CLIFFORD EXTENSIONS

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Abstract. In this note, we generalize the construction of Clifford algebras and introduce the notion of Clifford extensions. Clifford extensions are constructed as Frobenius extensions which are Auslander-Gorenstein rings if so is a base ring.

Key Words: Auslander-Gorenstein ring, Clifford extension, Frobenius extension.

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Clifford algebras play important roles in various fields and the construction of Clifford algebras contains that of complex numbers, quaternions, and so on (see e.g. [6]). In this note, we generalize the construction of Clifford algebras and introduce the notion of Clifford extensions. Clifford extensions are constructed as Frobenius extensions, the notion of which we recall below, and we have already known that Frobenius extensions of Auslander-Gorenstein rings (see Definition 2) are also Auslander-Gorenstein rings. It should be noted that little is known about constructions of Auslander-Gorenstein rings although Auslander-Gorenstein rings appear in various fields of current research in mathematics including noncommutative algebraic geometry, Lie algebras, and so on (see e.g. [2], [3], [4] and [11]).

Recall the notion of Frobenius extensions of rings due to Nakayama and Tsuzuku [9, 10] which we modify as follows (cf. [1, Section 1]). We use the notation A/R to denote that a ring A contains a ring R as a subring. We say that A/R is a Frobenius extension if the following conditions are satisfied: (F1) A is finitely generated as a left R-module; (F2) A is finitely generated projective as a right R-module; (F3) there exists an isomorphism $\phi : A \overset{\sim}{\to} \mathrm{Hom}_R(A, R)$ in Mod-A. Note that ϕ induces a unique ring homomorphism $\theta : R \to A$ such that $x\phi(1) = \phi(1)\theta(x)$ for all $x \in R$. A Frobenius extension A/R is said to be of first kind if $A \cong \mathrm{Hom}_R(A, R)$ as R-A-bimodules, and to be of second kind if there exists an isomorphism $\phi : A \overset{\sim}{\to} \mathrm{Hom}_R(A, R)$ in Mod-A such that the associated ring homomorphism $\theta : R \to A$ induces a ring automorphism of R. Note that a Frobenius extension of first kind is a special case of a Frobenius extension of second kind. Let A/R be a Frobenius extension. Then A is an Auslander-Gorenstein ring if so is R, and the converse holds true if A is projective as a left R-module, and if A/R is split, i.e., the inclusion $R \to A$ is a split monomorphism of R-R-bimodules. Note that A is projective as a left R-module if A/R is of second kind.

Let $n \geq 2$ be an integer. We fix a set of integers $I = \{0, 1, \dots, n-1\}$ and a ring R. First, we construct a split Frobenius extension Λ/R of second kind using a certain pair (σ, c) of $\sigma \in \mathrm{Aut}(R)$ and $c \in R$. Namely, we define an appropriate multiplication on a free right R-module Λ with a basis $\{v_i\}_{i \in I}$. We show that this construction can be iterated

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arbitrary times (Proposition 11). Then we deal with the case where $n = 2$ and study the iterated Frobenius extensions. For $m \geq 1$ we construct ring extensions Λ_m/R using the following data: a sequence of elements $c_1, c_2, \cdots$ in $\mathrm{Z}(R)$ and signs $\varepsilon(i,j)$ for $1 \leq i,j \leq m$. Namely, we define an appropriate multiplication on a free right R-module Λ_m with a basis $\{v_x\}_{x \in I^m}$. We show that Λ_m is obtained by iterating the construction above m times, that Λ_m/R is a split Frobenius extension of first kind, and that if $c_i \in \mathrm{rad}(R)$ for $1 \leq i \leq m$ then $R/\mathrm{rad}(R) \xrightarrow{\sim} \Lambda_m/\mathrm{rad}(\Lambda_m)$ (Theorem 13). We call Λ_m Clifford extensions of R because they have the following properties similar to Clifford algebras. For each $x = (x_1, \ldots, x_m) \in I^m$ we set $S(x) = \{i \mid x_i = 1\}$. Also we set $v_x = t_i$ for $x \in I^m$ with $S(x) = \{i\}$. Then the following hold: (C1) $t_i^2 = v_0 c_i$ for all $1 \leq i \leq m$; (C2) $t_i t_j + t_j t_i = 0$ unless $i = j$; (C3) $v_x = t_{i_1} \cdots t_{i_r}$ if $S(x) = \{i_1, \ldots, i_r\}$ with $i_1 < \cdots < i_r$.

1. PRELIMINARIES

For a ring R we denote by $\mathrm{rad}(R)$ the Jacobson radical of R, by $R^\times$ the set of units in R, by $\mathrm{Z}(R)$ the center of R and by $\mathrm{Aut}(R)$ the group of ring automorphisms of R. Usually, the identity element of a ring is simply denoted by 1. Sometimes, we use the notation 1_R to stress that it is the identity element of the ring R. We denote by Mod-R the category of right R-modules. Left R-modules are considered as right R^{op}-modules, where R^{op} denotes the opposite ring of R. In particular, we denote by inj dim R (resp., inj dim R^{op}) the injective dimension of R as a right (resp., left) R-module and by $\mathrm{Hom}_R(-,-)$ (resp., $\mathrm{Hom}_{R^{\mathrm{op}}}(-,-)$) the set of homomorphisms in Mod-R (resp., Mod-R^{op}). Sometimes, we use the notation X_R (resp., ${}_RX$) to stress that the module X considered is a right (resp., left) R-module.

We start by recalling the notion of Auslander-Gorenstein rings.

Proposition 1 (Auslander)**.** *Let R be a right and left noetherian ring. Then for any $n \geq 0$ the following are equivalent.*

1. *In a minimal injective resolution $I^\bullet$ of R in* Mod-R*,* flat dim $I^i \leq i$ *for all* $0 \leq i \leq n$.
2. *In a minimal injective resolution $J^\bullet$ of R in* Mod-R^{op}*,* flat dim $J^i \leq i$ *for all* $0 \leq i \leq n$.
3. *For any $1 \leq i \leq n+1$, any $M \in$* mod-R *and any submodule X of* $\mathrm{Ext}^i_R(M,R) \in$ mod-R^{op} *we have* $\mathrm{Ext}^j_{R^{\mathrm{op}}}(X,R) = 0$ *for all* $0 \leq j < i$.
4. *For any $1 \leq i \leq n+1$, any $X \in$* mod-R^{op} *and any submodule M of* $\mathrm{Ext}^i_{R^{\mathrm{op}}}(X,R) \in$ mod-R *we have* $\mathrm{Ext}^j_R(M,R) = 0$ *for all* $0 \leq j < i$.

Proof. See e.g. [5, Theorem 3.7]. □

Definition 2 ([4])**.** A right and left noetherian ring R is said to satisfy the Auslander condition if it satisfies the equivalent conditions in Proposition 1 for all $n \geq 0$, and to be an Auslander-Gorenstein ring if it satisfies the Auslander condition and inj dim $R =$ inj dim $R^{\mathrm{op}} < \infty$.

It should be noted that for a right and left noetherian ring R we have inj dim $R =$ inj dim R^{op} whenever inj dim $R < \infty$ and inj dim $R^{\mathrm{op}} < \infty$ (see [12, Lemma A]).

Next, we recall the notion of Frobenius extensions of rings due to Nakayama and Tsuzuku [9, 10], which we modify as follows.

Definition 3 ([7])**.** A ring A is said to be an extension of a ring R if A contains R as a subring, and the notation A/R is used to denote that A is an extension ring of R. A ring extension A/R is said to be Frobenius if the following conditions are satisfied:

(F1) A is finitely generated as a left R-module;
(F2) A is finitely generated projective as a right R-module;
(F3) $A \cong \mathrm{Hom}_R(A, R)$ as right A-modules.

In case R is a right and left noetherian ring, for any Frobenius extension A/R the isomorphism $A \xrightarrow{\sim} \mathrm{Hom}_R(A, R)$ in Mod-A yields an Auslander-Gorenstein resolution of A over R in the sense of [8, Definition 3.5].

The next proposition is well-known and easily verified.

Proposition 4. *Let A/R be a ring extension and $\phi : A \xrightarrow{\sim} \mathrm{Hom}_R(A, R)$ an isomorphism in* Mod-A*. Then the following hold.*

1. *There exists a unique ring homomorphism $\theta : R \to A$ such that $x\phi(1) = \phi(1)\theta(x)$ for all $x \in R$.*
2. *If $\phi' : A \xrightarrow{\sim} \mathrm{Hom}_R(A, R)$ is another isomorphism in* Mod-A*, then there exists $u \in A^\times$ such that $\phi'(1) = \phi(1)u$ and $\theta'(x) = u^{-1}\theta(x)u$ for all $x \in R$.*
3. *ϕ is an isomorphism of R-A-bimodules if and only if $\theta(x) = x$ for all $x \in R$.*

Definition 5 (cf. [9, 10])**.** A Frobenius extension A/R is said to be of first kind if $A \cong \mathrm{Hom}_R(A, R)$ as R-A-bimodules, and to be of second kind if there exists an isomorphism $\phi : A \xrightarrow{\sim} \mathrm{Hom}_R(A, R)$ in Mod-A such that the associated ring homomorphism $\theta : R \to A$ induces a ring automorphism $\theta : R \xrightarrow{\sim} R$.

Proposition 6 ([7, Proposition 1.6])**.** *If A/R is a Frobenius extension of second kind, then A is projective as a left R-module.*

Proposition 7 ([7, Proposition 1.7])**.** *For any Frobenius extensions Λ/A, A/R the following hold.*

1. *Λ/R is a Frobenius extension.*
2. *Assume Λ/A is of first kind. If A/R is of second (resp., first) kind, then so is Λ/R.*

Definition 8 ([1])**.** A ring extension A/R is said to be split if the inclusion $R \to A$ is a split monomorphism of R-R-bimodules.

Proposition 9 ([7, Proposition 1.9])**.** *For any Frobenius extension A/R the following hold.*

1. *If R is an Auslander-Gorenstein ring, then so is A with* $\mathrm{inj\ dim}\, A \le \mathrm{inj\ dim}\, R$.
2. *Assume A is projective as a left R-module and A/R is split. If A is an Auslander-Gorenstein ring, then so is R with* $\mathrm{inj\ dim}\, R = \mathrm{inj\ dim}\, A$.

2. Construction of Frobenius extensions

Throughout this section, we fix a set of integers $I = \{0, 1, \ldots, n-1\}$ with $n \geq 2$ arbitrary and a ring R together with a pair (σ, c) of $\sigma \in \mathrm{Aut}(R)$ and $c \in R$ satisfying the following condition:

$$(*) \quad \sigma^n = \mathrm{id}_R \quad \text{and} \quad c \in R^\sigma \cap \mathrm{Z}(R).$$

This is obviously satisfied if $\sigma = \mathrm{id}_R$ and $c \in \mathrm{Z}(R)$.

Let Λ be a free right R-module with a basis $\{v_i\}_{i\in I}$ and $\{\delta_i\}_{i\in I}$ the dual basis of $\{v_i\}_{i\in I}$ for the free left R-module $\mathrm{Hom}_R(\Lambda, R)$, i.e., $\lambda = \sum_{i\in I} v_i\delta_i(\lambda)$ for all $\lambda \in \Lambda$. We set

$$v_{i+kn} = v_i c^k$$

for $i \in I$ and $k \in \mathbb{Z}_+$, the set of non-negative integers, and define a multiplication on Λ subject to the following axioms:

(L1) $v_i v_j = v_{i+j}$ for all $i, j \in I$;

(L2) $a v_i = v_i \sigma^i(a)$ for all $a \in R$ and $i \in I$.

Lemma 10. *The following hold.*

1. $v_i v_j = v_j v_i$ *for all* $i, j \in I$ *and* $v_i^n = v_0 c^i$ *for all* $i \in I$.
2. *For any* $\lambda, \mu \in \Lambda$ *we have* $\lambda\mu = \sum_{i,j\in I} v_{i+j}\sigma^j(\delta_i(\lambda))\delta_j(\mu)$ *and hence* $\delta_0(\lambda\mu) = \delta_0(\lambda)\delta_0(\mu) + \sum_{i\in I\setminus\{0\}} \sigma^{n-i}(\delta_i(\lambda))\delta_{n-i}(\mu)c$.
3. *For any* $\lambda \in \Lambda$ *and* $i, j \in I$ *we have* $\delta_i(\lambda v_j) = \sigma^j(\delta_{i-j}(\lambda))$ *if* $i \geq j$ *and* $\delta_i(\lambda v_j) = \sigma^j(\delta_{i-j+n}(\lambda))c$ *if* $i < j$.

Proposition 11. *The following hold.*

1. Λ *is an associative ring with* $1 = v_0$ *and contains* R *as a subring via the injective ring homomorphism* $R \to \Lambda, a \mapsto v_0 a$.
2. Λ/R *is a split Frobenius extension of second kind.*
3. *If* $c \in \mathrm{rad}(R)$, *then* $R/\mathrm{rad}(R) \xrightarrow{\sim} \Lambda/\mathrm{rad}(\Lambda)$.
4. *For any* $\varepsilon \in R^\sigma \cap \mathrm{Z}(R)$ *with* $\varepsilon^n = 1$ *there exists* $\tilde{\sigma} \in \mathrm{Aut}(\Lambda)$ *such that* $\delta_i(\tilde{\sigma}(\lambda)) = \sigma(\delta_i(\lambda))\varepsilon^i$ *for all* $\lambda \in \Lambda$ *and* $i \in I$, *and for any* $c' \in R^\sigma \cap \mathrm{Z}(R)$ *the pair* $(\tilde{\sigma}, c')$ *satisfies the condition (*).*

It should be noted that Proposition 11(4) enables us to iterate the construction above arbitrary times. For instance, one may start from $\sigma = \mathrm{id}_R$.

Remark 12. Let $R[t;\sigma]$ be a right skew polynomial ring with trivial derivation, i.e., $R[t;\sigma]$ consists of all polynomials in an indeterminate t with right-hand coefficients in R and the multiplication is defined by the following rule: $at = t\sigma(a)$ for all $a \in R$. Then $(t^n - c) = (t^n - c)R[t;\sigma]$ is a two-sided ideal and the residue ring $R[t;\sigma]/(t^n - c)$ is isomorphic to Λ.

In the next section, we will deal with the case where $n = 2$ and denote by $Cl_1(R;\sigma, c)$ the ring Λ constructed above.

3. Clifford extensions

In this section, we fix a set of integers $I = \{0, 1\}$ and a ring R together with a sequence of elements $c_1, c_2, \dots$ in $\mathrm{Z}(R)$. Setting $0 + i = i + 0 = i$ for all $i \in I$ and $1 + 1 = 0$, we consider I as a cyclic group of order 2. For any $n \geq 1$ we denote by I^n the direct product of n copies of I and consider I^{n-1} as a subgroup of I^n via the injective group homomorphism

$$I^{n-1} \to I^n, (x_1, \dots, x_{n-1}) \mapsto (x_1, \dots, x_{n-1}, 0),$$

where $I^0 = \{0\}$ is the trivial group. According to Proposition 11(4), one can construct inductively various I^n-graded rings which are Frobenius extensions of R. However, in this note we restrict ourselves to the following particular case.

Let $n \geq 1$. For each $x = (x_1, \ldots, x_n) \in I^n$ we set $S(x) = \{i \mid x_i = 1\}$. Note that $S(x+y) = S(x) + S(y)$, the symmetric difference of $S(x)$ and $S(y)$, for all $x, y \in I^n$. We set

$$\varepsilon(i,j) = \begin{cases} +1 & \text{if } i \leq j, \\ -1 & \text{if } i > j \end{cases}$$

for $1 \leq i, j \leq n$ and

$$c(x,y) = \prod_{(i,j) \in S(x) \times S(y)} \varepsilon(i,j) \prod_{k \in S(x) \cap S(y)} c_k$$

for $x, y \in I^n$. We denote by s the element $x \in I^n$ with $S(x) = \{1, \ldots, n\}$.

Let Λ_n be a free right R-module with a basis $\{v_x\}_{x \in I^n}$. We denote by $\{\delta_x\}_{x \in I^n}$ the dual basis of $\{v_x\}_{x \in I^n}$ for the free left R-module $\mathrm{Hom}_R(\Lambda_n, R)$, i.e., $\lambda = \sum_{x \in I^n} v_x \delta_x(\lambda)$ for all $\lambda \in \Lambda_n$. We define a multiplication on Λ_n subject to the following axioms:

(M1) $v_x v_y = v_{x+y} c(x,y)$ for all $x, y \in I^n$;
(M2) $a v_x = v_x a$ for all $x \in I^n$ and $a \in R$.

In the following, we set $v_x = t_i$ for $x \in I^n$ with $S(x) = \{i\}$. It is easy to see the following:

(C1) $t_i^2 = v_0 c_i$ for all $1 \leq i \leq n$;
(C2) $t_i t_j + t_j t_i = 0$ unless $i = j$;
(C3) $v_x = t_{i_1} \cdots t_{i_r}$ if $S(x) = \{i_1, \ldots, i_r\}$ with $i_1 < \cdots < i_r$.

Theorem 13. *For any $n \geq 1$ the following hold.*

1. *Λ_n is an associative ring with $1 = v_0$ and contains R as a subring via the injective ring homomorphism $R \to \Lambda_n, a \mapsto v_0 a$.*
2. *Λ_n/R is a split Frobenius extension of first kind.*
3. *If $c_i \in \mathrm{rad}(R)$ for all $1 \leq i \leq n$, then $R/\mathrm{rad}(R) \xrightarrow{\sim} \Lambda_n/\mathrm{rad}(\Lambda_n)$.*

Remark 14. If $d(x,y) = |S(x) \times S(y)| - |S(x) \cap S(y)|$ is even, then $v_x v_y = v_y v_x$. In particular, $v_s \in \mathrm{Z}(\Lambda_n)$ if n is odd.

Denote by J^n the subset of I^n consisting of all $x \in I^n$ with $|S(x)|$ even. Then J^n is a subgroup of I^n and $\Lambda_n^0 = \oplus_{x \in J^n} v_x R$ is a subring of Λ_n.

Proposition 15. *Assume n is even. Then $v_s \in \Lambda_n^0$ and the following hold.*

1. *Λ_n^0/R is a split Frobenius extension of first kind.*
2. *If $c_i \in \mathrm{rad}(R)$ for all $1 \leq i \leq n$, then $R/\mathrm{rad}(R) \xrightarrow{\sim} \Lambda_n^0/\mathrm{rad}(\Lambda_n^0)$.*

We denote by $Cl_n(R; c_1, \ldots, c_n)$ (resp., $Cl_n^0(R; c_1, \ldots, c_n)$) the ring Λ_n (resp., Λ_n^0) constructed above, which we call Clifford extensions of R.

Remark 16. If $c_i \in R^\times$ for some $1 \leq i \leq n$, then $Cl_n^0(R; c_1, \ldots, c_n)/R$ is a split Frobenius extension of first kind.

Example 17. Let K be a commutative field and V a 3-dimensional K-space. Then $Cl_3^0(K;0,0,0) \cong K \ltimes V$, the trivial extension of K by V, which is not a Frobenius algebra.

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