Sage における degree 2 の Siegel 保型形式の計算

竹森 翔

Department of the Mathematics, Kyoto University

Siegel modular forms

n : a positive integer
Γ_n: the Siegel modular group of degree n.

$$\Gamma_n := \left\{ g \in \mathrm{GL}_{2n}(\mathbb{Z}) \;\middle|\; {}^t g w_n g = w_n \right\},$$

where $w_n = \begin{pmatrix} 0_n & -1_n \\ 1_n & 0_n \end{pmatrix}$. Note that $\Gamma_1 = \mathrm{SL}_2(\mathbb{Z})$. Define Siegel upper half space $\mathfrak{H}_n$ by

$$\mathfrak{H}_n := \left\{ Z = X + iY \;\middle|\; X,\ Y \in \mathrm{Sym}_n(\mathbb{R}),\ Y \text{ is positive definite} \right\}.$$

Siegel modular forms

k: a non-negative integer
$M_k(\Gamma_n)$: the set of holomorphic functions on $\mathfrak{H}_n$ satisfying the following condition.

$$F((AZ+B)(CZ+D)^{-1}) = \det(CZ+D)^k F(Z)$$

$\forall \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_n$. If $n = 1$, we add the cusp condition. By definition, we have

$$M_k(\Gamma_n) \cdot M_l(\Gamma_n) \subseteq M_{k+l}(\Gamma_n).$$

Then $M_k(\Gamma_n)$ is a finite dimensional vector space over $\mathbb{C}$. We call an element of $M_k(\Gamma_n)$ a Siegel modular form of weight k and degree n.

Fourier expansion

F: a Siegel modular form of degree 2.
Then Fourier coefficients of F are indexed by

$$\left\{ \begin{pmatrix} n & r/2 \\ r/2 & m \end{pmatrix} \;\middle|\; n, r, m \in \mathbb{Z},\ n, m, 4nm - r^2 \geq 0 \right\}.$$

We have the following Fourier expansion:

$$F\left(\begin{pmatrix} \tau & z \\ z & \omega \end{pmatrix}\right) = \sum_{\substack{n,\, r,\, m \in \mathbb{Z} \\ n,\, m,\, 4nm - r^2 \geq 0}} a((n, r, m), F) \mathrm{e}(n\tau + rz + m\omega),$$

for $\begin{pmatrix} \tau & z \\ z & \omega \end{pmatrix} \in \mathfrak{H}_2$. Here $\mathrm{e}(z) = \exp(2\pi i z)$ for $z \in \mathbb{C}$.

Siegel Φ operator

We define a linear map $\Phi : M_k(\Gamma_2) \to M_k(\Gamma_1)$ by

$$\Phi(F) = \sum_{n=0}^{\infty} a((n,0,0),F)e(nz).$$

We call Φ the Siegel operator.
The space of cusp forms $S_k(\Gamma_2)$ is defined by

$$S_k(\Gamma_2) = \ker(\Phi).$$

Siegel Eisenstein series

$k \geq 4$: an even integer
We define the Siegel Eisenstein series of degree 2 and weight k by

$$E_k = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_2} \det(CZ + D)^{-k},$$

where $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ with A, B, C, D in M_2 and $\Gamma_\infty = \left\{ \gamma \in \Gamma_2 \mid C = 0 \right\}$. Then this series converges absolutely to an element of $M_k(\Gamma_2)$. Fourier coefficients of E_k were explicitly calculated by Kaufhold.

Hecke operators

For $n = 1, 2$ and $m \geq 1$, $T^{(n)}(m)$ denotes the Hecke operator of degree n. Hecke operators acts on $M_k(\Gamma_n)$ and $S_k(\Gamma_n)$. It is known that $M_k(\Gamma_n)$ have a basis consisting eigenvector for all $T^{(n)}(m)$. We call such a modular form an eigenform.
We can compute the Fourier coefficients of $T^{(n)}(m)F$ if Fourier coefficients of F are known. For example, we have

$$\begin{aligned} a((n,r,m), T^{(2)}(2)F) = {} & a(2n, 2r, 2m) + 2^{2k-3} a(n/2, r/2, m/2) \\ & + 2^{k-2} \{a(m/2, -r, 2n) + a(n/2, r, 2m) \\ & + a((n+r+m)/2, r+2m, 2m)\}. \end{aligned}$$

Here we put $a(n,r,m) = a((n,r,m), F)$ and we understand $a(n,r,m) = 0$ unless $n, r, m \in \mathbb{Z}$.

Euler factor of spinor L-function

Let $n = 1$ or 2. Let $F \in M_k(\Gamma_n)$ be an eigenform and p a prime. We denote by $\lambda(m)$ the eigenvalue of $T^{(n)}(m)$. We define a polynomial $Q_p^{(n)}(F; X)$ as follows.

1. If $n = 1$, then we define

$$Q_p^{(1)}(F; X) = 1 - \lambda(p)X + p^{k-1}X^2.$$

2. If $n = 2$, then we define

$$\begin{aligned} Q_p^{(2)}(F; X) &= 1 - \lambda(p)X \\ &+ \left(\lambda(p)^2 - \lambda(p^2) - p^{2k-4}\right) X^2 - \lambda(p)p^{2k-3}X^3 + p^{4k-6}X^4. \end{aligned}$$

Remark 1

$Q_p^{(n)}(F; p^{-s})$ is the Euler factor of spinor L-function.

Algorithm for computing Fourier coefficients

Let

$$M(\Gamma_2) = \bigoplus_{k \in \mathbb{Z}_{\geq 0}} M_k(\Gamma_2)$$

be the ring of Siegel modular forms of degree 2. Put

$$x_{10} := E_4 E_6 - E_{10},$$

$$x_{12} := 3^2 \cdot 7^2 E_4^3 + 2 \cdot 5^3 E_6^2 - 691 E_{12}.$$

Then they are Siegel cusp forms of weight 10 and 12 respectively. For $k = 10,\ 12$, we put

$$X_k := \frac{1}{a((1,1,1), x_k)} x_k.$$

Algorithm for computing Fourier coefficients

By Igusa, it is known that

1. There exists a weight $\mathbf{35}$ cusp form X_{35} (we normalize X_{35} so that $a((2,-1,3), X_{35}) = 1$).
2. E_4, E_6, X_{10}, X_{12} and X_{35} generate $M(\Gamma_2)$ as a $\mathbb{C}$-algebra.
3. E_4, E_6, X_{10} and X_{12} are algebraically independent over $\mathbb{C}$.
4. E_4, E_6, X_{10}, X_{12} and X_{35} have integral Fourier coefficients.

Algorithm for computing Fourier coefficients

The cusp form X_{35} can be constructed as follows.

Theorem 2 (Ibukiyama)

For $Z \in \mathfrak{H}_2$, we put $Z = \begin{pmatrix} \tau & z \\ z & \omega \end{pmatrix}$ and

$$\partial_\tau = \frac{1}{2\pi i}\frac{\partial}{\partial\tau}, \qquad \partial_\omega = \frac{1}{2\pi i}\frac{\partial}{\partial\omega}, \qquad \partial_z = \frac{1}{2\pi i}\frac{\partial}{\partial z}.$$

Then

$$X_{35} = \frac{1}{2^9 \cdot 3^4} \det \begin{pmatrix} 4E_4 & 6E_6 & 10X_{10} & 12X_{12} \\ \partial_\tau E_4 & \partial_\tau E_6 & \partial_\tau X_{10} & \partial_\tau X_{12} \\ \partial_\omega E_4 & \partial_\omega E_6 & \partial_\omega X_{10} & \partial_\omega X_{12} \\ \partial_z E_4 & \partial_z E_6 & \partial_z X_{10} & \partial_z X_{12} \end{pmatrix}.$$

Simple implementation in Sage

We implement a class of Siegel modular forms and its multiplication method briefly.
We create a class **SiegelModularForm**. This class has two attributes, **prec** and **fc_dct**.
For a positive integer x, we put

$$S(x) = \left\{(n, r, m) \in \mathbb{Z}^3 \;\middle|\; n, m, 4nm - r^2 \geq 0, \max(n, m) \leq x\right\}.$$

The attribute **prec** is a positive integer. We compute the Fourier coefficients for $(n, r, m) \in S(\mathrm{prec})$. The attribute **fc_dct** is a dictionary whose keys are elements of $S(\mathrm{prec})$.

Simple implementation in Sage

```
class SiegelModularForm(object):
    def __init__(self, fc_dct, prec):
        self.fc_dct = fc_dct
        self.prec = prec

    # self == other calls self.__eq__(other)
    def __eq__(self, other):
        return self.fc_dct == other.fc_dct

    # self * other calls self.__mul__(other)
    def __mul__(self, other):
        prec = self.prec
        dct = mul_internal(self.fc_dct,
                           other.fc_dct,
                           prec)
        return SiegelModularForm(dct, prec)
```

Simple implementation in Sage

We define the function **mul_internal** later. We also define two functions, **semipos_mats** and **semipos_mats_lt**.
The function **semipos_mats** takes one positive integer x and returns the set $S(x)$.
The function **semipos_mats_lt** takes three integers n, r, m and returns the set

$$\left\{(a, b, c) \in S(\max(n, m)) \;\middle|\; (n - a, r - b, m - c) \in S(\max(n, m))\right\}.$$

Simple implementation in Sage

```
@cached_function
def semipos_mats(x):
    return [(n, r, m) for n in range(x + 1)\
                      for m in range(x + 1)\
                      for r in range(-2*x, 2*x + 1)\
                      if 4*n*m - r^2 >= 0]

@cached_function
def semipos_mats_lt(n, r, m):
    s = semipos_mats(max(n, m))
    return [(a, b, c) for a, b, c in s \
                      if (n-a, r-b, m-c) in s]
```

Simple implementation in Sage

We define **mul_internal** as follows.

```
def mul_internal(dct1, dct2, prec):
    s = semipos_mats(prec)
    dct = {(n, r, m):0 for n, r, m in s}
    for n, r, m in s:
        for a,b,c in semipos_mats_lt(n,r,m):
            dct[(n, r, m)] += dct1[(a, b, c)]
                * dct2[(n-a, r-b, m-c)]
    return dct
```

Computation of eigenforms

In the following computation, we use a package “degree2” for Siegel modular forms of degree 2. The source code can be found at `https://github.com/stakemori/degree2`.
The following code has been tested under Sage 5.11 and “degree2” (revision a701a39).

Computation of eigenforms

We want to calculate the basis of $M_k(\Gamma_2)$ consisting of eigenforms. Put

$$N_k(\Gamma_2) = \left\{ F \in M_k(\Gamma_2) \;\middle|\; a((0,0,0),F) = 0 \right\}.$$

Then we have

$$M_k(\Gamma_2) = \mathbb{C}E_k \oplus N_k(\Gamma_2).$$

Since E_k is an eigenform and $N_k(\Gamma_2)$ is stable under the action of Hecke operators, this is a decomposition as a Hecke module.

Computation of eigenforms

In “degree2”, we can obtain the space $N_k(\Gamma_2)$ by the function **KlingenEisensteinAndCuspForms**.

```
sage: N10 = KlingenEisensteinAndCuspForms(10)
sage: N10.dimension()
1
sage: F10 = N10.basis()[0]
```

$N_{10}(\Gamma_2)$ is one dimensional and spanned by X_{10}. We can also obtain X_{10} by the function **x10_with_prec**.

```
sage: X10 = x10_with_prec(2)
sage: X10 == F10
True
```

Computation of eigenforms

To obtain an eigenform of weight 12, we compute the characteristic polynomial of $T^{(2)}(2)$ on $N_{12}(\Gamma_2)$.

```
sage: N12 = KlingenEisensteinAndCuspForms(12, 5)
sage: N12.hecke_matrix(2).charpoly().factor()
(x - 2784) * (x + 24600)
sage: G12 = N12.eigenform_with_eigenvalue_t2(-24600)
sage: F12 = G12.normalize()
```

Here **G12** is an eigenform of $N_{12}(\Gamma_2)$ whose eigenvalue of $T^{(2)}(2)$ is equal to -24600 and **F12** is a constant multiple of **G12** whose Fourier coefficient at $(1, 0, 0)$ is 1.

Klingen Eisenstein series

The space $N_{12}(\Gamma_2)$ is spanned by F_{12} and X_{12}. Here F_{12} is an eigenform such that $\Phi(F_{12}) = \Delta$, where $\Delta \in S_{12}(\Gamma_1)$ is the Ramanujan's Δ. We can compute the polynomial $Q_p^{(2)}(F_{12}; X)$ for $p = 2$ as follows.

```
sage: F12.euler_factor_of_spinor_l(2).factor()
(1 + 24*x + 2048*x^2) *
   (1 + 24576*x + 2147483648*x^2)
```

The first factor is equal to $Q_2^{(1)}(\Delta; x)$ and the second factor is equal to $Q_2^{(1)}(\Delta; 2^{10}x)$.

Klingen Eisenstein series

THEOREM 3

There exists a $\mathbb{C}$-linear injective map $E : S_k(\Gamma_1) \hookrightarrow N_k(\Gamma_2)$ such that $\Phi \circ E = \mathbf{id}$. For a prime p and an eigenform $f \in S_k(\Gamma_1)$, $E(f)$ is also an eigenform and we have

$$Q_p^{(2)}(E(f); X) = Q_p^{(1)}(f; X) Q_p^{(1)}(f; p^{k-2} X).$$

For $f \in S_k(\Gamma_1)$, we call $E(f)$ the Klingen Eisenstein series associated to f. By the theorem above, we have

$$N_k(\Gamma_2) = E(S_k(\Gamma_1)) \oplus S_k(\Gamma_2).$$

Euler factor of elliptic cusp forms

For $f \in S_k(\Gamma_1)$ with $\mathbf{dim}\, S_k(\Gamma_1) = 1$ and $p = 2$, we compute $Q_p^{(1)}(f; X)$. The function **euler_factor_of_l** takes an eigenform and a prime p and returns $Q_p^{(1)}(f; X)$. **wts_of_one_dim** is the list of the positive integers k such that $12 \le k < 30$ and $\mathbf{dim}\, S_k(\Gamma_2) = 1$.

```
def euler_factor_of_l(f, p):
    wt = f.weight()
    return 1 - f[p]/f[1]*x + p^(wt-1)*x^2

wts_of_one_dim = \
    [k for k in range(12, 30) \
        if CuspForms(1, k).dimension() == 1]
```

Euler factor of elliptic cusp forms

eulerfactors_at_2 is a dictionary such that
$\mathbf{eulerfactors\_at\_2}[k] = Q_2^{(1)}(f_k; X)$ for $k \in \mathbf{wts\_of\_one\_dim}$.
Here $f_k \in S_k(\Gamma_1)$.

```
eulerfactors_at_2 = {}
for k in wts_of_one_dim:
    f = CuspForms(1, k).basis()[0]
    eulerfactors_at_2[k] = \
        euler_factor_of_l(f, 2)
```

Euler factor of elliptic cusp forms

```
sage: eulerfactors_at_2
{12: 1 + 24*X + 2048*X^2,
 16: 1 - 216*X + 32768*X^2,
 18: 1 + 528*X + 131072*X^2,
 20: 1 - 456*X + 524288*X^2,
 22: 1 + 288*X + 2097152*X^2,
 26: 1 + 48*X + 33554432*X^2}
```

Saito Kurokawa lift

We calculate $Q_p^{(2)}(F_k; X)$ for $p = 2$, $F_k \in S_k(\Gamma_2)$ and a small weight k.

```
sage: X10 = x10_with_prec(4)
sage: X10.euler_factor_of_spinor_l(2).factor()
(-1 + 256*x) * (-1 + 512*x) *
  (1 + 528*x + 131072*x^2)
```

The last factor is equal to $Q_2^{(1)}(f_{18}, x)$, where $f_{18} \in S_{18}(\Gamma_1)$ is an eigenform. We also have

$$Q_2^{(2)}(X_{12}; X) = (1 - 2^{10}X)(1 - 2^{11}X)Q_2^{(1)}(f_{22}, X).$$

But not every eigenform of $S_k(\Gamma_2)$ is related to an eigenform of $S_k(\Gamma_1)$.

Saito Kurokawa lift

For a cusp form $F \in S_k(\Gamma_2)$, we consider the following condition.

$$a(n,r,m) = \sum_{d>0, d|\gcd(n,r,m)} d^{k-1} a(1, r/d, mn/d^2),$$

for all $n, m, 4nm - r^2 \geq 0$. Here we put

$$a(n,r,m) = a((n,r,m), F).$$

We denote by $S_k^*(\Gamma_2)$ the set of Siegel cusp forms $F \in S_k(\Gamma_2)$ satisfying the condition above. We call $S_k^*(\Gamma_2)$ the Maass subspace.

Saito Kurokawa lift

THEOREM 4 (Maass, Andrianov, Zagier)

Let k be an even number . The Maass subspace $S_k^*(\Gamma_2)$ is stable under the action of Hecke operators. There exists a one-to-one correspondence between an eigenform $f \in S_{2k-2}(\Gamma_1)$ and an eigenform $F \in S_k^*(\Gamma_2)$ given by

$$Q_p^{(2)}(F; X) = (1 - p^{k-2}X)(1 - p^{k-1}X)Q_p^{(1)}(f; X).$$