No.

Jiang-Su alg. の吸収性について.

1. C^*-alg.
2. C.p. map.
3. Jiang-Su alg.
4. Main Thm.

§1. C^*-alg.

A : Banach alg.

$*$: involution

If $\|a^*a\| = \|a\|^2$, $\forall a \in A$, we say that A is a C^*-algebra.

$\|a^*a\| = \sup_{\lambda \in sp(a^*a)} |\lambda|$

<u>Ex.</u> $M_n(\mathbb{C})$, $B(H)$ $(A \subset B(H))$

$I(k, k-1) := \{ f \in C([0,1]) \otimes M_k \otimes M_{k-1}$

$: f(0) \in M_k \otimes \mathbb{C},\ f(1) \in \mathbb{C} \otimes M_{k-1} \}$. : projection-less.

v.N.の型は I, II, III 型. ちなみに $\overline{I(k, k-1)}^w$ は I 型 $(\simeq L^\infty([0,1]) \otimes M_k \otimes M_{k-1})$

<u>Intro.</u> $ab - ba =: [a, b]$

<u>Prob. (Halmas)</u>

$a_n, b_n \in M_n^1$, $\|[a_n, b_n]\| \to 0$,

$\overset{?}{\exists} a_n', b_n' \in M_n$, $\|[a_n', b_n']\| = 0$, $\|a_n' - a_n\| + \|b_n' - b_n\| \to 0$.

Ex. (1981, Voiculescu)

$$a_n := \begin{bmatrix} 0 & & & 1 \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{bmatrix}, \quad b_n := \begin{bmatrix} 1 & & & \\ & \lambda & & 0 \\ & & \lambda^2 & \\ & 0 & & \ddots \\ & & & & \lambda^{n-1} \end{bmatrix} \qquad \lambda = e^{\frac{2\pi i}{n}}$$

$a_n b_n a_n^* b_n^* = \lambda I \to I.$

$\log(a_n b_n a_n^* b_n^*) = \frac{2\pi i}{n} \neq 0. \longrightarrow$ Problem は No.

<u>Thm (1997, H. Lin)</u>

$a_n, b_n \in M_{n,+}^1$ のときは Problem は Yes!

$\prod_{n\in\mathbb{N}} M_n := \{(a_n)_n \mid \sup\|a_n\| < \infty\}.$

$\cup$

$\sum M_n := \{(a_n)_n \mid \|a_n\| \to 0\}.$

$\mathcal{M} := \prod M_n / \sum M_n$

$[\overline{(a_n)}, \overline{(b_n)}] = 0$ in $\mathcal{M}$

$\Rightarrow C^*(\{\overline{(a_n)}, \overline{(b_n)}\}) \cong C(X)$

$$X = \begin{cases} \mathbb{T}^2 \cdots \text{unitary} \\ [0,1]^2 \cdots \text{positive.} \end{cases}$$

$\iota : C(X) \hookrightarrow \mathcal{M}$

$\rightsquigarrow K_*(\iota) : K_*(X) \to K_*(\mathcal{M}) \qquad K_0(\mathbb{T}^2) = \mathbb{Z}^2,\ K_1(\mathbb{T}^2) = \mathbb{Z}^2.$

ピッタリ可換近似 $\Rightarrow \mathrm{Im}(K_0(\iota)) \cong \mathbb{Z},\ \mathrm{Im}(K_1(\iota)) \cong 0$

$K_0([0,1]^2)=\mathbb{Z}$, $K_1([0,1]^2)=0$ からピッタリ可換近似ができる.

§2. c.p. map.

A, B : C^*-alg.

$\varphi : A \to B$: $\overbrace{\text{linear}, \times, *}^{*\text{-homo.}}$ $\Rightarrow$ φ : norm continuous.

$\|\varphi(a)\|^2 = \sup_{\lambda \in sp(\varphi(a^*a))} |\lambda| \leq \sup_{\lambda \in sp(a^*a)} |\lambda| = \|a\|^2$.

$A \cong B$: $*$-iso. がある.

Def. $\varphi : A \to B$: lin.

If $\varphi \otimes id_n : A \otimes \mathbb{M}_n \to B \otimes \mathbb{M}_n$ is positive $\forall n \in \mathbb{N}$.

we say that φ is completely positive.

Ex. • $\varphi : A \to \mathbb{C}$: state is c.p.

• $v \in A$, $\varphi(a) := vav^*$ is c.p.

Thm (Gelfand-Naimark, Segal)

φ : state, $\mathcal{H} := \widetilde{(A / \{a : \varphi(a^*a)=0\})}$

$\pi : A \to \mathbb{B}(\mathcal{H})$: $*$-homo, 左かけ算で def.

$\exists \xi \in \mathcal{H}$, $\varphi(a) = (\pi(a)\xi | \xi)$

Thm (Stinespring)

$\varphi : A \to B \subset B(\mathcal{H})$: c.p. , $^{\exists}\pi : A \to B(\mathcal{K})$ $*$-homo.

$^{\exists} v \in B(\mathcal{K}, \mathcal{H})$ s.t. $\varphi(a) = v\pi(a)v^*$

・Kasparov, Wolf による拡張がある.

Thm (Choi-Effros, Kirchberg)

A : C^*-alg.

$A \otimes_{max} B \cong A \otimes_{min} B$ $(^{\forall}B : C^*\text{alg})$

$\Leftrightarrow$ $^{\exists}\varphi_n : A \to \oplus M_{k_i}$: c.p.

$^{\exists}\psi_n : \oplus M_{k_i} \to A$: c.p.

s.t. $\|\psi_n \circ \varphi_n(a) - a\| \to 0$ $^{\forall}a \in A$.

↪ だいたい $v(\oplus M_{k_i})v^* \subset A$ が A を近似.

$\overset{\text{def.}}{\Leftrightarrow}$ A is nuclear.

Thm (G. Elliott)

A : nuc. ψ_n : $*$-hom. $\Rightarrow$ A is classifiable by K_0.

§3. Jiang-Su alg. (江, 苏)

$\mathcal{Z} := \varinjlim I(k_n, k_n - 1)$ with a uniq. tr. simple.

$K_0(\mathcal{Z}) \cong \mathbb{Z}$, $K_1(\mathcal{Z}) \cong 0$

$\mathcal{C}$: a class of nuc. C^*-algs, classifiable.

$A, \mathcal{Z}$ belong to $\mathcal{C}$

$K_*(A \otimes \mathcal{Z}) \cong K_*(A)$

$\Rightarrow A \otimes \mathcal{Z} \cong A$.

Conjecture (A. Toms, W. Winter)

A : unital, separable, simple, inf-dim. C^*-alg. nuc.

T.F.A.E

(i) $A \otimes \mathcal{Z} \cong A$.

(ii) A has strict comparison.

(iii) $\dim_{nuc}(A) < \infty$

$d_\tau(a) = \lim_{n \to \infty} \tau(a^{\frac{1}{n}}) \quad a \in A_+$

(ii) $a, b \in A_+ \ d_\tau(a) < d_\tau(b)$

$\Rightarrow {}^\exists r_n \in A$ s.t. $\| r_n^* b r_n - a \| \to 0$

(iii) Thm A is nuclear

$\Leftrightarrow {}^\exists \varphi_n : A \to \bigoplus_{i=1}^{N_n} M_{n_i}$: c.p.

$\psi_{n_i} : M_{n_i} \to A$: c.p. disjoint order zero (直交性を保つ)

s.t. $\| (\bigoplus_{i=1}^{N_n} \psi_{n_i}) \circ \varphi_n(a) - a \| \to 0 \quad {}^\forall a \in A$.

$\dim_{nuc}(A) < \infty \Leftrightarrow {}^\exists N_n \ \sup N_n < \infty$

No.

(i) $\Rightarrow$ (ii) M. Rørdom.

(iii) $\Rightarrow$ (i) W. Winter.

(ii) $\Rightarrow$ (iii) open!

§4. Main Thm. (with. H. Matsui)

A : uni. sep. simp. inf-dim. nuc. C^*-alg. with a uniq. tr. τ.

T.F.A.E.

(i) $A \otimes \mathcal{Z} \cong A$.

(ii) A has strict comparison.

(iii) $\forall \varphi : A \to A$: c.p. can be excised in small central sequences.

(iv) A has property (SI)

Exsision. ($\overset{\text{pure state}}{\omega(x)} \approx axa \in A_+$)

$A_\infty := A' \cap \ell^\infty(\mathbb{N}, A)/c_0$

$(a_n)_n \in A_\infty \Leftrightarrow \|[a_n, a]\| \to 0 \quad \forall a \in A$.

Def $\varphi : A \to A$: c.p.

If for $(e_n)_n, (f_n)_n \in A^1_{\infty,+}$ with $\max_\tau \tau(e_n) \to 0$

$\lim_m \liminf_n \min_\tau \tau(f_n^m) > 0$

$\exists s_n \in A$.

$\|s_n^* x s_n - \varphi(x) e_n\| \to 0$, $\|f_n s_n - s_n\| \to 0$.

we say that φ can be excised in s.c.s.

<u>Prop.</u> $\omega \in S(A)$ $c_i, d_i \in A$, $i = 1, \cdots, N$

$\varphi(a) := \sum_{i\cdot j} \omega(d_i^* a d_j) c_i^* c_j$ can be exc. in s.c.s.

☺ $F \subset A'$: finite, $\varepsilon > 0$

$\exists e_n', f_n' \in A_+'$ s.t.

$\max_\tau d_\tau(e_n') \to 0$, $(e_n')_n = (e_n)_n$.

$\lim_m \liminf_n \min_\tau \tau(f_n'^m) > 0$, $(f_n' f_n)_n = (f_n') \in A_\infty$ が示せる.

W.M.A.T. $\|c_i\| \leq 1$, ω is pure. ← グリムの補題から.

$\exists a \in A_+$ s.t. $\|a\| = 1$, $\| a(\omega(x) - x)a \| < \frac{\varepsilon}{N^2}$

$x \in G := \{ d_i^* f d_j : f \in F \}$

(Akemann, Anderson, Pedersen).

A : simple から.

$\liminf_n \min_\tau \tau(\underbrace{f_n'^{\frac{1}{2}} a^2 f_n'^{\frac{1}{2}}}_{b_n}) > 0$

$d_\tau(e_n') < d_\tau(b_n)$ $\forall \tau$.

(ii) $\exists r_n \in A$, $\| r_n^* b_n r_n - e_n \| \to 0$, $\|r_n\| \leq 1$,

$S_n := \sum_{i=1}^N d_i a f_n'^{\frac{1}{2}} r_n c_i$ とすれば. $\|f_n S_n - S_n\| \to 0$ で条件の1つがいえた.

$x \in F$, $S_n^* x S_n = \sum_{i\cdot j} c_i^* r_n^* f_n'^{\frac{1}{2}} \underbrace{a d_i^* x d_j a}_{\approx\ \omega(d_i^* x d_j) \cdot a^2} f_n'^{\frac{1}{2}} r_n c_j$

$\overset{\varepsilon}{\approx} \sum \omega(d_i^* x d_j) c_i^* \underbrace{r_n^* b_n r_n}_{\approx\ e_n} c_j \approx \varphi(x) \cdot e_n$.

No.

⟨(ii) ⇒ (iii)⟩

A : nuclear. + Prop. ■

(SI)性.

Def. e_n, f_n は前の通り.

If $^\exists (s_n)_n \in A_\infty$ s.t.

$\|s_n^* s_n - e_n\| \to 0$, $\|f_n s_n - s_n\| \to 0$,

we say that A has property (SI)

⟨(iii) ⇒ (iv)⟩

id_A に対し. $^\exists s_n \in A$ s.t. $\|s_n^* a s_n - a e_n\| \to 0$

$(f_n s_n)_n = (s_n)_n$. であるが.

とくに. $(s_n^* s_n)_n = (e_n)_n$ となる. また.

$$\|[s_n, a]\|^2 = \|a^* \underbrace{s_n^* s_n}_{e_n} a - a^* \underbrace{s_n^* a s_n}_{a e_n} - \underbrace{s_n^* a^* s_n}_{a^* e_n} a + \underbrace{s_n^* a^* a s_n}_{a^* a e_n}\| \to 0.$$

⟨(iv) ⇒ (i)⟩

$\overline{\pi_\tau(A)}^w \cong R \supset M_{k-1}$

$\rightsquigarrow$ $^\exists c_{i,n} \in A$, $(c_{i,n} c_{j,n}^*)_n = \delta_{ij}\ (c_{1,n}^2)$

$\tau(\underbrace{1 - \sum c_{i,n}^* c_{i,n}}_{= e_n}) \to 0$

(SI) より $(s_n)_n \in A_\infty$ $(s_n^* s_n + \sum c_{i,n}^* c_{i,n})_n = 1$, $(c_{i,n} s_n)_n = (s_n)_n$.

$\underset{\mathcal{Z}}{I(k, k-1)} \subset_{\text{unital}} A_\infty$ ⇒ 一般論から $A \otimes \mathcal{Z} \cong A$ がいえる.