マルチンゲール調和解析と放物型方程式の最大正則性

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田中　仁 氏 (東大数理) との共同研究

Outline of the Talk

マルチンゲール調和解析の理論は、例えば、Doob の最大定理に見られるように、確率解析において有用であるだけでなく、調和解析の理論の抽象的な枠組みへの一般化とみなせる点で、興味深いと考えられる。また、その理論は、古典調和解析（すなわち、ユークリッド空間の上のフーリエ解析）の種々の一般化においても重要な役割を果たす。たとえば、Non-Doubling 測度を持つ空間上での特異積分作用素の L^2 有界性の証明、Banach 空間に値をとる函数への古典調和解析の理論の拡張、特異積分の有界性に関する、L^p 空間での Sharp な重みつき評価などへの応用などが近年の目覚ましい結果である。ここでは、Banach 空間に値をとる古典調和解析の理論とその最大正則性への応用について復習し、次に、最近、田中仁氏（東大数理）との共同研究で得られた、マルチンゲール調和解析の結果に関して述べる。最初に、両者の結果について述べるのに必要なマルチンゲール理論の基礎について復習する。最後に、両方のトピックを融合するという観点から、今後の課題についても述べたい。

- Setting:

Let $(\Omega, \mathcal{F}, \mu)$ be a σ-finite measure space. Denote by $\mathcal{F}^0$ the collection of sets in $\mathcal{F}$ with finite measure. The measure space $(\Omega, \mathcal{F}, \mu)$ is called σ-finite if there exist sets $E_i \in \mathcal{F}^0$ such that $\bigcup_{i=0}^{\infty} E_i = \Omega$. In this talk, all measure spaces are assumed to be σ-finite. An $\mathcal{F}$-measurable function $f : \Omega \to \mathbb{R}$ is called $\mathcal{A}$-integrable if it is integrable on all sets of $\mathcal{A}$, i.e.,

$$1_E f \in L^1(\mathcal{F}, \mu) \text{ for all } E \in \mathcal{A}.$$

We denote the collection of all such functions by $L^1_{\mathcal{A}}(\mathcal{F}, \mu)$.

Conditional Expectation

If $\mathcal{G} \subset \mathcal{F}$ is another σ-algebra, it is called a sub-σ-algebra of $\mathcal{F}$. A function $g \in L^1_{\mathcal{G}^0}(\mathcal{G}, \mu)$ is called the conditional expectation of $f \in L^1_{\mathcal{G}^0}(\mathcal{F}, \mu)$ with respect to $\mathcal{G}$ if there holds

$$\int_G f \, d\mu = \int_G g \, d\mu \text{ for all } G \in \mathcal{G}^0.$$

The conditional expectation of f with respect to $\mathcal{G}$ will be denoted by $E[f|\mathcal{G}]$, which exists uniquely in $L^1_{\mathcal{G}^0}(\mathcal{G}, \mu)$ due to σ-finiteness of $(\Omega, \mathcal{G}, \mu)$

σ-finite filtered measure space

A family of sub-σ-algebras $(\mathcal{F}_i)_{i\in\mathbb{Z}}$ is called a filtration of $\mathcal{F}$ if $\mathcal{F}_i \subset \mathcal{F}_j \subset \mathcal{F}$ whenever $i, j \in \mathbb{Z}$ and $i < j$. We call a quadruplet $(\Omega, \mathcal{F}, \mu; (\mathcal{F}_i)_{i\in\mathbb{Z}})$ a σ-finite filtered measure space. We write

$$\mathcal{L} := \bigcap_{i\in\mathbb{Z}} L^1_{\mathcal{F}_i^0}(\mathcal{F}, \mu).$$

Notice that $L^1_{\mathcal{F}_i^0}(\mathcal{F}, \mu) \supset L^1_{\mathcal{F}_j^0}(\mathcal{F}, \mu)$ whenever $i < j$. For a function $f \in \mathcal{L}$ we will denote $E[f|\mathcal{F}_i]$ by $\mathcal{E}_i f$. By the tower rule of conditional expectations, a family of functions $\mathcal{E}_i f \in L^1_{\mathcal{F}_i^0}(\mathcal{F}_i, \mu)$ becomes a martingale. (see the Definition of Martingales below).

Weights, Positive Operators and Maximal Operators

By a weight we mean a nonnegative function which belongs to $\mathcal{L}$ and, by a convention, we will denote the set of all weights by $\mathcal{L}^+$.

Let α_i, $i \in \mathbb{Z}$, be a nonnegative bounded $\mathcal{F}_i$-measurable function and set $\alpha = (\alpha_i)$. For a function $f \in \mathcal{L}$ we define a positive operator T_α by

$$T_\alpha f := \sum_{i \in \mathbb{Z}} \alpha_i \mathcal{E}_i f,$$

and, define a generalized Doob's maximal operator M_α by

$$M_\alpha f := \sup_{i \in \mathbb{Z}} \alpha_i |\mathcal{E}_i f|.$$

When $\alpha = (1_\Omega)$ this is Doob's maximal operator and we will write then $M_\alpha f =: f^*$.

Aim in the second part of the talk

Our aim in the second part of the talk to investigate the weights for which the boundedness of positive operators and maximal operators hold, when α is given. Namely, we are interested in the characterization of weights u, v for which the following holds:

$$\|T^{\alpha}(fv)\|_{L^q(ud\mu)} \leq C\|f\|_{L^p(vd\mu)}.$$

We are also interested in the characterization of weights u, v for which the following holds:

$$\|M^{\alpha}f\|_{L^q(ud\mu)} \leq C\|f\|_{L^p(vd\mu)}.$$

Properties of Conditional Expectation Operators I

(i) Let $f \in L^1_{\mathcal{G}^0}(\mathcal{F}, \mu)$ and g be an $\mathcal{G}$-measurable function. Then the two conditions $fg \in L^1_{\mathcal{G}^0}(\mathcal{F}, \mu)$ and $gE[f|\mathcal{G}] \in L^1_{\mathcal{G}^0}(\mathcal{G}, \mu)$ are equivalent and, assuming one of these conditions, we have

$$E[fg|\mathcal{G}] = gE[f|\mathcal{G}];$$

(ii) Let $f_1, f_2 \in L^1_{\mathcal{G}^0}(\mathcal{F}, \mu)$. Then the three conditions

$$E[f_1|\mathcal{G}]f_2 \in L^1_{\mathcal{G}^0}(\mathcal{G}, \mu), \quad E[f_1|\mathcal{G}]E[f_2|\mathcal{G}] \in L^1_{\mathcal{G}^0}(\mathcal{G}, \mu)$$

and

$$f_1 E[f_2|\mathcal{G}] \in L^1_{\mathcal{G}^0}(\mathcal{G}, \mu)$$

are all equivalent and, assuming one of these conditions, we have

$$E[E[f_1|\mathcal{G}]f_2|\mathcal{G}] = E[f_1|\mathcal{G}]E[f_2|\mathcal{G}] = E[f_1 E[f_2|\mathcal{G}]|\mathcal{G}].$$

Properties of Conditional Expectation Operators II

(iii) Let $\mathcal{G}_1 \subset \mathcal{G}_2$ ($\subset \mathcal{F}$) be two sub-σ-algebras of $\mathcal{F}$ and let $f \in L_{\mathcal{G}_2^0}(\mathcal{F}, \mu)$. Then

$$E[f|\mathcal{G}_1] = E[E[f|\mathcal{G}_2]|\mathcal{G}_1].$$

Definition of Martingale in Our Setting

Definition

Let $(\Omega, \mathcal{F}, \mu; (\mathcal{F}_i)_{i \in \mathbb{Z}})$ be a σ-finite filtered measure space. Let $(f_i)_{i \in \mathbb{Z}}$ be a sequence of $\mathcal{F}_i$-measurable functions. Then the sequence $(f_i)_{i \in \mathbb{Z}}$ is called a "martingale" if $f_i \in L^1_{\mathcal{F}_i^0}(\mathcal{F}_i, \mu)$ and $f_i = \mathcal{E}_i f_j$ whenever $i < j$.

Example of Martingale

(i) $\mathcal{E}_i f \in L^1_{\mathcal{F}_i^0}(\mathcal{F}_i, \mu)$ for $f \in \mathcal{L}$.

(ii) Let $(\Omega, \mathcal{F}, \mu)$ be a probability space. Let $(X_i)_{i\in\mathbb{N}}$ be a sequence of independent random variables in Ω, each of which has a mean value zero. Let $\mathcal{F}_i$ be a σ-algebra generated by $(X_k)_{1\le k\le i}$. Then the sequence $(X_i)_{i\in\mathbb{N}}$ becomes a martingale with respect to the filtration $(\mathcal{F}_i)_{i\in\mathbb{N}}$.

Conditional Expectation Operators for functions with values in Banach Space

Conditional Expectation Operator for functions with values in a Banach Space X can be defined, by first being defined on the set of simple functions, and then by being extended to $L^1(\Omega; X)$ using the density of simple functions in $L^1(\Omega; X)$. When considering functions with values in Banach spaces in this talk, we assume that Ω is a probability space. Martingale with values in Banach spaces can be defined in the same way as in the scalar case.

Martingale Transform

Let $(f_i)_{i\in\mathbb{N}}$ be an X-valued martingale with mean value zero. $\delta f_i = f_i - f_{i-1}$ $(f_i = 0)$ are called martingale differences. Let $\epsilon_i \in \{-1, +1\}$ for $i \in \mathbb{N}$ be arbitrarily taken. Then $g_i = \sum_{1\le k\le i} \epsilon_k \delta f_k$ is obviously an X-valued martingale with mean value zero. This transform mapping (f_i) to (g_i) is called a martingale transform. We are interested in L^p boundedness properties of martingale transforms. Namely,

$$\|g_i\|_{L^p(\Omega,\mu)} \le C_p \|f_i\|_{L^p(\Omega,\mu)}$$

for any (f_i), (ϵ_i) and $i \in \mathbb{N}$.
Burkholder (1966) showed that when X is $\mathbb{R}$ or $\mathbb{C}$, martingale transforms are bounded in L^p where $1 < p < \infty$.

The class UMD_p

Burkholder was next interested in the characterization of Banach spaces for which martingale transforms are bounded in L^p. If this is the case, a Banach space is said to have unconditional martingale differences in L^p $(1 < p < \infty)$, for short, UMD_p.

It can be easily seen, using the orthogonality of martingale differences, that every Hilbert space is UMD_2. One can also relatively easily see that if X is UMD_p, $(1 < p < \infty)$ and Γ is a σ-finite measure space, then $L^p(\Gamma; X)$ is UMD_p.

The class *UMD*

Burkholder (1981) showed that the class UMD_p, $(1 < p < \infty)$ does not depend on p.

UMD plays a crucial role in Harmonic Analysis with values in Banach spaces.

The class HT

The function space $\mathcal{S}(\mathbb{R}; X)$, which consists of X-valued rapidly decaying functions, can be defined similarly with scalar valued case.

Let $f \in \mathcal{S}(\mathbb{R}; X)$. The Hilbert transform of f is defined by

$$Hf = \frac{1}{\pi} \, P.V. \int \frac{f(s)}{t-s} \, ds$$

If the Hilbert transform is bounded in $L^p(\mathbb{R}; X)$ for some $(1 < p < \infty)$, then X is said to be in the class HT. The class HT is known to coincide with the class UMD. The direction UMD implies HT was proved by Burkholder and McConnell (1981). The opposite direction was proved by Bourgain (1986).

Fourier Multiplier Theorem of Weis

Theorem (Weis, 2001)

Let X and Y be UMD. Let $M(\tau) \in C^1(\mathbb{R} \setminus \{0\}, \mathcal{L}(X, Y))$. Let

$$\mathcal{R}(\{M(\tau) : \tau \in \mathbb{R} \setminus \{0\}\}) = c_0 < \infty,$$

$$\mathcal{R}(\{\tau M(\tau) : \tau \in \mathbb{R} \setminus \{0\}) = c_1 < \infty$$

and $1 < p < \infty$.
Then the Fourier Multiplier Operator T_M associated to M

$$[T_M f](t) := \mathcal{F}^{-1}[M(\tau)\mathcal{F}[f](\tau)](t), (f \in \mathcal{S}(\mathbb{R}; X))$$

is bounded from $L^p(\mathbb{R}; X)$ to $L^p(\mathbb{R}; Y)$. The operator norm is bounded from above by $C(c_0 + c_1)$ where C depends on p and X only.

$\mathcal{R}$-boundedness of an Operator Family

Definition

Let X and Y be Banach spaces. An operator family $\mathcal{T} \subset \mathcal{L}(X, Y)$ is called $\mathcal{R}$-bounded if there is $C > 0$ and $1 \leq p < \infty$ such that for each $N \in \mathbb{N}$, $T_j \in \mathcal{T}$, $x_j \in X$ and for all independent, symmetric , $\{-1, 1\}$-valued random variables ϵ_j on a probability space $(\Omega, \mathcal{F}, \mu)$ the inequality

$$| \sum_{j=1}^{N} \epsilon_j T_j x_j |_{L^p(\Omega;Y)} \leq C | \sum_{j=1}^{N} \epsilon_j x_j |_{L^p(\Omega;X)}$$

is valid. The smallest such C is called $\mathcal{R}$-bound of $\mathcal{T}$, we denote it by $\mathcal{R}(\mathcal{T})$.

This notion is originally due to Bourgain (80's).

Sectorial Operators

Definition

Let X be a complex Banach space, and A a closed linear operator in X. A is called sectorial if the following to conditions are satisfied.

$$(S1) \quad \overline{D(A)} = X, \ \ \overline{R(A)} = X, \ \ (-\infty, 0) \subset \rho(A)$$

$$(S2) \quad \text{There exists } M < \infty \text{ such that } |t(t+A)^{-1}| \le M \text{ for all } t > 0.$$

The class of sectorial operators are denoted by $\mathcal{S}(X)$.

The spectral angle of a sectorial operator

If $A \in \mathcal{S}(X)$ then $\rho(-A) \supset \Sigma_\theta$, for some $\theta > 0$, and

$$\sup\{|\lambda(\lambda + A)^{-1}| : |\arg\lambda| < \theta\} < \infty.$$

Hence we can define the *spectral angle* ϕ_A *of* $A \in \mathcal{S}(X)$ *by*

$$\phi_A = \inf\{\phi : \rho(-A) \supset \Sigma_{\pi-\phi}, \sup_{\lambda \in \Sigma_{\pi-\phi}} |\lambda(\lambda + A)^{-1}| < \infty\}.$$

Evidently, we have $\phi_A \in [0, \pi)$ *and*

$$\phi_A \leq \sup\{|\arg\lambda| : \lambda \in \sigma(A)\}.$$

$\mathcal{R}$-sectorial operators

Definition

A sectorial operator is called $\mathcal{R}$-sectorial if

$$\mathcal{R}_A(0) := \mathcal{R}\{t(t+A)^{-1} : t > 0\} < \infty.$$

The $\mathcal{R}$-angle $\phi_A^{\mathcal{R}}$ of A is defined by means of

$$\phi_A^{\mathcal{R}} := \inf\{\theta \in (0,\pi) : \mathcal{R}_A(\pi - \theta) < \infty\},$$

where

$$\mathcal{R}_A(\theta) := \mathcal{R}\{\lambda(\lambda+A)^{-1} : |\arg\lambda| \le \theta\}.$$

Evolution equation of parabolic type and solution formula

Consider the Cauchy problem

$$\frac{d}{dt}u(t) + Au(t) = f(t),\ \ t \geq 0,\ \ u(0) = 0, \tag{1}$$

where A denotes of a sectorial operator in a Banach space with spectral angle $\phi_A < \frac{\pi}{2}$. For a given function $f \in L_p(\mathbb{R}_+; X)$ the solution is represented by the variation of parameters formula

$$u(t) = \int_0^t e^{-As} f(t-s)ds,\ \ t \geq 0.$$

$\mathcal{R}$-Sectorial Operators and Maximal L^p- Regularity

Maximal regularity of type L^p on $\mathbb{R}_+$ means $Au \in L_p(\mathbb{R}_+; X)$ for each $f \in L_p(\mathbb{R}_+; X)$.
Looking at the problem on the whole line instead of the half-line, the question becomes whether the convolution operator with kernel

$$K(t) = Ae^{-At}\chi_{(0,\infty)}(t), \quad t \in \mathbb{R},$$

is L^p-bounded. The symbol of this convolution operator is given by

$$M(\rho) = A(i\rho + A)^{-1}, \quad \rho \in \mathbb{R}.$$

Maximal Regularity Theorem of Lutz Weis

Using this, we have the following theorem

Theorem (Weis, 2001)

Let X be a UMD, $1 < p < \infty$ and let A be a sectorial operator in X with spectral angle $\phi_A < \frac{\pi}{2}$. Then (1) has maximal regularity of type L^p on $\mathbb{R}_+$ if and only if A is $\mathcal{R}$-sectorial with $\phi_A^{\mathcal{R}} < \frac{\pi}{2}$.

Previously, only sufficient conditions for maximal regularity such as $\mathcal{BIP}$ is known. This type of theorem or a variant of it is now widely used to show maximal regularity of a linear system of parabolic type:

Stokes equations with various boundary condition,
Generalized Stokes equation with various boundary condition etc.

These have several important applications to corresponding nonlinear problems.

Positive operators and maximal operators

We recall the definition of Positive operators and Maximal operators:
Let α_i, $i \in \mathbb{Z}$, be a nonnegative bounded $\mathcal{F}_i$-measurable function and set $\alpha = (\alpha_i)$. For a function $f \in \mathcal{L}$ we define a positive operator T_α by

$$T_\alpha f := \sum_{i \in \mathbb{Z}} \alpha_i \mathcal{E}_i f,$$

and, define a generalized Doob's maximal operator M_α by

$$M_\alpha f := \sup_{i \in \mathbb{Z}} \alpha_i |\mathcal{E}_i f|.$$

When $\alpha = (1_\Omega)$ this is Doob's maximal operator and we will write then $M_\alpha f =: f^*$.

Weighted Inequality for Positive Operators I

For Positive operator case, we assume the following condition for $\alpha = (\alpha_i)$. $\overline{\alpha_i} \in \mathcal{L}^+$ where $\overline{\alpha_i} := \sum_{j \geq i} \alpha_j$.

Moreover,

$$\mathcal{E}_i \overline{\alpha_i} \approx \overline{\alpha_i}$$

holds.

Weighted Inequality for Positive Operators II

Theorem (Tanaka-T., 2012)

Let $1 < p \le q < \infty$, α *satisfy the previous condition and* $w \in \mathcal{L}^+$ *be a weight. Then the following statements are equivalent:*

(a) *There exists a constant* $C_1 > 0$ *such that*

$$\|T_\alpha f\|_{L^q(wd\mu)} \le C_1 \|f\|_{L^p(d\mu)};$$

(b) *For any* $E \in \mathcal{F}_i^0$, $i \in \mathbb{Z}$, *there exists a constant* $C_2 > 0$ *such that*

$$\left(\int_E \left(\sum_{j \ge i} \alpha_j \mathcal{E}_j w \right)^{p'} d\mu \right)^{\frac{1}{p'}} \le C_2 [wd\mu](E)^{\frac{1}{q'}}.$$

Moreover, the least possible C_1 *and* C_2 *are equivalent.*

Carleson Measure Theorem I

The proof of the previous theorem for positive operators uses the following Carleson measure theorem, which is new in the point that we treat different exponents in the embedding.

Let $(\Omega, \mathcal{F}, \mu; (\mathcal{F}_i)_{i \in \mathbb{Z}})$ be a σ-finite filtered measure space. We also let f_i, $i \in \mathbb{Z}$, be an $\mathcal{F}_i$-measurable nonnegative real-valued function and ν_i be a measure on $\mathcal{F}_i$. Set a maximal function of $f = (f_i)$ by $f^* := \sup_i f_i$.

Carleson Measure Theorem II

Theorem

Let $\theta \geq 1$ be arbitrarily taken and be fixed. Then the following conditions are equivalent:

(i) *For any $E \in \mathcal{F}_i$, $i \in \mathbb{Z}$, there exists a constant $C_0 > 0$ such that*

$$\sum_{j \geq i} \nu_j(E) \leq C_0 \mu(E)^\theta;$$

(ii) *For "any" $p \in (0, \infty)$ there exists a constant $C_p > 0$ such that*

$$\left(\sum_{i \in \mathbb{Z}} \int_\Omega f_i^{p\theta} \, d\nu_i \right)^{\frac{1}{p\theta}} \leq C_p \| f^* \|_{L^p(d\mu)};$$

Moreover, the least possible C_0 and C_p enjoy

$$C_p \leq (C_0 \theta)^{\frac{1}{p\theta}}, \qquad C_0 \leq C_p^{p\theta}.$$

Weighted Inequality for Positive Operators III

We can also have a weighted inequality for positive operators for $p < q$ case. In that case, we have to use Carleson embedding result for $\theta < 1$.

Weighted Inequality for Generalized Maximal operators I

The two-weight inequality for generalized maximal operators can be obtained. (cf. Long-Peng (1986), Chen-Liu (2011)).
Using this we can have,

Theorem

Let $1 < p < \infty$, $w \in \mathcal{L}^+$ be a weight and $\sigma = w^{1-p'} \in \mathcal{L}^+$. Then, one-weight norm inequality

$$\|f^*\|_{L^p(wd\mu)} \leq C_1 \|f\|_{L^p(wd\mu)}$$

holds "if and only if"

$$\sup_{i \in \mathbb{Z}} \|(\mathcal{E}_i w)(\mathcal{E}_i \sigma)^{p-1}\|_{L^\infty(d\mu)} < C_2 < \infty.$$

Moreover, the least possible C_1 and C_2 enjoy

$$C_2 \leq C_1^p, \qquad C_1 \leq C C_2^{\frac{1}{p-1}}.$$

Weighted Inequality for Generalized Maximal operators II

The quantity $\sup_{i \in \mathbb{Z}} \|(\mathcal{E}_i w)(\mathcal{E}_i \sigma)^{p-1}\|_{L^\infty(d\mu)}$ is called A_p norm of $w \in \mathcal{L}^+$.
(cf. Izumisawa-Kazamaki (1977), Jawerth (1986).)

Open Problem

(i) Characterize Two Weights for Positive Operator Inequality, or, in other words, find necessary and sufficient condition of u, v for which

$$\|T^{\alpha}(fv)\|_{L^q(ud\mu)} \leq C\|f\|_{L^p(vd\mu)}.$$

to hold.

(ii) Two-Weight Inequality for Martingale Transform and Sharp One-Weight Estimate.

(iii) The extension of (i) and (ii) to a Banach-space setting and its application to maximal regularities of various parabolic equations.