v, v' \in Q_0\}$.
(4) (a) For each $\widetilde{r} = (i, r) \in \mathbb{Z}_{\geq 0} \times Q_1 \subset \widetilde{Q}_1$, let $\widetilde{s}(\widetilde{r}) = (i, s(r))$, $\widetilde{t}(\widetilde{r}) = (i, t(r))$, and $\widetilde{u}(\widetilde{r}) = u(r)$.
   (b) For each $\widetilde{r} = r^i_{v,v'} \in \widetilde{Q}_1$, let $\widetilde{s}(\widetilde{r}) = (i, v)$, $\widetilde{t}(\widetilde{r}) = (i+1, v')$, and $\widetilde{u}(\widetilde{r}) = c^i_{v,v'}$.

The colored quiver $\widetilde{\Gamma}$ is represented by the diagram

$$\Gamma \Longrightarrow \Gamma \Longrightarrow \cdots .$$

Lemma 3.8. *Let Γ be a colored quiver. Let $\widetilde{\Gamma} = (\widetilde{Q}_0, \widetilde{Q}_1, \widetilde{C}, \widetilde{s}, \widetilde{t}, \widetilde{u})$ be the colored quiver*

$$\Gamma \Longrightarrow \Gamma \Longrightarrow \cdots .$$

Then we have

$$\operatorname{ASpec} \mathcal{A}_{\widetilde{\Gamma}} = \{\overline{M_{\widetilde{\Gamma}}}\} \amalg \operatorname{ASpec} \mathcal{A}_\Gamma$$

as a subset of $\operatorname{ASpec}(\operatorname{Mod} S_{\widetilde{C}})$, *where* $\overline{M_{\widetilde{\Gamma}}}$ *is the smallest element of* $\operatorname{ASpec} \mathcal{A}_{\widetilde{\Gamma}}$.

We refer the reader to [Kan13] for the explicit construction of Grothendieck categories to show Theorem 1.2.

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