THE ALMOST GORENSTEIN PROPERTY OF ASSOCIATED GRADED RINGS

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1. Almost Gonrestein associated graded rings

This paper purposes to expore the question of how the almost Gorenstein property of base local rings is inherited from that of the associated graded rings. Let us begin with our definition.

Definition 1.1. ([GTT]) Let R be a Noetherian local ring with maximal ideal $\mathfrak{m}$. Then R is said to be an almost Gorenstein local ring, if the following conditions are satisfied.

(1) R is a Cohen-Macaulay local ring, which possesses the canonical module K_R and
(2) there exists an exact sequence

$$0 \to R \to \mathrm{K}_R \to C \to 0$$

of R-modules such that $\mu_R(C) = \mathrm{e}^0_{\mathfrak{m}}(C)$.

Here $\mu_R(C)$ (resp. $\mathrm{e}^0_{\mathfrak{m}}(C)$) denotes the number of elements in a minimal system of generators for C (resp. the multiplicity of C with respect to $\mathfrak{m}$).

Similarly a Noetherian graded ring $R = \bigoplus_{n \ge 0} R_n$ with R_0 a local ring is called an almost Gorenstein graded ring, if R is a Cohen-Macaulay ring, possessing the graded canonical module K_R, such that there exists an exact sequence

$$0 \to R \to \mathrm{K}_R(-a) \to C \to 0$$

of graded R-modules with $\mu_R(C) = \mathrm{e}^0_{\mathfrak{M}}(C)$, where $a = \mathrm{a}(R)$ is the a–invariant of R and $\mathfrak{M}$ is the graded maximal ideal of R.

The main result of this paper is the following.

Theorem 1.2. *Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring with infinite residue class field, possessing the canonical module K_R of R. Let I be an $\mathfrak{m}$-primary ideal of R and $\mathrm{gr}_I(R) = \bigoplus_{n \ge 0} I^n/I^{n+1}$ the associated graded ring of I. If $\mathrm{gr}_I(R)$ is an almost Gorenstein graded ring with $\mathrm{r}(\mathrm{gr}_I(R))) = \mathrm{r}(R)$, then R is an almost Gorenstein local ring.*

Theorem 1.2 is reduced to the case where $\dim R = 1$ by induction on $\dim R$. We begin with the key result of dimension one. Let us fix our notation.

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Setting 1.3. Let R be a Cohen-Macaulay local ring of dimension one. We consider a filtration $\mathcal{F} = \{I_n\}_{n\in\mathbb{Z}}$ of ideals of R. Therefore $\{I_n\}_{n\in\mathbb{Z}}$ is a family of ideals of R which satisfies the following three conditions: (1) $I_0 = R$ but $I_1 \neq R$, (2) $I_n \supseteq I_{n+1}$ for all $n \in \mathbb{Z}$, and (3) $I_m I_n \subseteq I_{m+n}$ for all $m, n \in \mathbb{Z}$. We set

$$\mathcal{R} = \mathcal{R}(\mathcal{F}) = \sum_{n\geq 0} I_n t^n \subseteq R[t],$$

$$\mathcal{R}' = \mathcal{R}'(\mathcal{F}) = \sum_{n\in\mathbb{Z}} I_n t^n \subseteq R[t, t^{-1}], \quad \text{and}$$

$$G = G(\mathcal{F}) = \mathcal{R}'(\mathcal{F})/t^{-1}\mathcal{R}'(\mathcal{F}),$$

which we call them, the Rees algebra, the extended Rees algebra, and the associated graded ring of $\mathcal{F}$, respectively (here t stands for an indeterminate). We assume the following three conditions are satisfied:

(1) R is a homomorphic image of a Gorenstein ring,
(2) $\mathcal{R}$ is a Noetherian ring, and
(3) G is a Cohen-Macaulay ring.

Let $\mathrm{a}(G) = \max\{n \in \mathbb{Z} \mid [\mathrm{H}^1_{\mathfrak{M}}(G)]_n \neq (0)\}$ ([GW]), where $\{[\mathrm{H}^1_{\mathfrak{M}}(G)]_n\}_{n\in\mathbb{Z}}$ stands for the homogeneous components of the first graded local cohomology module $\mathrm{H}^1_{\mathfrak{M}}(G)$ of G with respect to the graded maximal ideal $\mathfrak{M} = \mathfrak{m}G + G_+$ of G. We set $c = \mathrm{a}(G) + 1$ and $K = \mathrm{K}_R$. Then by [GI, Theorem 1.1] we have a unique family $\omega = \{\omega_n\}_{n\in\mathbb{Z}}$ of R-submodules of K satisfying the following four conditions:

(i) $\omega_n \supseteq \omega_{n+1}$ for all $n \in \mathbb{Z}$,
(ii) $\omega_n = K$ for all $n \leq -c$,
(iii) $I_m \omega_n \subseteq \omega_{m+n}$ for all $m, n \in \mathbb{Z}$, and
(iv) $\mathrm{K}_{\mathcal{R}'} \cong \mathcal{R}'(\omega)$ and $\mathrm{K}_G \cong G(\omega)(-1)$ as graded $\mathcal{R}'$-modules,

where $\mathcal{R}'(\omega) = \sum_{n\in\mathbb{Z}} \omega_n t^n \subset K[t, t^{-1}]$ and $G(\omega) = \mathcal{R}'(\omega)/t^{-1}\mathcal{R}'(\omega)$, and $\mathrm{K}_{\mathcal{R}'}$ and K_G denote respectively the graded canonical modules of $\mathcal{R}'$ and G. Notice that $[G(\omega)]_n = (0)$ if $n < -c$ (see condition (ii)).

With this notation we have the following.

Lemma 1.4. *There exist integers $d > 0$ and $k \geq 0$ such that $\omega_{dn-c} = I_d^{n-k}\omega_{dk-c}$ for all $n \geq k$.*

Proof. Let $L = \mathcal{R}(\omega)(-c)$, where $\mathcal{R}(\omega) = \sum_{n\geq 0} \omega_n t^n \subseteq K[t]$. Then L is a finitely generated graded $\mathcal{R}$-module such that $L_n = (0)$ for $n < 0$. We choose an integer $d \gg 0$ so that the Veronesean subring $\mathcal{R}^{(d)} = \sum_{n\geq 0} \mathcal{R}_{dn}$ of $\mathcal{R}$ with order d is standard, whence $\mathcal{R}^{(d)} = R[\mathcal{R}_d]$. Then, because $L^{(d)} = \sum_{n\geq 0} L_{dn}$ is a finitely generated graded $\mathcal{R}^{(d)}$-module, we may choose a homogeneous system $\{f_i\}_{1\leq i\leq \ell}$ of generators of $L^{(d)}$ so that for each $1 \leq i \leq \ell$

$$f_i \in [L^{(d)}]_{k_i} = [\mathcal{R}(\omega)]_{dk_i - c}$$

with $k_i \geq \frac{c}{d}$. Setting $k = \max\{k_i \mid 1 \leq i \leq \ell\}$, for all $n \geq k$ we get

$$\omega_{dn-c} \subseteq \sum_{i=1}^{\ell} I_{d(n-k_i)}\omega_{dk_i-c} \subseteq I_d^{n-k}\omega_{dk-c},$$

as asserted. $\square$

Let us fix an element $f \in K$ and let $\xi = \overline{ft^{-c}} \in G(\omega)(-c)$ denote the image of $ft^{-c} \in \mathcal{R}'(\omega)$ in $G(\omega)$. Assume $(0) :_G \xi = (0)$ and consider the following short exact sequence

$$(E) \quad 0 \to G \xrightarrow{\psi} G(\omega)(-c) \to C \to 0,$$

of graded G-modules, where $\psi(1) = \xi$. Then $C_\mathfrak{p} = (0)$ for all $\mathfrak{p} \in AssG$, because $[G(\omega)]_\mathfrak{p} \cong [\mathrm{K}_G]_\mathfrak{p} \cong \mathrm{K}_{G_\mathfrak{p}}$ as $G_\mathfrak{p}$-modules by condition (iv) above and $\ell_{G_\mathfrak{p}}(G_\mathfrak{p}) = \ell_{G_\mathfrak{p}}(\mathrm{K}_{G_\mathfrak{p}})$ ([HK, Korollar 6.4]). Therefore $\ell_R(C) = \ell_G(C) < \infty$ since $\dim G = 1$, so that C is finitely graded. We now consider the exact sequence

$$\mathcal{R}' \xrightarrow{\varphi} \mathcal{R}'(\omega)(-c) \to D \to 0$$

of graded $\mathcal{R}'$-modules defined by $\varphi(1) = ft^{-c}$. Then $C \cong D/uD$ as a G-module, where $u = t^{-1}$. Notice that $\dim \mathcal{R}'/\mathfrak{p} = 2$ for all $\mathfrak{p} \in \mathrm{Ass}\mathcal{R}'$, because $\mathcal{R}'$ is a Cohen-Macaulay ring of dimension 2. We then have $D_\mathfrak{p} = (0)$ for all $\mathfrak{p} \in \mathrm{Ass}\mathcal{R}'$, since $\dim_{\mathcal{R}'} D \leq 1$. Hence the homomorphism φ is injective, because $\mathcal{R}'(\omega) \cong \mathrm{K}_{\mathcal{R}'}$ by condition (iv) and $\ell_{\mathcal{R}'_\mathfrak{p}}([\mathcal{R}'(\omega)]_\mathfrak{p}) = \ell_{\mathcal{R}'\mathfrak{p}}([\mathrm{K}_{\mathcal{R}'}]_\mathfrak{p}) = \ell_{\mathcal{R}'\mathfrak{p}}(\mathrm{K}_{\mathcal{R}'_\mathfrak{p}}) = \ell_{\mathcal{R}'_\mathfrak{p}}(\mathcal{R}'_\mathfrak{p})$ for all $\mathfrak{p} \in \mathrm{Ass}\mathcal{R}'$. The snake lemma shows u acts on D as a non-zerodivisor, since u acts on $\mathcal{R}'(\omega)$ as a non-zerodivisor.

Let us suppose that $C \neq (0)$ and set $S = \{n \in \mathbb{Z} \mid C_n \neq (0)\}$. We write $S = \{n_1 < n_2 < \cdots < n_\ell\}$, where $\ell = \sharp S > 0$. We then have the following.

Lemma 1.5. *$D_n = (0)$ if $n > n_\ell$ and $D_n \cong K/Rf$ if $n \leq 0$. Consequently, $\ell_R(K/Rf) = \ell_R(C)$.*

Proof. Let $n > n_\ell$. Then $C_n = (0)$. By exact sequence (E) above, we get $I_n/I_{n+1} \cong \omega_{n-c}/\omega_{n+1-c}$, whence $\omega_{n-c} = I_nf + \omega_{n+1-c}$. Therefore $\omega_n - c \subseteq I_nf + \omega_q$ for all $q \in \mathbb{Z}$. By Lemma 1.4 we may choose integers $d \gg 0$ and $k \geq 0$ so that

$$\omega_{n-c} \subseteq I_nf + \omega_{dm-c} \subseteq I_nf + I_d^{m-k}f$$

for all $m \geq k$. Consequently, $\omega_{n-c} = I_nf$. Hence $D_n = (0)$ for all $n \geq n_\ell$. If $n \leq 0$, then $D_n \cong [\mathcal{R}'(\omega)(-c)]_n/\mathcal{R}'_nf \cong K/Rf$ (see condition (ii) above). To see the last assertion, notice that because $S = \{n_1 < n_2 < \cdots < n_\ell\}$, $D_n = u^{n-n_1}D_{n_1} \cong D_{n_1}$ if $n \leq n_1$ and $D_n = u^{n_{i+1}-n}D_{n_{i+1}} \cong D_{n_{i+1}}$ if $1 \leq i < \ell$ and $n_i < n \leq n_{i+1}$. Therefore since $D_n = (0)$ for $n > n_\ell$, we get

$$\ell_R(K/Rf) = \ell_R(D_0) = \ell_R(D_{n_1}) = \sum_{i=1}^{\ell} \ell_R(C_{n_i}) = \ell_R(C).$$

$\square$

Exact sequence (E) above now shows the following estimations. Remember that $\mathrm{r}(R) \le \mathrm{r}(G)$, because $\mathrm{K}_G[t] = K[t, t^{-1}]$ so that $\mu_R(K) \le \mu_G(\mathrm{K}_G)$.

Proposition 1.6. $\mathrm{r}(R) - 1 \le \mathrm{r}(G) - 1 \le \mu_G(C) \le \ell_G(C) = \ell_R(C) = \ell_R(K/Rf)$.

We are now back to a general situation of Setting 1.3.

Theorem 1.7. *Let G be as in Setting 1.3 and assume that G is an almost Gorenstein graded ring with $\mathrm{r}(G) = \mathrm{r}(R)$. Then R is an almost Gorenstein local ring.*

Proof. We may assume that G is not a Gorenstein ring. We choose an exact sequence

$$0 \to G \xrightarrow{\psi} G(\omega)(-c) \to C \to 0$$

of graded G-modules so that $C \ne (0)$ and $\mathfrak{M}C = (0)$. Then $\mu_G(C) = \ell_G(C)$. We set $\xi = \psi(1)$ and write $\xi = \overline{ft^{-c}}$ with $f \in K$. Hence $(0) :_G \xi = (0)$. We now look at the estimations stated in Proposition 1.6. If $\mathrm{r}(R) - 1 = \ell_G(C)$, then $\ell_R(K/Rf) = \mu_R(K/Rf)$ because $\mathrm{r}(R) - 1 = \mu_R(K) - 1 \le \mu_R(K/Rf) \le \ell_R(K/Rf) = \ell_G(C)$, so that $\mathfrak{m}{\cdot}(K/Rf) = (0)$. Consequently, we get the exact sequence

$$0 \to R \xrightarrow{\varphi} K \to K/Rf \to 0$$

of R-modules with $\varphi(1) = f$, whence R is an almost Gorenstein local ring. If $\mathrm{r}(R) - 1 < \ell_G(C)$, then $\psi(1) \in \mathfrak{M}{\cdot}[G(\omega)(-c)]$ because $\mathrm{r}(G) - 1 < \mu_G(C)$, so that $G_{\mathfrak{M}}$ is a discrete valuation ring. This is impossible, since G is not a Gorenstein ring. $\square$

The converse of Theorem 1.7 is also true when G satisfies some additional conditions. To see this, we need the following. Recall that our graded ring G is said to be level, if $\mathrm{K}_G = G{\cdot}[\mathrm{K}_G]_{-a}$, where $a = \mathfrak{a}(G)$.

Lemma 1.8. *Suppose that $\mathrm{Q}(\widehat{R})$ is a Gorenstein ring and the field $R/\mathfrak{m}$ is infinite. Let us choose a canonical ideal K of R so that $R \subseteq K \subseteq \overline{R}$. Let $a \in \mathfrak{m}$ be a regular element of R such that $I = aK \subsetneq R$. We then have the following.*

1. *Suppose that G is an integral domain. Then there is an element $f \in K \setminus \omega_{1-c}$ so that $af \in I$ generates a minimal reduction of I. Hence $(0) :_G \xi = (0)$, where $\xi = \overline{ft^{-c}} \in G(\omega)(-c)$.*
2. *Suppose that $\mathrm{Q}(G)$ is a Gorenstein ring and G is a level ring. Then there is an element $f \in K$ such that $af \in I$ generates a reduction of I and $G_{\mathfrak{p}}{\cdot}\frac{\xi}{1} = [G(\omega)(-c)]_{\mathfrak{p}} \cong G_{\mathfrak{p}}$ for all $\mathfrak{p} \in \mathrm{Ass}G$, where $\xi = \overline{ft^{-c}} \in G(\omega)(-c)$. Hence $(0) :_G \xi = (0)$.*

Proof. (1) Let $L = \omega_{1-c}$. Then $aL \subsetneq aK = I$, since $L \subsetneq K$. We write $I = (x_1, x_2, \ldots, x_n)$ such that each x_i generates a minimal reduction of I. Choose $f = x_i$ so that $x_i \notin L$, which is the required one.

(2) Let $M = G(\omega)(-c)$. Then since $M = G{\cdot}M_0$ and $M_{\mathfrak{p}} \ne (0)$, $M_0 \not\subseteq \mathfrak{p}M_{\mathfrak{p}} \cap M$ for any $\mathfrak{p} \in \mathrm{Ass}G$. Choose an element $f \in K$ so that af generates a reduction of I and

$\xi = \overline{ft^{-c}} \notin \mathfrak{p}M_{\mathfrak{p}} \cap M$ for any $\mathfrak{p} \in \mathrm{Ass}G$. Then $G_{\mathfrak{p}} \cdot \frac{\xi}{1} = M_{\mathfrak{p}}$ for all $\mathfrak{p} \in \mathrm{Ass}G$, because $M_{\mathfrak{p}} \cong G_{\mathfrak{p}}$. $\square$

Theorem 1.9. *Suppose that R is an almost Gorenstein local ring and the field $R/\mathfrak{m}$ is infinite. Assume that one of the following conditions is satisfied:*

(1) *G is an integral domain;*
(2) *$\mathrm{Q}(G)$ is a Gorenstein ring and G is a level ring.*

Then G is an almost Gorenstein graded ring with $\mathrm{r}(G) = \mathrm{r}(R)$.

Proof. The ring $\mathrm{Q}(\widehat{R})$ is Gorenstein, since R is an almost Gorenstein local ring. Let K be a canonical ideal of R such that $R \subseteq K \subseteq \overline{R}$. We choose an element $f \in K$ and $a \in \mathfrak{m}$ as in Lemma 1.8. Then $\mu_R(K/Rf) = \mathrm{r}(R) - 1$, since f is a part of a minimal system of generators of K (recall that af generates a minimal reduction of $I = aK$). Therefore by Proposition 1.6, $\mathrm{r}(G) = \mathrm{r}(R)$ and $\mathfrak{M}{\cdot}C = (0)$, whence G is an almost Gorenstein graded ring. $\square$

To prove Theorem 1.2, we need one more result.

Proposition 1.10. *Let $G = G_0[G_1]$ be a Noetherian standard graded ring. Assume that G_0 is an Artinian local ring with infinite residue class field. If G is an almost Gorenstein graded ring with $\dim G \geq 2$, then $G/(x)$ is an almost Gorenstein graded ring for some non-zerodivisor $x \in G_1$.*

Proof. We may assume that G is not a Gorenstein ring. Let $\mathfrak{m}$ be the maximal ideal of G_0 and set $\mathfrak{M} = \mathfrak{m}G + G_+$. We consider the sequence

$$0 \to G \to \mathrm{K}_G(-a) \to C \to 0$$

of graded G-modules such that $\mu_G(C) = \mathrm{e}^0_{\mathfrak{M}}(C)$, where $a = \mathrm{a}(G)$ is the a-invariant of G. Then because the field $G_0/\mathfrak{m}$ is infinite and the ideal $G_+ = (G_1)$ of G is a reduction of $\mathfrak{M}$, we may choose an element $x \in G_1$ so that x is G-regular and superficial for C with respect to $\mathfrak{M}$. We set $\overline{G} = G/(x)$ and remember that x is C-regular, as $\dim_G C = \dim G - 1 > 0$. We then have the exact sequence

$$0 \to G/(x) \to (\mathrm{K}_G/x\mathrm{K}_G)(-a) \to C/xC \to 0$$

of graded $\overline{G}$-modules. We now notice that $\mathrm{a}(\overline{G}) = a + 1$ and that

$$(\mathrm{K}_G/x\mathrm{K}_G)(-a) \cong \mathrm{K}_{\overline{G}}(-(a+1))$$

as a graded $\overline{G}$-module, while we see

$$\mathrm{e}^0_{\mathfrak{M}/(x)}(C/xC) = \mathrm{e}^0_{\mathfrak{M}}(C) = \mu_G(C) = \mu_G(C/xC),$$

since x is superficial for C with respect to $\mathfrak{M}$. Thus $\overline{G}$ is an almost Gorenstein graded ring. $\square$

Proof of Theorem 1.2. We set $d = \dim R$ and $G = \mathrm{gr}_I(R)$. We may assume that G is not a Gorenstein ring. Hence $d = \dim G \geq 1$. By Theorem 1.7 we may also assume that $d > 1$ and that our assertion holds true for $d-1$. Let us consider an exact sequence

$$0 \to G \to \mathrm{K}_G(-a) \to C \to 0$$

of graded G-modules with $\mu_G(C) = \mathrm{e}^0_{\mathfrak{M}}(C)$, where $\mathfrak{M} = \mathfrak{m}G + G_+$ and $a = \mathrm{a}(G)$. We choose an element $a \in I$ so that the initial form $a^* = a + I^2 \in G_1 = I/I^2$ of a is G-regular and $G/a^*G = \mathrm{gr}_{I/(a)}(R/(a))$ is an almost Gorenstein graded ring (this choice is possible; see Proposition 1.10). Then the hypothesis on d shows $R/(a)$ is an almost Gorenstein local ring. Therefore R is an almost Gorenstein local ring, because a is R-regular. $\square$

We readily get the following, since $\mathrm{r}(R) \leq \mathrm{r}(\mathrm{gr}_I(R))$.

Corollary 1.11. *Let $(R, \mathfrak{m})$ be a Noetherian local ring with infinite residue class field. Suppose that R is a homomorphic image of a Gorenstein local ring. Let I be an $\mathfrak{m}$-primary ideal of R and assume that $\mathrm{gr}_I(R) = \bigoplus_{n \geq 0} I^n/I^{n+1}$ is an almost Gorenstein graded ring. Then R is an almost Gorenstein local ring, if* $\mathrm{r}(\mathrm{gr}_I(R))) = 2$

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