

# ON THE STRUCTURE OF COHEN-MACAULAY MODULES OVER HYPERSURFACES OF COUNTABLE COHEN-MACAULAY REPRESENTATION TYPE

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ABSTRACT. Let  $R$  be a complete local hypersurface over an algebraically closed field of characteristic different from two, and suppose that  $R$  has countable Cohen-Macaulay representation type. In this paper, it is proved that the maximal Cohen-Macaulay  $R$ -modules which are locally free on the punctured spectrum are dominated by the maximal Cohen-Macaulay  $R$ -modules which are not locally free on the punctured spectrum. More precisely, there exists a single  $R$ -module  $X$  such that the indecomposable maximal Cohen-Macaulay  $R$ -modules not locally free on the punctured spectrum are  $X$  and its syzygy  $\Omega X$  and that any other maximal Cohen-Macaulay  $R$ -module is obtained from some extension of  $X$  and  $\Omega X$ .

## 1. INTRODUCTION

Let  $R$  be a complete local hypersurface over an algebraically closed field  $k$ .

Assume (for simplicity) that  $k$  has characteristic zero, and suppose that  $R$  has finite Cohen-Macaulay representation type, namely, there exist only finitely many isomorphism classes of indecomposable maximal Cohen-Macaulay  $R$ -modules. Then  $R$  is isomorphic to the residue ring  $k[[x_0, x_1, x_2, \dots, x_d]]/(f)$  where  $f$  is one of the following polynomials:

$$\begin{aligned} (A_n)(n \geq 1) & x_0^2 + x_1^{n+1} + x_2^2 + \cdots + x_d^2, \\ (D_n)(n \geq 4) & x_0^2 x_1 + x_1^{n-1} + x_2^2 + \cdots + x_d^2, \\ (E_6) & x_0^3 + x_1^4 + x_2^2 + \cdots + x_d^2, \\ (E_7) & x_0^3 + x_0 x_1^3 + x_2^2 + \cdots + x_d^2, \\ (E_8) & x_0^3 + x_1^5 + x_2^2 + \cdots + x_d^2. \end{aligned}$$

In this case, all objects and morphisms in the category  $\text{CM}(R)$  of maximal Cohen-Macaulay  $R$ -modules have been classified completely, that is to say, the Auslander-Reiten quiver of the stable category  $\underline{\text{CM}}(R)$  of  $\text{CM}(R)$  has been obtained. For the details, see [3, 6, 11].

Now, assume that  $k$  has characteristic different from two, and suppose that  $R$  has countable Cohen-Macaulay representation type, namely, there exist infinitely but only countably many isomorphism classes of indecomposable maximal Cohen-Macaulay  $R$ -modules. Then  $R$  is isomorphic to  $k[[x_0, x_1, x_2, \dots, x_d]]/(f)$  where  $f$  is either of the following:

$$\begin{aligned} (A_\infty^d) & x_0^2 + x_2^2 + \cdots + x_d^2, \\ (D_\infty^d) & x_0^2 x_1 + x_2^2 + \cdots + x_d^2. \end{aligned}$$

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In this case, all objects in  $\text{CM}(R)$  have been classified completely [3, 4, 6], but morphisms in  $\text{CM}(R)$  have not. The purpose of this paper is to investigate the relationships among objects in  $\text{CM}(R)$  by focusing on the objects that are locally free on the punctured spectrum of  $R$ . Modules which are locally free on the punctured spectrum have recently been studied in relation to whose nonfree loci; see [9, 10]. We also make use of nonfree loci to get our results.

Let us introduce here some notation. We denote by  $\mathcal{P}(R)$  the full subcategory of  $\text{CM}(R)$  consisting of all modules that are locally free on the punctured spectrum of  $R$ . Let  $\mathcal{M}(R)$  be the set of nonisomorphic indecomposable maximal Cohen-Macaulay  $R$ -modules that are *not* locally free on the punctured spectrum of  $R$ , and let  $\mathcal{V}(M)$  be the nonfree locus of a finitely generated  $R$ -module  $M$ . (Recall that the nonfree locus of  $M$  is defined as the set of prime ideals  $\mathfrak{p}$  of  $R$  such that the  $R_{\mathfrak{p}}$ -module  $M_{\mathfrak{p}}$  is nonfree.) Let  $F_1 \xrightarrow{\partial} F_0 \rightarrow M \rightarrow 0$  be part of a minimal free resolution of  $M$ . Then the (first) syzygy of  $M$  is by definition the image of the map  $\partial$  and denoted by  $\Omega M$ . (Note that it is uniquely determined up to isomorphism.)

The main theorem of this paper is the following.

**Theorem 1.1.** *Let  $k$  be an algebraically closed field of characteristic different from 2. Let  $R$  be a complete local hypersurface over  $k$  of countable Cohen-Macaulay representation type. Then, as we stated above,  $R$  is (isomorphic to) a residue ring  $k[[x_0, x_1, x_2, \dots, x_d]]/(f)$ , where  $f$  is either  $(A_{\infty}^d)$  or  $(D_{\infty}^d)$ . Let  $\mathfrak{p}_R = (x_0, x_2, \dots, x_d)$  and  $\mathfrak{m}_R = (x_0, x_1, x_2, \dots, x_d)$  be ideals of  $R$ . The following hold.*

- (1) *There exists an  $R$ -module  $X_R$  such that*
  - (a)  $\mathcal{M}(R) = \{X_R, \Omega(X_R)\}$ ,
  - (b)  $\mathcal{V}(X_R) = \{\mathfrak{p}_R, \mathfrak{m}_R\} = \mathcal{V}(\Omega(X_R))$ .
- (2) *For each indecomposable  $R$ -module  $M \in \mathcal{P}(R)$ , there is an exact sequence*

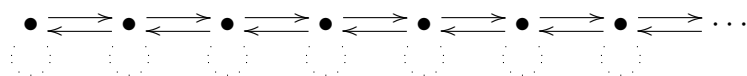
$$0 \rightarrow L \rightarrow M \oplus R^n \rightarrow N \rightarrow 0$$

*of  $R$ -modules with  $L, N \in \mathcal{M}(R)$  and  $n \geq 0$ .*

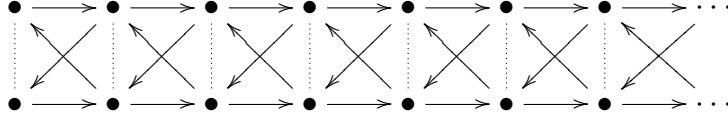
The first assertion of this theorem especially says that there exist at most two indecomposable maximal Cohen-Macaulay  $R$ -modules which are *not* locally free on the punctured spectrum of  $R$ . The second assertion of the theorem especially says that the maximal Cohen-Macaulay  $R$ -modules which are locally free on the punctured spectrum of  $R$  are dominated by the maximal Cohen-Macaulay  $R$ -modules which are *not* locally free on the punctured spectrum of  $R$ . On the other hand, the structure of the maximal Cohen-Macaulay modules locally free on the punctured spectrum has been clarified by Schreyer [8]:

**Theorem 1.2** (Schreyer). *Let  $n$  be a positive integer. The Auslander-Reiten quiver of the stable category of  $\mathcal{P}(R)$  has the following form.*

- (1) *When  $f = (A_{\infty}^{2n-1})$ :*



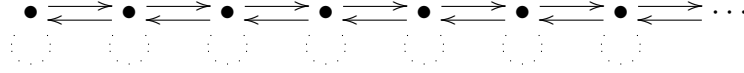
- (2) *When  $f = (D_{\infty}^{2n-1})$ :*



(3) When  $f = (A_\infty^{2n})$ :



(4) When  $f = (D_\infty^{2n})$ :



Consequently, combining our Theorem 1.1 with this result due to Schreyer, we get a new understanding of the structure of the category of maximal Cohen-Macaulay modules over hypersurfaces of countable Cohen-Macaulay representation type.

We give two applications of our Theorem 1.1. Rouquier [7] introduces the dimension of a triangulated category. Using Theorem 1.1, we are able to calculate the dimension of the triangulated category  $\underline{\text{CM}}(R)$ . We can also obtain a system of generators of the Grothendieck group of  $\text{CM}(R)$ .

**Corollary 1.3.** *With the notation of Theorem 1.1 the following hold.*

- (1) *The triangulated category  $\underline{\text{CM}}(R)$  has dimension one.*
- (2) *The Grothendieck group of  $\text{CM}(R)$  is generated by  $[R]$  and  $[X_R]$ .*

## 2. ONE AND TWO DIMENSIONAL CASES

In this section, we prove Theorem 1.1 in the cases where the ring  $R$  is of dimension 1 and 2. The following proposition includes Theorem 1.1 in the 1-dimensional case.

**Proposition 2.1.** *Let  $S$  be 2-dimensional regular local ring and let  $x_0, x_1$  be a regular system of parameters of  $S$ .*

- (1) *Let  $R = S/(x_0^2)$ . Then  $R/(x_0) \cong \Omega(R/(x_0))$ , and the following statements hold.*
  - (a) *For every indecomposable maximal Cohen-Macaulay  $R$ -module  $M$ , there is an exact sequence*

$$0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$$

*of  $R$ -modules with  $L, N \in \{0, R/(x_0)\}$ .*

- (b) *One has  $\mathcal{M}(R) = \{R/(x_0)\}$ .*
  - (c) *One has  $\mathcal{V}(R/(x_0)) = \{(x_0), (x_0, x_1)\}$ .*
- (2) *Let  $R = S/(x_0^2 x_1)$ . Then  $R/(x_0 x_1) \cong \Omega(R/(x_0))$ , and the following statements hold.*
  - (a) *For every indecomposable maximal Cohen-Macaulay  $R$ -module  $M$ , there is an exact sequence*

$$0 \rightarrow L \rightarrow M \oplus R^n \rightarrow N \rightarrow 0$$

*of  $R$ -modules with  $L, N \in \{0, R/(x_0), R/(x_0 x_1)\}$  and  $n = 0, 1$ .*

- (b) *One has  $\mathcal{M}(R) = \{R/(x_0), R/(x_0 x_1)\}$ .*
  - (c) *One has  $\mathcal{V}(R/(x_0)) = \{(x_0), (x_0, x_1)\} = \mathcal{V}(R/(x_0 x_1))$*

*Proof.* (1) By [3, Proposition 4.1], all the nonisomorphic indecomposable maximal Cohen-Macaulay  $R$ -modules are:

$$R, R/(x_0), \text{Coker } \varphi_n \ (n = 1, 2, \dots),$$

where  $\varphi_n = \begin{pmatrix} x_0 & x_1^n \\ 0 & -x_0 \end{pmatrix}$ . There is a commutative diagram

$$\begin{array}{ccccccc} & & 0 & & 0 & & 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ (\dots & \xrightarrow{x_0} & R & \xrightarrow{x_0} & R & \xrightarrow{x_0} & R \longrightarrow 0) \\ & & \begin{pmatrix} 1 \\ 0 \end{pmatrix} \downarrow & & \begin{pmatrix} 1 \\ 0 \end{pmatrix} \downarrow & & \begin{pmatrix} 1 \\ 0 \end{pmatrix} \downarrow \\ (\dots & \xrightarrow{\varphi_n} & R^2 & \xrightarrow{\varphi_n} & R^2 & \xrightarrow{\varphi_n} & R^2 \longrightarrow 0) \\ & & \begin{pmatrix} 0 & 1 \end{pmatrix} \downarrow & & \begin{pmatrix} 0 & 1 \end{pmatrix} \downarrow & & \begin{pmatrix} 0 & 1 \end{pmatrix} \downarrow \\ (\dots & \xrightarrow{-x_0} & R & \xrightarrow{-x_0} & R & \xrightarrow{-x_0} & R \longrightarrow 0) \\ & & \downarrow & & \downarrow & & \downarrow \\ & & 0 & & 0 & & 0 \end{array}$$

whose rows are acyclic complexes and whose columns are exact sequences of modules. This forms a short exact sequence of complexes, and taking the long exact sequence of its homology modules, we see that there is an exact sequence

$$0 \rightarrow R/(x_0) \rightarrow \text{Coker } \varphi_n \rightarrow R/(x_0) \rightarrow 0$$

of  $R$ -modules. Also, the first (or third) row in the above diagram makes an exact sequence

$$0 \rightarrow R/(x_0) \rightarrow R \rightarrow R/(x_0) \rightarrow 0.$$

We have

$$\mathcal{V}(R/(x_0)) = \{\mathfrak{p}, \mathfrak{m}\} = \text{Spec } R,$$

where  $\mathfrak{p} = (x_0)$  and  $\mathfrak{m} = (x_0, x_1)$  (cf. [10, Proposition 1.15(4)]). This especially says that  $R/(x_0)$  belongs to  $\mathcal{M}(R)$ . There is an equality of matrices with entries being in  $R_{\mathfrak{p}}$ :

$$\begin{pmatrix} -\frac{x_0}{x_1^n} & -1 \\ \frac{1}{x_1^n} & 0 \end{pmatrix} \begin{pmatrix} x_0 & x_1^n \\ 0 & -x_0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{x_0}{x_1^n} & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Hence  $(\text{Coker } \varphi_n)_{\mathfrak{p}}$  is isomorphic to  $R_{\mathfrak{p}}$ , which implies that  $\text{Coker } \varphi_n$  is not in  $\mathcal{M}(R)$ . Therefore the equality  $\mathcal{M}(R) = \{R/(x_0)\}$  holds.

(2) By [3, Proposition 4.2], all the nonisomorphic indecomposable maximal Cohen-Macaulay  $R$ -modules are:

$$R, R/(x_0), R/(x_0x_1), R/(x_0^2), R/(x_1), \\ \text{Coker } \varphi_n^+, \text{Coker } \varphi_n^-, \text{Coker } \psi_n^+, \text{Coker } \psi_n^- \ (n = 1, 2, \dots),$$

where

$$\begin{aligned}\varphi_n^+ &= \begin{pmatrix} x_0 & x_1^n \\ 0 & -x_0 \end{pmatrix}, \quad \varphi_n^- = \begin{pmatrix} x_0x_1 & x_1^{n+1} \\ 0 & -x_0x_1 \end{pmatrix}, \\ \psi_n^+ &= \begin{pmatrix} x_0x_1 & x_1^n \\ 0 & -x_0 \end{pmatrix}, \quad \psi_n^- = \begin{pmatrix} x_0 & x_1^n \\ 0 & -x_0x_1 \end{pmatrix}.\end{aligned}$$

Setting  $x_1^0 = 1$ , for  $n \geq 0$  we have commutative diagrams

$$\begin{array}{cccccccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ R & \xrightarrow{x_0} R & \xrightarrow{x_0x_1} R & \xrightarrow{x_0} R & R & \xrightarrow{x_0x_1} R & \xrightarrow{x_0} R & \xrightarrow{x_0x_1} R \\ \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) \downarrow & \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) \downarrow & \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) \downarrow & \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) \downarrow & \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) \downarrow & \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) \downarrow & \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) \downarrow & \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) \downarrow \\ R^2 & \xrightarrow{\varphi_n^+} R^2 & \xrightarrow{\varphi_n^-} R^2 & \xrightarrow{\varphi_n^+} R^2 & R^2 & \xrightarrow{\psi_n^+} R^2 & \xrightarrow{\psi_n^-} R^2 & \xrightarrow{\psi_n^+} R^2 \\ (0 \ 1) \downarrow & (0 \ 1) \downarrow & (0 \ 1) \downarrow & (0 \ 1) \downarrow & (0 \ 1) \downarrow & (0 \ 1) \downarrow & (0 \ 1) \downarrow & (0 \ 1) \downarrow \\ R & \xrightarrow{-x_0} R & \xrightarrow{-x_0x_1} R & \xrightarrow{-x_0} R & R & \xrightarrow{-x_0} R & \xrightarrow{-x_0x_1} R & \xrightarrow{-x_0} R \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}$$

with exact rows and columns. Also we have equalities of matrices over  $R$ :

$$\begin{aligned}\begin{pmatrix} x_0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_0 & 1 \\ 0 & -x_0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -x_0 & 1 \end{pmatrix} &= \begin{pmatrix} x_0^2 & 0 \\ 0 & 1 \end{pmatrix}, \\ \begin{pmatrix} 1 & 0 \\ x_0 & 1 \end{pmatrix} \begin{pmatrix} x_0x_1 & x_1 \\ 0 & -x_0x_1 \end{pmatrix} \begin{pmatrix} -x_0 & 1 \\ 1 & 0 \end{pmatrix} &= \begin{pmatrix} x_1 & 0 \\ 0 & 0 \end{pmatrix}.\end{aligned}$$

Hence there are exact sequences

$$\begin{aligned}0 &\rightarrow R/(x_0x_1) \rightarrow R \rightarrow R/(x_0) \rightarrow 0, \\ 0 &\rightarrow R/(x_0) \rightarrow R/(x_0^2) \rightarrow R/(x_0) \rightarrow 0, \\ 0 &\rightarrow R/(x_0x_1) \rightarrow R/(x_1) \oplus R \rightarrow R/(x_0x_1) \rightarrow 0, \\ 0 &\rightarrow R/(x_0) \rightarrow \text{Coker } \varphi_n^+ \rightarrow R/(x_0) \rightarrow 0, \\ 0 &\rightarrow R/(x_0x_1) \rightarrow \text{Coker } \varphi_n^- \rightarrow R/(x_0x_1) \rightarrow 0, \\ 0 &\rightarrow R/(x_0x_1) \rightarrow \text{Coker } \psi_n^+ \rightarrow R/(x_0) \rightarrow 0, \\ 0 &\rightarrow R/(x_0) \rightarrow \text{Coker } \psi_n^- \rightarrow R/(x_0x_1) \rightarrow 0.\end{aligned}$$

Put  $\mathfrak{p} = (x_0)$ ,  $\mathfrak{q} = (x_1)$  and  $\mathfrak{m} = (x_0, x_1)$ . Then we have  $\text{Spec } R = \{\mathfrak{p}, \mathfrak{q}, \mathfrak{m}\}$ , and easily see that the equalities

$$\begin{aligned}\mathcal{V}(R/(x_0)) &= \mathcal{V}(R/(x_0x_1)) = \{\mathfrak{p}, \mathfrak{m}\}, \\ \mathcal{V}(R/(x_0^2)) &= \mathcal{V}(R/(x_1)) = \{\mathfrak{m}\}\end{aligned}$$

hold (cf. [10, Proposition 1.15(4)]). Therefore  $R/(x_0), R/(x_0x_1) \in \mathcal{M}(R)$  and  $R/(x_0^2), R/(x_1) \notin \mathcal{M}(R)$ . Since  $R_{\mathfrak{q}}$  is a field, all  $R_{\mathfrak{q}}$ -modules are free. There are equalities

of matrices whose entries are in  $R_{\mathfrak{p}}$ :

$$\begin{aligned} & \begin{pmatrix} \frac{x_0}{x_1^n} & 1 \\ \frac{1}{x_1^n} & 0 \end{pmatrix} \begin{pmatrix} x_0 & x_1^n \\ 0 & -x_0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{x_0}{x_1^n} & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \\ & \begin{pmatrix} \frac{x_0}{x_1^n} & 1 \\ \frac{1}{x_1^n} & 0 \end{pmatrix} \begin{pmatrix} x_0x_1 & x_1^n \\ 0 & -x_0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{x_0x_1}{x_1^n} & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

Hence we obtain isomorphisms  $(\text{Coker } \varphi_n^+)_{\mathfrak{p}} \cong R_{\mathfrak{p}} \cong (\text{Coker } \psi_n^+)_{\mathfrak{p}}$ . Note that  $\text{Coker } \varphi_n^-$  and  $\text{Coker } \psi_n^-$  are the syzygies of  $\text{Coker } \varphi_n^+$  and  $\text{Coker } \psi_n^+$ , respectively. Thus  $\text{Coker } \varphi_n^+$ ,  $\text{Coker } \varphi_n^-$ ,  $\text{Coker } \psi_n^+$  and  $\text{Coker } \psi_n^-$  are not in  $\mathcal{M}(R)$ . Consequently, we have  $\mathcal{M}(R) = \{R/(x_0), R/(x_0x_1)\}$ .  $\square$

Next, let us consider the case where the base ring has dimension 2.

**Proposition 2.2.** *Let  $k$  be an algebraically closed field.*

- (1) *Let  $R = k[[x_0, x_1, x_2]]/(x_0x_2)$ . Then  $R/(x_2) \cong \Omega(R/(x_0))$ , and the following hold.*
- (a) *For any indecomposable maximal Cohen-Macaulay  $R$ -module  $M$ , there is an exact sequence of  $R$ -modules*

$$0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$$

*with  $L, N \in \{0, R/(x_0), R/(x_2)\}$ .*

- (b) *One has  $\mathcal{M}(R) = \{R/(x_0), R/(x_2)\}$ .*
- (c) *One has  $\mathcal{V}(R/(x_0)) = \{(x_0, x_2), (x_0, x_1, x_2)\} = \mathcal{V}(R/(x_2))$ .*
- (2) *Let  $R = k[[x_0, x_1, x_2]]/(x_0^2x_1 - x_2^2)$ . Then  $(x_0, x_2) \cong \Omega(x_0, x_2)$ , and the following hold.*
- (a) *For any indecomposable maximal Cohen-Macaulay  $R$ -module  $M$ , there is an exact sequence of  $R$ -modules*

$$0 \rightarrow L \rightarrow M \oplus R^n \rightarrow N \rightarrow 0$$

*with  $L, N \in \{0, (x_0, x_2)\}$  and  $n = 0, 1$ .*

- (b) *One has  $\mathcal{M}(R) = \{(x_0, x_2)\}$ .*
- (c) *One has  $\mathcal{V}((x_0, x_2)) = \{(x_0, x_2), (x_0, x_1, x_2)\}$ .*

*Proof.* (1) By [4, Theorem 5.3], all indecomposable maximal Cohen-Macaulay  $R$ -modules are:

$$R, R/(x_0), R/(x_2), \text{Coker } \varphi_n^+, \text{Coker } \varphi_n^- \quad (n = 1, 2, \dots),$$

where

$$\varphi_n^+ = \begin{pmatrix} x_2 & x_1^n \\ 0 & x_0 \end{pmatrix}, \quad \varphi_n^- = \begin{pmatrix} x_0 & -x_1^n \\ 0 & x_2 \end{pmatrix}.$$

Since we have a commutative diagram

$$\begin{array}{ccccccc}
 & 0 & & 0 & & 0 & & 0 \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 & R & \xrightarrow{x_2} & R & \xrightarrow{x_0} & R & \xrightarrow{x_2} & R \\
 (1) & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 & R^2 & \xrightarrow{\varphi_n^+} & R^2 & \xrightarrow{\varphi_n^-} & R^2 & \xrightarrow{\varphi_n^+} & R^2 \\
 (0 \ 1) & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 & R & \xrightarrow{x_0} & R & \xrightarrow{x_2} & R & \xrightarrow{x_0} & R \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 & 0 & & 0 & & 0 & & 0
 \end{array}$$

with exact rows and columns, there are exact sequences

$$\begin{aligned}
 0 &\rightarrow R/(x_2) \rightarrow R \rightarrow R/(x_0) \rightarrow 0, \\
 0 &\rightarrow R/(x_2) \rightarrow \text{Coker } \varphi_n^+ \rightarrow R/(x_0) \rightarrow 0, \\
 0 &\rightarrow R/(x_0) \rightarrow \text{Coker } \varphi_n^- \rightarrow R/(x_2) \rightarrow 0.
 \end{aligned}$$

We have  $\mathcal{V}(R/(x_0)) = \{(x_0, x_2), (x_0, x_1, x_2)\} = \mathcal{V}(R/(x_2))$ , which implies that  $R/(x_0)$  and  $R/(x_2)$  belong to  $\mathcal{M}(R)$ . Let  $\mathfrak{p}$  be any nonmaximal prime ideal of  $R$ . Then at least one of  $x_0, x_1, x_2$  is not in  $\mathfrak{p}$ , and we have an equality of matrices whose entries are in  $R_{\mathfrak{p}}$ :

$$\begin{aligned}
 \begin{pmatrix} x_0 & -x_1^n \\ 0 & \frac{1}{x_0} \end{pmatrix} \begin{pmatrix} x_2 & x_1^n \\ 0 & x_0 \end{pmatrix} &= \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{if } x_0 \notin \mathfrak{p}, \\
 \begin{pmatrix} \frac{x_0}{x_1^n} & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_2 & x_1^n \\ 0 & x_0 \end{pmatrix} \begin{pmatrix} -x_1^n & x_0 \\ x_2 & \frac{1}{x_1^n} \end{pmatrix} &= \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{if } x_1 \notin \mathfrak{p}, \text{ and} \\
 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_2 & x_1^n \\ 0 & x_0 \end{pmatrix} \begin{pmatrix} -x_1^n & \frac{1}{x_2} \\ x_2 & 0 \end{pmatrix} &= \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{if } x_2 \notin \mathfrak{p}.
 \end{aligned}$$

In each case  $(\text{Coker } \varphi_n^+)_{\mathfrak{p}}$  is isomorphic to  $R_{\mathfrak{p}}$ . Since  $\text{Coker } \varphi_n^-$  is the syzygy of  $\text{Coker } \varphi_n^+$ , we see that  $\text{Coker } \varphi_n^+$  and  $\text{Coker } \varphi_n^-$  are not in  $\mathcal{M}(R)$ . Therefore the equality  $\mathcal{M}(R) = \{R/(x_0), R/(x_2)\}$  holds.

(2) By [4, Theorem 5.7] the following three assertions hold.

(i) All indecomposable maximal Cohen-Macaulay  $R$ -modules are:

$$\begin{aligned}
 &R, \text{Coker } \alpha^+, \text{Coker } \alpha^-, \text{Coker } \beta^+, \text{Coker } \beta^-, \\
 &\text{Coker } \varphi_n^+, \text{Coker } \varphi_n^-, \text{Coker } \psi_n^+, \text{Coker } \psi_n^- \quad (n = 1, 2, \dots),
 \end{aligned}$$

where

$$\begin{aligned}\alpha^+ &= \begin{pmatrix} x_2 & x_0x_1 \\ x_0 & x_2 \end{pmatrix}, \quad \alpha^- = \begin{pmatrix} -x_2 & x_0x_1 \\ x_0 & -x_2 \end{pmatrix}, \\ \beta^+ &= \begin{pmatrix} x_0^2 & x_2 \\ x_2 & x_1 \end{pmatrix}, \quad \beta^- = \begin{pmatrix} x_1 & -x_2 \\ -x_2 & x_0^2 \end{pmatrix}, \\ \varphi_n^+ &= \begin{pmatrix} x_2 & x_0x_1 & 0 & -x_1^{n+1} \\ x_0 & x_2 & x_1^n & 0 \\ 0 & 0 & x_2 & x_0x_1 \\ 0 & 0 & x_0 & x_2 \end{pmatrix}, \quad \varphi_n^- = \begin{pmatrix} -x_2 & x_0x_1 & 0 & -x_1^{n+1} \\ x_0 & -x_2 & x_1^n & 0 \\ 0 & 0 & -x_2 & x_0x_1 \\ 0 & 0 & x_0 & -x_2 \end{pmatrix}, \\ \psi_n^+ &= \begin{pmatrix} x_2 & x_0x_1 & -x_1^n & 0 \\ x_0 & x_2 & 0 & x_1^n \\ 0 & 0 & x_2 & x_0x_1 \\ 0 & 0 & x_0 & x_2 \end{pmatrix}, \quad \psi_n^- = \begin{pmatrix} -x_2 & x_0x_1 & -x_1^n & 0 \\ x_0 & -x_2 & 0 & x_1^n \\ 0 & 0 & -x_2 & x_0x_1 \\ 0 & 0 & x_0 & -x_2 \end{pmatrix}.\end{aligned}$$

(ii) There are isomorphisms

$$\begin{aligned}\text{Coker } \alpha^+ &\cong \text{Coker } \alpha^-, & \text{Coker } \beta^+ &\cong \text{Coker } \beta^-, \\ \text{Coker } \varphi_n^+ &\cong \text{Coker } \varphi_n^-, & \text{Coker } \psi_n^+ &\cong \text{Coker } \psi_n^-.\end{aligned}$$

(iii) The modules  $\text{Coker } \beta^+$ ,  $\text{Coker } \varphi_n^+$  and  $\text{Coker } \psi_n^+$  are locally free on the punctured spectrum of  $R$ .

We can easily check that the sequence

$$R^2 \xrightarrow{\alpha^+} R^2 \xrightarrow{(x_0, -x_2)} R \longrightarrow R/(x_0, x_2) \longrightarrow 0$$

is exact. Hence  $\text{Coker } \alpha^+$  is isomorphic to the prime ideal  $\mathfrak{p} := (x_0, x_2)$  of  $R$ . For an ideal  $I$  of  $R$ , denote by  $V(I)$  the set of prime ideals of  $R$  containing  $I$ . It is easy to see that  $\mathcal{V}(\mathfrak{p})$  is contained in  $V(\mathfrak{p})$ , while  $\mathfrak{p}$  belongs to  $\mathcal{V}(\mathfrak{p})$  because the local ring  $R_{\mathfrak{p}}$  is not regular. Thus we obtain

$$\mathcal{V}(\mathfrak{p}) = V(\mathfrak{p}) = \{\mathfrak{p}, \mathfrak{m}\},$$

where  $\mathfrak{m} = (x_0, x_1, x_2)$  is the maximal ideal of  $R$ , and we have  $\mathcal{M}(R) = \{\mathfrak{p}\}$ .

Setting  $x_1^0 = 1$ , for  $n \geq 0$  we have commutative diagrams

$$\begin{array}{cccccccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ R^2 & \xrightarrow{\alpha^+} & R^2 & \xrightarrow{\alpha^-} & R^2 & \xrightarrow{\alpha^+} & R^2 & \xrightarrow{\alpha^+} & R^2 \\ \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \downarrow & \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \downarrow & \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \downarrow & \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \downarrow & \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \downarrow & \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \downarrow & \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \downarrow & \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \downarrow \\ R^4 & \xrightarrow{\varphi_n^+} & R^4 & \xrightarrow{\varphi_n^-} & R^4 & \xrightarrow{\varphi_n^+} & R^4 & \xrightarrow{\varphi_n^+} & R^4 \\ \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \downarrow & \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \downarrow & \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \downarrow & \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \downarrow & \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \downarrow & \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \downarrow & \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \downarrow & \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \downarrow \\ R^2 & \xrightarrow{\alpha^+} & R^2 & \xrightarrow{\alpha^-} & R^2 & \xrightarrow{\alpha^+} & R^2 & \xrightarrow{\alpha^+} & R^2 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}$$

with exact rows and columns. There is an equality of matrices over  $R$

$$\begin{pmatrix} 0 & -x_0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ x_0 & -x_2 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_2 & x_0x_1 & 0 & -x_1 \\ x_0 & x_2 & 1 & 0 \\ 0 & 0 & x_2 & x_0x_1 \\ 0 & 0 & x_0 & x_2 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ x_0 & 0 & -x_2 & 1 \\ 0 & 1 & x_0 & 0 \end{pmatrix} = \begin{pmatrix} x_0^2 & x_2 & 0 & 0 \\ x_2 & x_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

which gives an isomorphism  $\text{Coker } \varphi_0^+ \cong \text{Coker } \beta^+ \oplus R$ . Consequently, we obtain exact sequences

$$\begin{aligned} 0 &\rightarrow \text{Coker } \alpha^+ \rightarrow \text{Coker } \beta^+ \oplus R \rightarrow \text{Coker } \alpha^+ \rightarrow 0, \\ 0 &\rightarrow \text{Coker } \alpha^+ \rightarrow \text{Coker } \varphi_n^+ \rightarrow \text{Coker } \alpha^+ \rightarrow 0, \\ 0 &\rightarrow \text{Coker } \alpha^+ \rightarrow \text{Coker } \psi_n^+ \rightarrow \text{Coker } \alpha^+ \rightarrow 0. \end{aligned}$$

This completes the proof of the second assertion of the proposition.  $\square$

As an application of the previous propositions, we can calculate the Grothendieck group  $K_0(\text{CM}(R))$  of  $\text{CM}(R)$  for the hypersurfaces  $R$  of types  $(A_\infty^1)$ ,  $(D_\infty^1)$ ,  $(A_\infty^2)$  and  $(D_\infty^2)$ .

**Proposition 2.3.** *With the notation of Theorem 1.1, we have*

$$K_0(\text{CM}(R)) \cong \begin{cases} \mathbb{Z} & \text{if } f = (A_\infty^1), \\ \mathbb{Z}^2 & \text{if } f = (D_\infty^1) \text{ or } f = (A_\infty^2), \\ \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} & \text{if } f = (D_\infty^2). \end{cases}$$

*Proof.* Let  $R$  be as in Theorem 1.1, and suppose  $d = 1, 2$ . By Propositions 2.1 and 2.2, there is a surjective homomorphism  $\mathbb{Z}^2 \rightarrow K_0(\text{CM}(R))$ . Indeed, sending the canonical basis  $\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$  of  $\mathbb{Z}^2$  to

$$\begin{aligned} \{[R], [R/(x_0)]\} &\text{ if } f = (A_\infty^1), (D_\infty^1) \text{ or } (A_\infty^2), \\ \{[R], [(x_0, x_2)]\} &\text{ if } f = (D_\infty^2) \end{aligned}$$

makes such a surjection. We get a short exact sequence

$$0 \rightarrow \mathbb{Z}^{2-r} \rightarrow \mathbb{Z}^2 \rightarrow K_0(\text{CM}(R)) \rightarrow 0$$

of  $\mathbb{Z}$ -modules, where  $r$  denotes the rank of  $K_0(\text{CM}(R))$ .

Let  $Q$  be the total quotient ring of  $R$ . For a commutative ring  $A$ , denote by  $\text{mod } A$  the category of finitely generated  $A$ -modules. Define homomorphisms  $f : K_0(\text{CM}(R)) \rightarrow K_0(\text{mod } R)$  and  $g : K_0(\text{mod } R) \rightarrow K_0(\text{mod } Q)$  by  $f([M]) = [M]$  and  $g([N]) = [N \otimes_R Q]$  for  $M \in \text{CM}(R)$  and  $N \in \text{mod } R$ . For  $N \in \text{mod } R$ , there exists an exact sequence

$$0 \rightarrow X_n \rightarrow X_{n-1} \rightarrow \cdots \rightarrow X_0 \rightarrow N \rightarrow 0$$

of  $R$ -modules with  $X_i \in \text{CM}(R)$  for  $0 \leq i \leq n$ . (For instance, a free resolution of  $N$  gives such an exact sequence.) Then in  $K_0(\text{mod } R)$  the equality

$$[N] = \sum_{i=0}^n (-1)^i [X_i]$$

holds, which shows that the map  $f$  is surjective. It is obvious that  $g$  is also surjective. Taking the composition, we have a surjective homomorphism

$$K_0(\text{CM}(R)) \rightarrow K_0(\text{mod } Q)$$

of  $\mathbb{Z}$ -modules. Note that  $Q$  is an Artinian ring. Hence every finitely generated  $Q$ -module has finite length, and  $K_0(\text{mod } Q)$  is the free  $\mathbb{Z}$ -module with basis

$$\{[Q/\mathfrak{M}] \mid \mathfrak{M} \text{ is a maximal ideal of } Q\}$$

(cf. [1, Theorem 1.7]).

When  $f = (A_\infty^1)$ , the ring  $Q$  has a unique maximal ideal. Hence  $K_0(\text{mod } Q) \cong \mathbb{Z}$ . Since there is an exact sequence

$$0 \rightarrow R/(x_0) \rightarrow R \rightarrow R/(x_0) \rightarrow 0,$$

the equality  $[R] = 2[R/(x_0)]$  holds in  $K_0(\text{CM}(R))$ . Therefore we have  $r = 1$ . There is a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{\begin{pmatrix} 1 \\ -2 \end{pmatrix}} & \mathbb{Z}^2 & \longrightarrow & K_0(\text{CM}(R)) \longrightarrow 0 \\ & & \parallel & & \cong \downarrow \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} & & \downarrow \\ 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{\begin{pmatrix} 1 \\ 0 \end{pmatrix}} & \mathbb{Z}^2 & \longrightarrow & \mathbb{Z} \longrightarrow 0 \end{array}$$

with exact rows. Thus, the  $\mathbb{Z}$ -module  $K_0(\text{CM}(R))$  is isomorphic to  $\mathbb{Z}$ .

When either  $f = (D_\infty^1)$  or  $f = (A_\infty^2)$ , the ring  $Q$  has two maximal ideals, and we have  $K_0(\text{mod } Q) \cong \mathbb{Z}^2$ . The  $\mathbb{Z}$ -module  $K_0(\text{CM}(R))$  is also isomorphic to  $\mathbb{Z}^2$ .

When  $f = (D_\infty^2)$ , the ring  $Q$  is a field. Hence  $K_0(\text{mod } Q) \cong \mathbb{Z}$ . With the notation of the proof of Proposition 2.2(2), we have isomorphisms

$$\text{Coker } \alpha^- \cong \text{Coker } \alpha^+ \cong (x_0, x_2)$$

and an exact sequence

$$0 \rightarrow \text{Coker } \alpha^- \rightarrow R^2 \rightarrow \text{Coker } \alpha^+ \rightarrow 0.$$

Thus  $K_0(\text{CM}(R))$  has a relation  $2[R] = 2[(x_0, x_2)]$ , and we see that  $r = 1$ . There is a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{\begin{pmatrix} 2 \\ -2 \end{pmatrix}} & \mathbb{Z}^2 & \longrightarrow & K_0(\text{CM}(R)) \longrightarrow 0 \\ & & \parallel & & \cong \downarrow \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} & & \downarrow \\ 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{\begin{pmatrix} 0 \\ 2 \end{pmatrix}} & \mathbb{Z}^2 & \longrightarrow & \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \longrightarrow 0 \end{array}$$

with exact rows. Therefore  $K_0(\text{CM}(R))$  is isomorphic to  $\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ .  $\square$

### 3. GENERAL CASE

In this section, we give a proof of Theorem 1.1 in the general case. Throughout this section, we assume that  $k$  is an algebraically closed field of characteristic different from two. First of all, let us recall the definition of a matrix factorization.

**Definition 3.1.** Let  $S$  be the formal power series ring  $k[[x_0, x_1, \dots, x_n]]$ , and let  $f$  be a nonzero element in the maximal ideal  $(x_0, x_1, \dots, x_n)S$ . A pair of square matrices  $(A, B)$  with entries being in  $S$  is called a *matrix factorization* of  $f$  if it satisfies  $AB = BA = fE$ , where  $E$  is an identity matrix.

Let  $\text{MF}(f)$  be the category of matrix factorizations of  $f$ . Eisenbud [5, Section 6] (see also [11, Chapter 7]) proved that taking the cokernel induces a category equivalence

$$\text{MF}(f)/\langle(1, f)\rangle \xrightarrow{\cong} \text{CM}(S/(f)),$$

where  $\langle(1, f)\rangle$  denotes the ideal of the category  $\text{MF}(f)$  generated by  $(1, f)$ . We identify  $\text{CM}(S/(f))$  with  $\text{MF}(f)/\langle(1, f)\rangle$ .

The following lemma is called Knörrer's periodicity (cf. [6, Theorem 3.1], [11, Chapter 12]) and it will play a key role in this section.

**Lemma 3.2.** *Let  $S = k[[x_0, x_1, \dots, x_n]]$  and  $T = k[[x_0, x_1, \dots, x_n, y, z]]$  be formal power series rings and  $f \in (x_0, x_1, \dots, x_n)S$  a nonzero element. The functor  $\text{MF}(f) \rightarrow \text{MF}(f + yz)$  given by*

$$(A, B) \mapsto \left( \begin{pmatrix} A & yE \\ zE & -B \end{pmatrix}, \begin{pmatrix} B & yE \\ zE & -A \end{pmatrix} \right)$$

*induces a triangle equivalence between the stable categories*

$$F : \underline{\text{CM}}(S/(f)) \xrightarrow{\cong} \underline{\text{CM}}(T/(f + yz)).$$

Here we recall some basic properties of the stable category of maximal Cohen-Macaulay modules. Let  $R$  be a Henselian Gorenstein local ring. For maximal Cohen-Macaulay  $R$ -modules  $M$  and  $N$ , the conditions

- (1)  $M \cong N$  in  $\underline{\text{CM}}(R)$ ,
- (2)  $M \oplus R^m \cong N \oplus R^n$  in  $\text{CM}(R)$  for some  $m, n \geq 0$

are equivalent to each other. Since  $R$  is Henselian,  $\text{mod } R$  is a Krull-Schmidt category. Hence for any object  $M \in \underline{\text{CM}}(R)$  there exists a unique object  $M_0 \in \text{CM}(R)$  such that  $M_0$  has no nonzero free summand and that  $M_0$  is isomorphic to  $M$  in  $\underline{\text{CM}}(R)$ . This means that for any object  $M \in \underline{\text{CM}}(R)$ , we may assume that  $M$  has no nonzero free summand. If  $M$  is an indecomposable object of  $\underline{\text{CM}}(R)$ , then  $M_0$  is a nonfree indecomposable maximal Cohen-Macaulay  $R$ -module.

**Proposition 3.3.** *Let  $S, T, f$  and  $F$  be as in Lemma 3.2. Put  $R = T/(f + yz)$  and  $R' = S/(f)$ . For a nonfree maximal Cohen-Macaulay  $R'$ -module  $M$ , the following statements hold.*

- (1) *One has an inclusion  $\mathcal{V}_R(FM) \subseteq \mathcal{V}_R(y, z)$ .*
- (2) *One has a bijection  $\Phi : \mathcal{V}_R(FM) \rightarrow \mathcal{V}_{R'}(M)$  which sends  $\mathfrak{P}$  to  $\mathfrak{P}/(y, z)$ .*

*Proof.* (1) Let  $(A, B) \in \text{MF}(f)$  be a matrix factorization corresponding to  $M$ . Then the  $R$ -module  $\text{Coker} \begin{pmatrix} A & yE \\ zE & -B \end{pmatrix}$  is isomorphic to the  $R$ -module  $FM$  up to free summand, and  $FM$  is a nonfree maximal Cohen-Macaulay  $R$ -module. Let  $\mathfrak{P}$  be a prime ideal of  $R$  which does not contain  $y$ . Then there is an equality

$$\begin{pmatrix} \frac{1}{y}B & E \\ \frac{1}{y}E & 0 \end{pmatrix} \begin{pmatrix} A & yE \\ zE & -B \end{pmatrix} \begin{pmatrix} yE & 0 \\ -A & E \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & E \end{pmatrix}$$

of matrices over  $R_{\mathfrak{P}}$ . Hence  $(FM)_{\mathfrak{P}}$  is a free  $R_{\mathfrak{P}}$ -module. Therefore  $\mathcal{V}_R(FM)$  is contained in  $\mathcal{V}_R(y)$ . Similarly,  $\mathcal{V}_R(FM)$  is contained in  $\mathcal{V}_R(z)$ .

(2) The assignment  $\mathfrak{P} \mapsto \mathfrak{P}/(y, z)$  makes a bijection  $\mathcal{V}_R(y, z) \rightarrow \text{Spec } R'$ . For  $\mathfrak{P} \in \mathcal{V}_R(y, z)$ , set  $\mathfrak{p} = \mathfrak{P}/(y, z) \in \text{Spec } R'$ . It is enough to show that  $\mathfrak{P}$  belongs to  $\mathcal{V}_R(FM)$  if and only if  $\mathfrak{p}$  belongs to  $\mathcal{V}_{R'}(M)$ . There is a ring isomorphism  $R/(y, z) \cong R'$ . Also, there

is an  $R'$ -module isomorphism  $FM/(y, z)FM \cong M \oplus \Omega_{R'}M$  up to free  $R'$ -summand. We have  $R'_\mathfrak{p}$ -module isomorphisms

$$\begin{aligned} (FM)_{\mathfrak{p}}/(y, z)(FM)_{\mathfrak{p}} &\cong (FM/(y, z)FM)_{\mathfrak{p}} \\ &\cong M_{\mathfrak{p}} \oplus (\Omega_{R'}M)_{\mathfrak{p}} \cong M_{\mathfrak{p}} \oplus (\Omega_{R'}M)_{\mathfrak{p}} \cong M_{\mathfrak{p}} \oplus \Omega_{R'_\mathfrak{p}}M_{\mathfrak{p}} \end{aligned}$$

up to free  $R'_\mathfrak{p}$ -summand. Since  $y, z$  is an  $R$ -regular sequence and  $FM$  is a (nonzero) maximal Cohen-Macaulay  $R$ -module, the sequence  $y, z$  is  $FM$ -regular. Hence  $(FM)_{\mathfrak{p}}$  is  $R_{\mathfrak{p}}$ -free if and only if  $M_{\mathfrak{p}} \oplus \Omega_{R'_\mathfrak{p}}M_{\mathfrak{p}}$  is  $R'_\mathfrak{p}$ -free (cf. [2, Lemma 1.3.5]). Therefore  $\mathfrak{P} \in \mathcal{V}_R(FM)$  if and only if  $\mathfrak{p} \in \mathcal{V}_{R'}(M)$ .  $\square$

Note that in general a nonfree finitely generated module  $M$  over a commutative local ring  $(R, \mathfrak{m})$  is locally free on the punctured spectrum of  $R$  if and only if the equality  $\mathcal{V}_R(M) = \{\mathfrak{m}\}$  holds. Thus Proposition 3.3 yields the corollary below.

**Corollary 3.4.** *Keep the notation of Proposition 3.3. Then  $M$  is locally free on the punctured spectrum of  $R'$  if and only if  $FM$  is locally free on the punctured spectrum of  $R$ .*

Now we can prove Theorem 1.1 in the general case.

*Proof of Theorem 1.1.* Let  $R = k[[x_0, x_1, x_2, \dots, x_d]]/(f)$ , where  $f$  is either  $(A_\infty^d)$  or  $(D_\infty^d)$ . We prove the theorem by induction on  $d = \dim R$ , and we may assume  $d \geq 3$  thanks to Propositions 2.1 and 2.2. Put  $R' = k[[x_0, x_1, x_2, \dots, x_{d-2}]]/(f')$  such that  $f = f' + x_{d-1}^2 + x_d^2$ . Then  $f'$  is either  $(A_\infty^{d-2})$  or  $(D_\infty^{d-2})$ . Since the characteristic of  $k$  is not equal to 2, the ring  $R$  is isomorphic to  $k[[x_0, x_1, x_2, \dots, x_{d-2}, y, z]]/(f' + yz)$ , where  $y, z$  are indeterminates over  $k[[x_0, x_1, x_2, \dots, x_{d-2}]]$ . Let  $F : \underline{\mathbf{CM}}(R') \rightarrow \underline{\mathbf{CM}}(R)$  be the equivalence functor given by Lemma 3.2.

(1) By induction hypothesis, there exists a nonfree indecomposable maximal Cohen-Macaulay  $R'$ -module  $X_{R'}$  satisfying

$$\begin{aligned} \mathcal{M}(R') &= \{X_{R'}, \Omega_{R'}(X_{R'})\}, \\ \mathcal{V}_{R'}(X_{R'}) &= \{(x_0, x_2, \dots, x_{d-2})R', (x_0, x_1, x_2, \dots, x_{d-2})R'\} = \mathcal{V}_{R'}(\Omega_{R'}(X_{R'})). \end{aligned}$$

Let  $X_R$  be the nonfree direct summand of  $FX_{R'}$ . Recall that the shift functor on the triangulated category  $\underline{\mathbf{CM}}(R)$  (respectively,  $\underline{\mathbf{CM}}(R')$ ) is the cosyzygy functor  $\Omega_R^{-1}$  (respectively,  $\Omega_{R'}^{-1}$ ). Hence  $F$  commutes with the syzygy functor, and we have

$$F(\Omega_{R'}(X_{R'})) \cong \Omega_R(FX_{R'}) \cong \Omega_R(X_R)$$

in  $\underline{\mathbf{CM}}(R)$ . It follows from this that  $\Omega_R(X_R)$  is the nonfree direct summand of  $F(\Omega_{R'}(X_{R'}))$ . By Proposition 3.3 and Corollary 3.4, we see that  $\mathcal{M}(R) = \{X_R, \Omega_R(X_R)\}$  and that  $\mathcal{V}_R(X_R) = \{(x_0, x_2, \dots, x_d)R, (x_0, x_1, x_2, \dots, x_d)R\} = \mathcal{V}_R(\Omega_R(X_R))$ .

(2) Let  $M \in \mathcal{P}(R)$  be an indecomposable  $R$ -module. Then there exists  $M' \in \mathbf{CM}(R')$  such that  $FM'$  is isomorphic to  $M$  up to free summand. Since the  $R$ -module  $M$  is indecomposable,  $M'$  can be chosen as an indecomposable  $R'$ -module. Corollary 3.4 implies that  $M'$  is in  $\mathcal{P}(R')$ . By induction hypothesis, there is an exact sequence

$$0 \rightarrow L' \rightarrow M' \oplus R'^n \rightarrow N' \rightarrow 0,$$

where  $L', N' \in \mathcal{M}(R')$  and  $n \geq 0$ . Then we obtain an exact triangle  $L' \rightarrow M' \rightarrow N' \rightarrow \Omega_{R'}^{-1}L'$  in  $\underline{\mathbf{CM}}(R')$ , which gives an exact triangle  $FL' \rightarrow M \rightarrow FN' \rightarrow \Omega_R^{-1}FL'$  in  $\underline{\mathbf{CM}}(R)$ .

Let  $L$  and  $N$  be the nonfree direct summands of  $FL'$  and  $FN'$ , respectively. Then we have an exact triangle  $L \rightarrow M \rightarrow N \rightarrow \Omega_R^{-1}L$ , which gives a short exact sequence

$$0 \rightarrow L \rightarrow M \oplus R^m \rightarrow N \rightarrow 0$$

of  $R$ -modules. Since  $L'$  and  $N'$  belong to  $\mathcal{M}(R')$ , the modules  $L$  and  $N$  are in  $\mathcal{M}(R)$  by Corollary 3.4.  $\square$

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