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§1 Feynman 1948, 1951, Feynman-Hibb 1965.

Ichinose 2010.

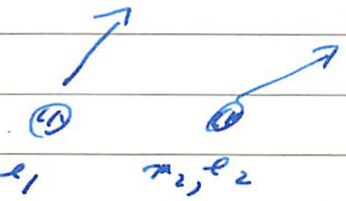
$^{\forall} T > 0$, $[0, T]$, $x^{(j)} \in R^3$ $(j = 1, 2, \dots, n)$.

$m_j > 0$, $e_j \in R$.

$$\begin{cases} \dfrac{d}{dt} m_j \dot{x}^{(j)} = e_j E(t, x^{(j)}) + e_j \dot{x}^{(j)} \times B(t, x^{(j)}), \\ \text{Maxwell eq.} \end{cases}$$

$\phi \in R$, $A \in R^3$, $E = -\frac{\partial A}{\partial t} - \phi$, $B = \nabla \times A$

$\vec{x} = (x^{(1)}, \dots, x^{(n)}) \in R^{3n}$



(1) $\mathcal{L}(t, \vec{x}, \dot{\vec{x}}, \phi, \frac{\partial \phi}{\partial x}, A, \dot{A}, \frac{\partial A}{\partial x})$

$$= \left\{ \sum_{j=1}^{n} \frac{m_j}{2} |\dot{x}^{(j)}|^2 - e_j \phi(t, x^{(j)}) + e_j \dot{x}^{(j)} \cdot A(t, x^{(j)}) \right\}$$

$$+ \frac{1}{8\pi} \int_{R^3} \left(\frac{1}{c^2} |E(t, x)|^2 - |B(t, x)|^2 \right) dx + C.$$

変分 $\vec{x} \to \vec{x} + \delta \vec{x}$, $\phi \to \phi + \delta \phi$, $A \to A + \delta \phi$.

§2. $L_1, L_2, L_3 \gg 1$.

$$V = \left[-\frac{L_1}{2}, \frac{L_1}{2}\right] \times \left[-\frac{L_2}{2}, \frac{L_2}{2}\right] \times \left[-\frac{L_3}{2}, \frac{L_3}{2}\right].$$

$\phi(t, x)$, $A(t, x)$ periodic in $x \in V$.

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(2) $\int_V \phi(t,x)\,dx = 0$, $\int_V A(t,x)\,dx = 0$

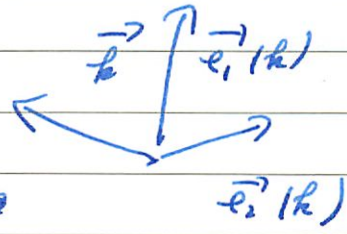
を仮定. $k = \left(\frac{2\pi}{L_1}s_1, \frac{2\pi}{L_2}s_2, \frac{2\pi}{L_3}s_3\right)$, $s_j \in \mathbb{Z}$

(3) $\phi = \frac{1}{|V|}\sum_{k\neq 0} \phi_k e^{ik\cdot x}, \in R,$

(4) $A = \frac{1}{|V|}\sum_{k\neq 0}\left\{\sum_{l=0}^{2} a_{lk}\,\vec{e}_l(k)\right\} \in R^3.$

$\vec{e}_0(k) = \frac{k}{|k|}$, $\vec{e}_1(k)$, $\vec{e}_2(k) \in R^3$ $(k \neq 0)$

ONS in R^3



$\Lambda = \{ \overset{\neq 0}{k} \in R^3;\ s_1, s_2, s_3$

$= 0, \pm 1, \pm 2, \cdots, \pm M \}$, $M >> 1$.

$\Lambda = {}^{\exists}\Lambda' \cup (-\Lambda')$, $\Lambda' \cap (-\Lambda') = \emptyset$ empty set.

$\cancel{\phi \in R, A}$ $\phi^* = \phi$, $A^* = A$.

$\phi_k = \phi_k^{(1)} - i\,\phi_k^{(2)}$, $a_{lk} = a_{lk}^{(1)} - i\,a_{lk}^{(2)}$

$\phi_{\Lambda'} := \{\phi_k^{(i)};\ i = 1, 2,\ k \in \Lambda'\} \in R^{2N}$, $N = {}^{\#}\{\Lambda'\}$,

$\tilde{a}_{\Lambda'} = \{a_{lk}^{(i)};\ i = 1, 2,\ l = 0, 1, 2,\ k \in \Lambda'\} \in R^{6N}$

independent variables.

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$\phi_\Lambda, \quad \hat{a}_\Lambda$

$a_{\Lambda'} = \{ a^{(i)}_{\ell k} \; ; \; i=1,2, \; \ell=1,2, \; k\in\Lambda' \} \in R^{4N}.$

(3), (4) を (1) に代入して

$\mathcal{L}'(\bar{x}^{j}, \dot{\bar{x}}^{j}, \phi_\Lambda, \hat{a}_\Lambda, \dot{\hat{a}}_\Lambda)$

$= \sum_{j=1}^{n} \frac{m_j}{2} |\dot{x}^{(j)}|^2 + \frac{1}{8\pi c^2 |V|} \sum_{k\in\Lambda} \{ |k|^2 (\phi_k^{(1)} - \frac{\dot{a}^{(2)}_{0k}}{|k|})^2$

$+ \cdots \}$

$+ \sum_{j=1}^{n} e_j \dot{x}^{(j)} \cdot A(x^{(j)}, \hat{a}_{\Lambda'}) + \mathcal{L}_{rad}(a_\Lambda, \dot{a}_\Lambda)$

(5) $\mathcal{L}_{rad}(a_\Lambda, \dot{a}_\Lambda) = \frac{1}{4\pi c^2} \sum_{k\in\Lambda} \sum_{i,\ell=1}^{2} \{ \frac{1}{2|V|} (\dot{a}^{(i)}_{\ell k})^2$

$- \frac{(c|k|)^2}{2|V|} (a^{(i)}_{\ell k})^2 + \frac{\hbar c|k|}{2} \}$ (harmonic osc.)

§3. Dirac constraints (Canad. J. Math. 1950, 菅野 2007 (吉岡)).

$\pi_k^{(i)} := \frac{\partial \mathcal{L}}{\partial \dot{\phi}_k^{(i)}} = 0$ (primary constraints)

$(i=1,2, \; k\in\Lambda')$.

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Legendre transf.

$$\mathcal{H} = p \cdot \dot{x} + \cdots \qquad - \mathcal{L}.$$

$$\chi_k^{(j)} := \{ \pi_k^{(j)}, \mathcal{H} + \sum_{s \in \Lambda'} \sum_{\ell=1}^{2} {}^{\exists}\lambda_{s\ell}\, \pi_s^{(\ell)} \} \left(= \frac{d\pi_k^{(j)}}{dt} \right) = 0$$

in $\pi_s^{(\ell)} = 0$ $(\ell = 1, 2,\ s \in \Lambda')$ (secondary constraints).

$$\{ \chi_k^{(j)}, \mathcal{H} + \sum_{s \in \Lambda'} \sum_{\ell=1}^{2} \lambda_{s\ell}\, \pi_s^{(\ell)} + \sum_{s \in \Lambda'} \sum_{\ell=1}^{2} {}^{\exists}\tau_{s\ell}\, \chi_s^{(\ell)} \} = 0$$

OK

$$(6) \quad \begin{cases} \phi_k^{(1)} - \dfrac{\dot{a}_{0k}^{(2)}}{|k|} - 4\pi c^2 \dfrac{\rho_k^{(1)}(\vec{x})}{|k|^2} = 0, \\[2ex] \phi_k^{(2)} + \dfrac{\dot{a}_{0k}^{(1)}}{|k|} - 4\pi c^2 \dfrac{\rho_k^{(2)}(\vec{x})}{|k|^2} = 0 \end{cases}$$

$\Longleftrightarrow$ equiv. $\quad 0 = \dfrac{\partial \mathcal{L}}{\partial \phi_k^{(j)}} \left(= \dfrac{d}{dt} \dfrac{\partial \mathcal{L}}{\partial \dot{\phi}_k^{(j)}} \right)$

$\Longleftrightarrow$ equiv. $\quad \nabla \cdot E(t, x) = 4\pi c^2 \sum_{j=1}^{n} \delta(x - x^{(j)})$ (Gauss's law).

~~(6)~~

$\nabla \cdot A = 0$ (Coulomb gauge).

$\Longleftrightarrow$

(7). $\quad a_{0k}^{(j)} = 0 \quad (j = 1, 2,\ k \in \Lambda).$

$$(\because)\ \nabla \cdot A = \frac{1}{|V|} \sum_{k \in \Lambda} \sum_{\ell=0}^{2} a_{\ell k}\, k \cdot \vec{e}_\ell(k)\, e^{ik \cdot x}$$

$$= \frac{1}{|V|} \sum_{k \in \Lambda} a_{0k} \frac{|k|^2}{|k|} e^{ik \cdot x}.$$

~~(注)~~

~~$$\frac{1}{c^2} \frac{\partial \phi}{\partial t} + \nabla \cdot A = 0 \quad \text{(Lorentz gauge).}$$~~

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(6), (7) を仮定すると、$\mathcal{L}'$ より

$$\frac{c^2}{2}\sum_{j,l=1, j\neq l}^{n} \frac{e_j e_l}{|x^{(j)}-x^{(l)}|}$$

(8) $\mathcal{L}(\vec{x}, \dot{\vec{x}}, a_\Delta, \dot{a}_\Delta)$

$$= \sum_{j=1}^{n} \frac{m_j}{2} |\dot{x}^{(j)}|^2 - \cancel{\frac{2\pi c^2}{|V|} \sum_{k\in\Delta} \sum_{j,l=1, j\neq l}^{n} \frac{e_j e_l \cos k\cdot(x^{(j)}-x^{(l)})}{|k|^2}}$$

$$+ \underbrace{\sum_{j=1}^{n} e_j \dot{x}^{(j)} \cdot A_c(x^{(j)}, a_{\Delta'})}_{\text{interaction}} + \mathcal{L}_{rad}(a_\Delta, \dot{a}_\Delta),$$

(9) $$A_c(x, a_{\Delta'}) = \frac{1}{|V|} g(k) \sum_{k\in\Delta} \sum_{l=1}^{2} \left(4 \dot{a}^{(1)}_{lk} \cos k\cdot x + 4 \dot{a}^{(2)}_{lk} \sin k\cdot x \right)$$

§4. $\Delta : 0 = \tau_0 < \tau_1 < \cdots < \tau_w = T$.

$$|\Delta| = \max_j (\tau_j - \tau_{j-1})$$

折れ線 $\vec{\gamma}_\Delta(\theta; \vec{x}^{(0)}, \ldots, \vec{x}^{(w-1)}, \vec{x}) \quad (0 \leq \theta \leq T)$, $\vec{x}^{(0)} \in R^{3n}$, $\vec{x} \in R^{3n}$

$a_{\Delta'\Delta}(\theta; a_{\Delta'}^{(0)}, \ldots, a_{\Delta'}^{(w-1)}, a_{\Delta'}) \in R^{4N}$, $a_{\Delta'}^{(0)} \in R^{4N}$

<u>Theorem.</u> ${}^\exists \delta_l > 0 \ (l = 1, 2, \ldots)$, ${}^\exists \delta_\alpha > 0$ $(b\alpha)$.

$$|\partial_\theta^l \varphi(\theta)| \leq C_l (1 + |\theta|)^{-(1+\delta_l)},$$

$$|\partial_x^\alpha g(x)| \leq C_\alpha (1 + |x|)^{-(1+\delta_\alpha)}.$$

を仮定. Then, $f(\vec{x}, a_{\Delta'}) \in L^2(R^{3n+4N})$

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$\exists$

$$N(\Delta,\Lambda)\int\cdots\int e^{i\hbar^{-1}\int_0^t \mathcal{L}(\vec{q}_\Delta(\theta),\,\dot{\vec{q}}_\Delta(\theta),\,a_{\Lambda\Delta}(\theta),\,\dot{a}_{\Lambda\Delta}(\theta))d\theta}$$

$$\times f(\vec{q}_\Delta(0),\, a_{\Lambda\Delta}(0))\, d\vec{x}^{(0)}\cdots d\vec{x}^{(\nu-1)}\, da_{\Lambda'}^{(0)}\cdots da_{\Lambda'}^{(\nu-1)}$$

in $\mathcal{E}_t^0([0,T];L^2)$.

$\Longrightarrow u = U(t,0)f$ in as $|\Delta|\to 0$.

$$i\hbar\frac{\partial u}{\partial t} = Hf, \qquad u(0)=f$$

(10) $$H=\sum_{j=1}^{n}\frac{1}{2m_j}\left|\frac{\hbar}{i}\frac{\partial}{\partial x^{(j)}} - e_j A_c(x^{(j)}, a_{\Lambda'})\right|^2 + \frac{2\pi c^2}{|V|}\sum_{k\in\Lambda}\sum_{j,l=1,\ j\neq l}^{n}\frac{e_j e_l \cos k\cdot(x^{(j)}-x^{(l)})}{|k|^2} + H_{rad}.$$

(11) $$H_{rad} = \sum_{k\in\Lambda}\sum_{j,l=1}^{2}\left\{\frac{1}{2}(4\pi c^2|V|)\left(\frac{\hbar}{i}\frac{\partial}{\partial a_{lk}^{(j)}}\right)^2 + \frac{1}{2}\left(\frac{1}{4\pi c^2|V|}\right)(c|k|)^2(a_{lk}^{(j)})^2 - \frac{\hbar c|k|}{2}\right\}.$$

$$H_{rad} = \sum_{k\in\Lambda}\sum_{j,l=1}^{2} \hbar c|k|\, \hat{a}_{lk}^{(j)+}\hat{a}_{lk}^{(j)},$$

$$\hat{a}_{lk}^{(1)+}\hat{a}_{lk}^{(1)} + \hat{a}_{lk}^{(2)+}\hat{a}_{lk}^{(2)}$$

$$= (\hat{a}_{lk}^{(1)} - i\,\hat{a}_{lk}^{(2)})^{+}(\hat{a}_{lk}^{(1)} - i\,\hat{a}_{lk}^{(2)})$$

$$=: \hat{a}_{lk}^{+}\,\hat{a}_{lk}.$$

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$$\Psi_0 = \prod_{k \in \Lambda'} \prod_{l=1}^{2} \sqrt{\frac{c|k|}{\pi \hbar |V|}}\, e^{-\frac{1}{2\hbar|V|}\left(a_{lk}^{(1)\,2} + a_{lk}^{(2)\,2}\right)}.$$

Then, $\hat{a}_{lk}\Psi_0 = 0$. (vacuum)

$$\hat{a}_{lk}^{+}\Psi_0 = \sqrt{\frac{2c|k|}{\hbar|V|}}\, a_{lk}^{*}\Psi_0,$$

(運動量 k, 偏光 l の1粒子状態, Feynman-Hibbs)

interaction term

$$A_c(x, a_{\Lambda'}) = \frac{1}{|V|}\sum_{k \in \Lambda}\sum_{l=1}^{2}\left(\underline{a_{lk}^{(1)}}\cos k\cdot x + \underline{a_{lk}^{(2)}}\sin k\cdot x\right)\vec{e}_l(k)$$

$$= \sqrt{\frac{4\pi\hbar}{|V|}}\, c\sum_{k\in\Lambda}\sum_{l=1}^{2}\frac{1}{\sqrt{2c|k|}}\left(\underline{\hat{a}_{lk}}\, e^{ik\cdot x} + \underline{\hat{a}_{lk}^{+}}\, e^{-ik\cdot x}\right)\vec{e}_l(k).$$