Noncommutative prob. sp.

$(\mathcal{A}, \varphi)$: noncomm. prob. sp.

$$\begin{cases} \mathcal{A} : *\text{-alg.}, \ni 1 \\ \varphi : \mathcal{A} \longrightarrow \mathbb{C} \text{ linear ftional } \begin{cases} \varphi(a^*a) \geq 0, \ \forall a \in \mathcal{A} \\ \varphi(1) = 1 \end{cases} \end{cases}$$

このとき φ : state という

e.g.

(1) $(\Omega, \mathcal{F}, P)$: meas. sp.

$\Rightarrow (L^\infty(\Omega), \mathbb{E})$: noncomm. prob. sp.

$$\mathbb{E}(f) = \int f \, dP.$$

(2) $(M_N(\mathbb{C}), tr_N)$: noncomm. prob. sp.

$$tr_N(A) := \frac{1}{N} \sum_{i=1}^{N} A_{ii} \quad , A = [A_{ij}] \in M_N(\mathbb{C}).$$

(3) $\mathcal{H}$: Hilbert sp., $\langle \cdot, \cdot \rangle$

$\xi_0 \in \mathcal{H}$, $\|\xi_0\| = 1$ をとって

$$\varphi(a) := \langle a\xi_0, \xi_0 \rangle$$

$\Rightarrow (B(\mathcal{H}), \varphi)$: noncomm. prob. sp.

(4) (Discrete group から作られる例)

G : countable discrete gr.

$\mathcal{H} := \ell^2(G) = \{ \xi : G \to \mathbb{C} \mid \sum_{g \in G} |\xi(g)|^2 < +\infty \}$: Hilbert sp.

$$\langle \xi, \eta \rangle := \sum_{g \in G} \xi(g) \overline{\eta(g)} \quad (\in \mathbb{C})$$

(通常の $\ell^2(\mathbb{N})$ と同様)

$$\delta_g(h) := \begin{cases} 1 & (h = g) \\ 0 & (h \neq g) \end{cases}, \quad \{\delta_g\}_{g\in G} : \ell^2(G) の ON-basis$$

$\forall g \in G$ に対し,

$$(\lambda(g)\xi)(h) := \xi(g^{-1}h)$$

$\Rightarrow \lambda : G \longrightarrow B(\ell^2(G))$ unitary repres.

λ : G の left regular repres.

$\because$ ・$\|\lambda(g)\xi\| = \left(\sum_h |\xi(g^{-1}h)|^2\right)^{1/2} = \left(\sum_h |\xi(h)|^2\right)^{1/2} = \|\xi\|$ より

$\lambda(g) \in B(\ell^2(G))$

・$\lambda(g_1)\lambda(g_2) = \lambda(g_1 g_2)$

・$\lambda(e) = 1$

・$\lambda(g)^* = \lambda(g^{-1})$

$$\left(\begin{aligned} \because \ \langle \lambda(g)\xi, \eta\rangle &= \sum_h \xi(g^{-1}h)\overline{\eta(h)} \\ &= \sum_h \xi(h)\overline{\eta(gh)} = \langle \xi, \lambda(g^{-1})\eta\rangle \end{aligned}\right) //$$

$$L(G) := \overline{\left\{ \sum_{\text{finite}} \alpha_g \lambda(g) \,\middle|\, \alpha_g \in \mathbb{C},\ g \in G \right\}}^{w} : \text{group v.N. alg.}$$

$$\qquad = \overline{\lambda(\mathbb{C}G)}^{w}$$

$\tau(x) := \langle x\delta_e, \delta_e\rangle$ とすると

τ : faith. normal tracial state on $L(G)$

$\because$ $\tau(\lambda(g_1)\lambda(g_2)) = \langle \lambda(g_1)\lambda(g_2)\delta_e, \delta_e\rangle$

$= \langle \lambda(g_2)\delta_e, \lambda(g_1^{-1})\delta_e\rangle = \langle \delta_{g_2}, \delta_{g_1^{-1}}\rangle = \begin{cases} 1 & g_2 = g_1^{-1} \\ 0 & g_2 \neq g_1^{-1} \end{cases}$

$= \tau(\lambda(g_2)\lambda(g_1)) \Rightarrow \tau(xy) = \tau(yx),\ \forall x, y \in L(G)$ //

$\Rightarrow (L(G), \tau)$: noncomm. prob. sp. (tracial W^*-prob. sp.)

$(\mathcal{A}, \varphi)$: noncomm. prob. sp.

- $a \in \mathcal{A}$: noncomm. random variable
- $\varphi(a^k)$ $(k=1,2,\cdots)$: k^{th} moment ($\varphi(a)$: expectation)
- $a = a^*$ のとき a : self-adjoint

Free independence

$(\mathcal{A}, \varphi)$: noncomm. prob. sp.

$\mathcal{A}_i$ $(i \in I)$: $\mathcal{A}$ の subalg.

- $(\mathcal{A}_i)_{i \in I}$: free or freely independent

$\underset{def.}{\Longleftrightarrow}$ $\forall i_1 \neq i_2 \neq \cdots \neq i_n$ in I ($i_1 = i_3$ etc は O.K.)

$\forall a_k \in \mathcal{A}_{i_k}$ $(k=1,\cdots,n)$

$\varphi(a_k) = 0$ $(k=1,\cdots,n) \Longrightarrow \varphi(a_1 \cdots a_n) = 0$.

- $a_i \in \mathcal{A}$ $(i \in I)$

$(a_i)_{i \in I}$: free $\underset{def.}{\Longleftrightarrow}$ $(\mathrm{Alg}\langle 1, a_i \rangle)_{i \in I}$: free

$\uparrow$

a_i (と1) により生成される subalg.

$\{ p(a_i) \mid p \in \mathbb{C}[x] \}$

多項式環.

Prop. 1 $(a_i)_{i \in I}$: freely indep.

$(a_i)_{i \in I}$ の joint moments は a_i の moments で決まる

$$\begin{bmatrix} \varphi(a_{i_1} a_{i_2} \cdots a_{i_k}) \\ (i_1 \neq i_2 \neq \cdots \neq i_k) \end{bmatrix} \qquad \begin{bmatrix} \varphi(a_i^m) \\ i \in I,\ m = 1, 2, \cdots \end{bmatrix}$$

∵ $a_1, \cdots, a_n$: freely indep.

$$0 = \varphi\big((a_1 - \varphi(a_1)1)(a_2 - \varphi(a_2)1) \cdots (a_n - \varphi(a_n)1)\big)$$

$$= \varphi(a_1 a_2 \cdots a_n) + \sum_{r=1}^{n} (-1)^r \sum_{1 \le i_1 < \cdots < i_r \le n} \varphi(a_{i_1}) \cdots \varphi(a_{i_r}) \underbrace{\varphi(a_1 \overset{i_1}{\vee} \cdots \overset{i_r}{\vee} a_n)}_{\substack{a_1 \cdots a_n \text{から} \\ a_{i_1}, \cdots, a_{i_r} \text{を除いた} \\ \text{積}}}$$

$$\therefore \varphi(a_1 \cdots a_n) = \sum_{r=1}^{n} (-1)^{r+1} \sum_{1 \le i_1 < \cdots < i_r \le n} \varphi(a_{i_1}) \cdots \varphi(a_{i_r}) \varphi(a_1 \cdots \underset{\hat{i_1}}{} \cdots \underset{\hat{i_r}}{} a_n)$$

⇒ 帰納法 //

<u>e.g.</u> a, b : freely indep.

- $0 = \varphi\big((a - \varphi(a)1)(b - \varphi(b)1)\big)$

$$= \varphi(ab) - \varphi(a)\varphi(b) - \cancel{\varphi(b)\varphi(a)} + \cancel{\varphi(a)\varphi(b)}$$

$$\therefore \varphi(ab) = \varphi(a)\varphi(b).$$

- $0 = \varphi\big((a - \varphi(a)1)(b - \varphi(b)1)(a - \varphi(a)1)\big)$

$$= \varphi(aba) - \varphi(b)\varphi(a^2)$$

$$\therefore \varphi(aba) = \varphi(a^2)\varphi(b)$$

- $\varphi(abab) = \varphi(a^2)\varphi(b)^2 + \varphi(a)^2\varphi(b^2) - \varphi(a)^2\varphi(b)^2$

☆ free indep. = joint moments を各 moments から計算する規則.

Classical (or tensor) :

$$\mathbb{E}(X_1^{m_1} X_2^{m_2} \cdots X_n^{m_n}) = \mathbb{E}(X_1^{m_1})\mathbb{E}(X_2^{m_2}) \cdots \mathbb{E}(X_n^{m_n})$$

indep.

☆ free prob. theory = noncomm. prob. sp. + free indep.

Fock sp.

$\mathcal{H}$: Hilbert sp.

$\mathcal{H}^{\otimes n} := \underbrace{\mathcal{H}\otimes\cdots\otimes\mathcal{H}}_{n} \longleftarrow$ $\mathcal{H}\otimes_{alg}\cdots\otimes_{alg}\mathcal{H}$ に内積

$\langle \xi_1\otimes\cdots\otimes\xi_n, \zeta_1\otimes\cdots\otimes\zeta_n\rangle := \prod_{i=1}^{n}\langle\xi_i,\zeta_i\rangle$

を入れて完備化.

$\mathcal{F}(\mathcal{H}) := \mathbb{C}\Omega \oplus \bigoplus_{n=1}^{\infty}\mathcal{H}^{\otimes n}$: full Fock sp.

$\uparrow$ vacuum vector $(\|\Omega\|=1)$.

$h\in\mathcal{H}$ に対し,

- (left) creation operator

$$\begin{cases} l(h)\Omega := h \\ l(h)(\zeta_1\otimes\cdots\otimes\zeta_n) := h\otimes\zeta_1\otimes\cdots\otimes\zeta_n \end{cases}$$

- (left) annihilation operator

$$\begin{cases} l^*(h)\Omega := 0 \\ l^*(h)(\zeta_1\otimes\cdots\otimes\zeta_n) = \langle\zeta_1, h\rangle\, \zeta_2\otimes\cdots\otimes\zeta_n \quad \left(= \Omega \text{ if } n=1\right) \end{cases}$$

⑦ (linear extension して)

$l(h)\in B(\mathcal{F}(\mathcal{H}))$, $\|l(h)\| = \|h\|$, $l^*(h) = l(h)^*$

$l^*(h)\,l(k) = \langle k, h\rangle 1$, 特に $\|h\|=1 \Rightarrow l(h)$: isometry

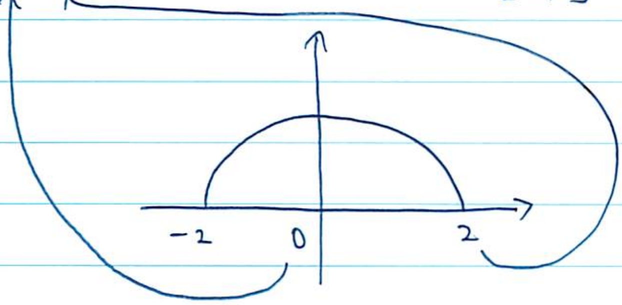
Theorem 2　$h \in \mathcal{H}$, $\|h\| = 1$ のとき,

$l(h) + l^*(h)$ の $\langle \cdot\, \Omega, \Omega \rangle$ に関する prob. dist. は

vacuum state

$w_{0,2}(x) := \frac{1}{2\pi}\sqrt{4-x^2}\,\chi_{[-2,2]}(x)\,dx$: (standard) semicircular law

variance = 1.



Proof　$l := l(h)$, $l^* := l^*(h)$ とおく

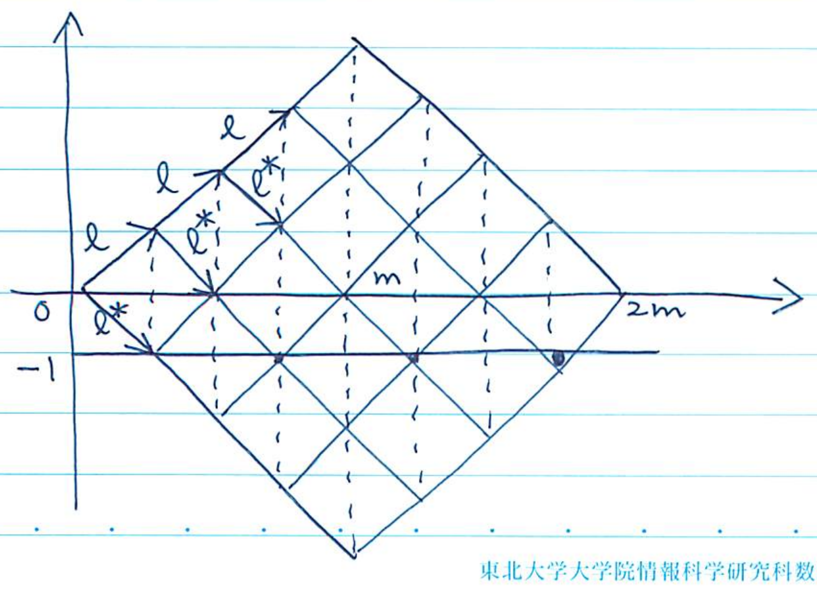
$$\langle (l+l^*)^n \Omega, \Omega \rangle = \frac{1}{2\pi}\int_{-2}^{2} x^n \sqrt{4-x^2}\,dx \qquad (n = 0, 1, 2, \cdots)$$

$$= \int x^n w_{0,2}(x)\,dx.$$

を示す.

$$\langle (l+l^*)^n \Omega, \Omega \rangle = \sum_{l_j = l \text{ or } l^*} \langle l_n l_{n-1} \cdots l_1 \Omega, \Omega \rangle$$

$l_n l_{n-1} \cdots l_1 \Omega$ に注目し, 次の傾き ±1 の格子を考える:



格子上 (i,j) $\begin{cases} i：\Omega \text{ から生成・消滅をくり返した回数} \\ \quad (l_i l_{i-1} \cdots l_1 \Omega) \\ j：\text{このとき得られる元 } \underbrace{h\otimes\cdots\otimes h}_{j} \text{ の tensor の回数} \end{cases}$

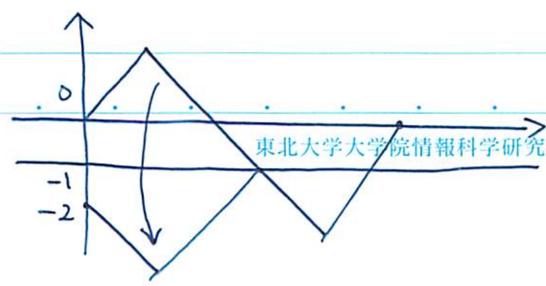
$\Longrightarrow$

$$\begin{cases} \circ\ (i,-1) \text{ を通るとき}, \quad l_n l_{n-1} \cdots l_1 \Omega = 0. \\ \quad (0 < i \leq n) \\ \circ\ (n,j),\ j \neq 0 \Longrightarrow l_n l_{n-1} \cdots l_1 \Omega \neq \Omega \\ \qquad\qquad \Longrightarrow \langle l_n l_{n-1} \cdots l_1 \Omega, \Omega \rangle = 0 \\ \qquad \text{この case は } n：\text{奇数のときを含む}. \\ \circ\ (0,0) \longrightarrow (n,0) \text{ ただし}, (i,-1) \text{ を通らない} \\ \qquad \Longrightarrow l_n l_{n-1} \cdots l_1 \Omega = \Omega. \end{cases}$$

以上より $n = 2m$ として,

$$\begin{cases} \circ\ (0,0) \longrightarrow (2m,0) \text{ の折線の個数} = \binom{2m}{m}. \\ \circ\ (0,0) \longrightarrow (2m,0) \text{ で } (i,-1) \text{ を通るものの個数} \\ \quad = (-2,0) \longrightarrow (2m,0) \text{ の折線の個数} = \binom{2m}{m-1}. \end{cases}$$

(∵ $(0,0)$ から $y=-1$ 上の最初の点までの折線を $y=-1$ で対称に折り返す：



)

よって、

$$\langle (l^*+l)^n\Omega, \Omega\rangle = \begin{cases} 0 & (n=2m-1) \\ \binom{2m}{m} - \binom{2m}{m-1} = \dfrac{(2m)!}{m!(m+1)!} & (n=2m) \end{cases}$$

↑ Catalan 数

$$\alpha_m := \frac{1}{2\pi}\int_{-2}^{2} x^{2m}\sqrt{4-x^2}\,dx$$

$$= -\frac{1}{2\pi}\int_{-2}^{2} \frac{-x}{\sqrt{4-x^2}}\cdot x^{2m-1}(4-x^2)\,dx.$$

部分積分.

$$= \frac{1}{2\pi}\int_{-2}^{2} \sqrt{4-x^2}\left(x^{2m-1}(4-x^2)\right)'dx.$$

$$= 4(2m-1)\alpha_{m-1} - (2m+1)\alpha_m.$$

$$\therefore\ \alpha_m = \frac{2(2m-1)}{m+1}\alpha_{m-1} = \frac{2^m(2m-1)\cdot(2m-3)\cdots\cdot 1}{(m+1)!}$$

$$= \frac{2^m(2m)!}{2^m m!(m+1)!} = \frac{(2m)!}{m!(m+1)!} = \langle (l+l^*)^{2m}\Omega, \Omega\rangle$$

//

R-transform

$\mathcal{F}(\mathcal{H})$ において、

$l := l(h)$, $l^* := l^*(h)$, $h\in\mathcal{H}$, $\|h\|=1$. $(l^*l=1, l^*\Omega=0)$

として、

$$T := l^* + \sum_{k=0}^{\infty}\alpha_{k+1}l^k \quad (\text{formal sum}).$$

Lem. 3 $\langle T^n\Omega, \Omega\rangle = \alpha_n + p_n(\alpha_1, \dots, \alpha_{n-1})$

universal polyn.

∵ 形式的に展開して計算

- l^k $(k \geq n)$ を含む項はすべて 0

  l^*は高々 n 個しかでてこない

  l^n を消せるのは l^{*n}だけだが、l^{*n} は l との積ででてこない

- l^{n-1}を含む項で生き残るのは、$\langle l^{*n-1} l^{n-1}\Omega, \Omega\rangle$ だけで係数は α_n.

$\therefore \langle T^n\Omega, \Omega\rangle = \alpha_n + p_n(\alpha_1, \dots, \alpha_{n-1})$ //

Lem. 4 $(\mathcal{A}, \varphi)$: noncomm. prob. sp., $a \in \mathcal{A}$: r.v.

$\Rightarrow \exists_1\, T = l^* + \sum_{k=0}^{\infty} \alpha_{k+1} l^k$: formal series

s.t. $\varphi(a^n) = \langle T^n\Omega, \Omega\rangle$, $n \in \mathbb{N}$.

∵ Lem. 3 より α_{k+1} は moment seq. $(\varphi(a^n))_{n=1}^{\infty}$ より帰納的に計算できる.

たとえば、

$$\alpha_1 = \varphi(a)$$
$$\alpha_2 = \varphi(a^2) - \varphi(a)^2$$
$$\alpha_3 = \varphi(a^3) - 3\varphi(a^2)\varphi(a) + 2\varphi(a)^3$$
$$\vdots$$

//

$a \in \mathcal{A}$: r.v. に対して, $(\alpha_k)_{k+1}^{\infty}$ を上のように定め,

$$R_a(z) := \sum_{k=0}^{\infty} \alpha_{k+1} z^k \quad (\text{formal power series})$$

R_a : R-transform of a という.

<u>Lem. 5</u> $a = a^*$, dist. of $a = w_{0,2}$

$\Longrightarrow R_a(z) = z.$

$\because$ dist. of a = dist of $\ell^* + \ell \Longrightarrow \alpha_2 = 1$, $\alpha_1, \alpha_3, \alpha_4, \dots = 0$.

//

<u>Prop. 6</u>.

(1) a : r.v., $\lambda \in \mathbb{C}$ に対し, $R_{\lambda a}(z) = \lambda R_a(\lambda z)$.

(2) $a, a_n (n \in \mathbb{N})$: r.v.s

$a_n \longrightarrow a$ in the dist. i.e., $\lim_{n\to\infty} \varphi(a_n^k) = \varphi(a^k)$, $k = 1, 2, \dots$

$\Longleftrightarrow \lim_{n\to\infty} R_{a_n}(z) = R_a(z)$ as formal power series

i.e., $\lim_{n\to\infty} \alpha_k^{(n)} = \alpha_k$.

係数.

(3) a, b : r.v.s, freely indep.

$\Longrightarrow R_{a+b}(z) = R_a(z) + R_b(z)$

Proof

(1) $R_{\lambda a}(z) = \sum_{k=0}^{\infty} \tilde{\alpha}_{k+1} z^k$ とすると

$$\varphi((\lambda a)^n) = \lambda^n \varphi(a^n) = \lambda^n \alpha_n + (\alpha_1, \dots, \alpha_{n-1} \text{の多項式})$$

よって $\tilde{\alpha}_n = \lambda^n \alpha_n$ $\therefore R_{\lambda a}(z) = \sum_{k=0}^{\infty} \lambda^{k+1} \alpha_{k+1} z^k = \lambda R_a(\lambda z)$ ┘

(2). $\varphi(a_n^k)$ は $\alpha_1^{(n)}, \dots, \alpha_k^{(n)}$ の多項式であり，逆に $\alpha_k^{(n)}$ は，

$\varphi(a_n^k)$ と $\alpha_1^{(n)}, \dots, \alpha_{k-1}^{(n)}$ の多項式より明らか ┘

Lem. 3 & Lem. 4.

(3). $R_a(z) = \sum_{k=0}^{\infty} \alpha_{k+1} z^k$, $R_b(z) = \sum_{k=0}^{\infty} \beta_{k+1} z^k$

(概略).

$h_1, h_2 \in \mathcal{H}$, $h_1 \perp h_2$, $\|h_1\| = \|h_2\| = 1$.

$l_i := l(h_i)$, $l_i^* := l^*(h_i)$ $i = 1, 2$.

a, b : free $\Longrightarrow$

dist. of (a, b)

$$= \text{dist. of } \Big(\overbrace{l_1^* + \sum_{k=0}^{\infty} \alpha_{k+1} l_1^k}^{T_1},\ \overbrace{l_2^* + \sum_{k=0}^{\infty} \beta_{k+1} l_2^k}^{T_2}\Big)$$

w.r.t. $\langle \cdot\, \Omega, \Omega \rangle$

free.

$$T_1 + T_2 = (l_1^* + l_2^*) + \sum_{k=0}^{\infty} \alpha_{k+1} l_1^k + \sum_{k=0}^{\infty} \beta_{k+1} l_2^k$$

について，

$$T = l^* + \sum_{k=0}^{\infty} \alpha_{k+1} l^k + \sum_{k=0}^{\infty} \beta_{k+1} l^k$$

$\langle (T_1+T_2)^n\Omega, \Omega\rangle = \langle T^n\Omega, \Omega\rangle$ を示せばよい.

$$a_j' = \begin{cases} l^* & \text{if } a_j = l_1^* + l_2^* \\ l & \text{if } a_j = l_1 \text{ or } l_2 \end{cases}$$ として $\langle a_1\cdots a_n\Omega, \Omega\rangle = \langle a_1'\cdots a_n'\Omega, \Omega\rangle$

を示せばよい. これは,

$$\begin{cases} (l_1^*+l_2^*)l_1 = (l_1^*+l_2^*)l_2 = 1, \quad (l_1^*+l_2^*)\Omega = 0 \\ l^*l = 1, \quad l^*\Omega = 0 \end{cases}$$

より O.K.

以上より,

$$\varphi((a+b)^n) = \langle T^n\Omega, \Omega\rangle.$$

$$\therefore R_{a+b}(z) = \sum_{k=0}^{\infty}(\alpha_{k+1}+\beta_{k+1})z^k = R_a(z) + R_b(z)$$

//

Theorem 7 (Free CLT).

$(\mathcal{A}, \varphi)$: noncomm. prob. sp.

$(a_i)_{i=1}^{\infty}$: free in $(\mathcal{A}, \varphi)$, $\varphi(a_i) = 0$

(1) $\displaystyle\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\varphi(a_i^2) = 1$.

(2) $\displaystyle\sup_{i\geq 1}|\varphi(a_i^k)| < +\infty$, $\forall k\in\mathbb{N}$.

$$\Rightarrow \lim_{n\to\infty}\varphi\left(\left(\frac{a_1+\cdots+a_n}{\sqrt{n}}\right)^k\right) = \frac{1}{2\pi}\int_{-2}^{2}x^k\sqrt{4-x^2}\,dx, \quad k=1,2,\cdots$$

<u>Proof</u> $R_{a_i}(z) = \sum_{k=0}^{\infty} \alpha_{k+1}^{(i)} z^k$, $b_n := \dfrac{a_1 + \cdots + a_n}{\sqrt{n}}$ とおくと，

$$R_{b_n}(z) \underset{\substack{\uparrow \\ \text{Prop. 6 (3) \& free}}}{=} \sum_{i=1}^{n} R_{\frac{a_i}{\sqrt{n}}}(z) \underset{\substack{\uparrow \\ \text{Prop. 6 (1)}}}{=} \sum_{i=1}^{n} \frac{1}{\sqrt{n}} R_{a_i}\left(\frac{z}{\sqrt{n}}\right).$$

$$= \sum_{i=1}^{n} \frac{1}{\sqrt{n}} \sum_{k=0}^{\infty} \alpha_{k+1}^{(i)} \frac{z^k}{(\sqrt{n})^k} = \sum_{k=0}^{\infty} \overbrace{\frac{1}{n^{\frac{k+1}{2}}} \left(\sum_{i=1}^{n} \alpha_{k+1}^{(i)} \right)}^{\text{係数}} z^k.$$

Lem. 4 (Proof) から

$$\begin{cases} \alpha_1^{(i)} = \varphi(a_i) = 0 \\ \alpha_2^{(i)} = \varphi(a_i^2) - \underbrace{\varphi(a_i)^2}_{0} = \varphi(a_i^2) \end{cases}.$$

(1) より

$$\frac{1}{n} \sum_{i=1}^{n} a_2^{(i)} \longrightarrow 1 \quad \text{as } n \longrightarrow \infty.$$

(2) より $\sup_{i \geq 1} |\alpha_k^{(i)}| < +\infty$ であるから $k \geq 3$ のとき，

$$\frac{1}{n^{\frac{k+1}{2}}} \sum_{i=1}^{n} \alpha_{k+1}^{(i)} \longrightarrow 0 \quad \text{as } n \longrightarrow \infty.$$

$$\therefore \lim_{n \to \infty} R_{b_n}(z) = z.$$

Lem. 5, Prop. 6 (2) より o.k. //