NAGATA CRITERION FOR SERRE'S (R_n) AND (S_n)-CONDITIONS

RYO TAKAHASHI *

Graduate School of Natural Science and Technology, Okayama University, Okayama 700-8530, Japan

1. Introduction

Throughout the present paper, we assume that all rings are noetherian commutative rings.

First of all, we recall Serre's (R_n) and (S_n)-conditions for a ring A. These are defined as follows. Let n be an integer.

(R_n) : If $\mathfrak{p} \in \mathrm{Spec}(A)$ and $\mathrm{ht}(\mathfrak{p}) \leqq n$, then $A_{\mathfrak{p}}$ is regular.

(S_n) : $\mathrm{depth}(A_{\mathfrak{p}}) \geqq \inf(n, \mathrm{ht}(\mathfrak{p}))$ for all $\mathfrak{p} \in \mathrm{Spec}(A)$.

Let $\mathbb{P}$ be a property of local rings. For a ring A we put

$$\mathbb{P}(A) = \{\mathfrak{p} \in \mathrm{Spec}(A) \mid \mathbb{P} \text{ holds for } A_{\mathfrak{p}}\}$$

and call it the $\mathbb{P}$-locus of A. The following statement is called the (ring-theoretic) Nagata criterion for the property $\mathbb{P}$, and we abbreviate it to (NC).

(NC) : If A is a ring and if $\mathbb{P}(A/\mathfrak{p})$ contains a non-empty open subset of $\mathrm{Spec}(A/\mathfrak{p})$ for every $\mathfrak{p} \in \mathrm{Spec}(A)$, then $\mathbb{P}(A)$ is open in $\mathrm{Spec}(A)$.

This statement was invented by Nagata in 1959. In algebraic geometry, there is a problem asking when the regular locus (that is, the non-singular locus) of a ring is open. He proposed the above criterion to consider this problem, and he proved that (NC) holds for $\mathbb{P}$ = regular ([6]). There are some other properties $\mathbb{P}$ for which (NC) holds, for example, $\mathbb{P}$ = Cohen-Macaulay ([3], [4]), Gorenstein ([2], [4]), and complete intersection ([2]). On the other hand, it is easy to see that (NC) holds for $\mathbb{P}$ = (integral) domain, coprimary (a ring A is called coprimary if $\sharp\mathrm{Ass}(A) = 1$), (R_0), (S_1), reduced, and normal. Moreover, as corollaries of these results, we easily see that the following proposition is true for $\mathbb{P}$ = Cohen-Macaulay ([3], [4]), Gorenstein ([4]), domain, coprimary, (R_0), (S_1), and reduced.

Let $\mathbb{P}$ be a property for which (NC) holds. Then, for a ring A satisfying $\mathbb{P}$, the $\mathbb{P}$-locus of a homomorphic image of A is open.

It is known that the properties "regular", "Cohen-Macaulay", "reduced", and "normal" are described by using (R_n) and (S_n). Since (NC) holds for each of these properties,

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we naturally expect that (NC) may hold for (R_n) and (S_n) for every $n \geqq 0$. This is in fact true, and the main purpose of this paper is to give its complete proof.

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2. (NC) for (S_n)-condition

The following lemma should be referred to [3] §22.

Lemma 2.1. *Let A be a domain, B an A-algebra of finite type, and M a finite B-module. Then there exists $f(\neq 0) \in A$ such that M_f is A_f-free (where A_f is the localization of A with respect to the multiplicatively closed set $\{1, f, f^2, \cdots\}$).*

Now we can prove the main result of this section.

Theorem 2.2. (NC) *holds for* $\mathbb{P} = (\mathrm{S}_n)$.

Proof. We prove the theorem by induction on n. It is easy to see that (NC) holds for $\mathbb{P} = (\mathrm{S}_0)$ and (S_1) respectively, hence we assume $n \geqq 2$ in the rest. Suppose that a ring A satisfies the assumption in (NC). We want to prove that the locus $\mathrm{S}_n(A)$ is open in $\mathrm{Spec}(A)$. Since (S_n) implies (S_{n-1}), the locus $\mathrm{S}_{n-1}(A)$ is open in $\mathrm{Spec}(A)$ by induction hypothesis. Therefore we can write $\mathrm{S}_{n-1}(A) = \bigcup_{i=1}^{s} \mathrm{D}(f_i)$ with $f_i \in A$, hence $\mathrm{S}_n(A) = \bigcup_{i=1}^{s}(\mathrm{S}_n(A) \cap \mathrm{D}(f_i)) = \bigcup_{i=1}^{s} \mathrm{S}_n(A_{f_i})$. Since $\mathrm{S}_{n-1}(A_{f_i}) = \mathrm{S}_{n-1}(A) \cap \mathrm{D}(f_i) = \mathrm{D}(f_i) = \mathrm{Spec}(A_{f_i})$, the condition (S_{n-1}) holds for A_{f_i}. Thus, replacing A by A_{f_i}, to prove the openness of $\mathrm{S}_n(A)$ we may assume that

(∗) the condition (S_{n-1}) holds for A.

Put $\mathcal{I} = \{I \mid I \text{ is an ideal of } A \text{ and } \mathrm{S}_n(A)^c \subseteqq \mathrm{V}(I)\}$, where $\mathrm{S}_n(A)^c$ is the complement set $\mathrm{Spec}(A) - \mathrm{S}_n(A)$. We have $\mathcal{I} \neq \varnothing$ because $(0) \in \mathcal{I}$. Since A is noetherian, $\mathcal{I}$ has maximal elements. Let I be one of them. If $I = A$ then $\mathrm{S}_n(A) = \mathrm{Spec}(A)$ which is open in $\mathrm{Spec}(A)$. Therefore we assume that $I \subsetneqq A$. It is easy to see from the maximality that $\sqrt{I} = I$ and that $\overline{\mathrm{S}_n(A)^c} = \mathrm{V}(I)$. It follows from this that I has a primary decomposition of the form $I = \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_t$, where each $\mathfrak{p}_i$ is a prime ideal, and we may assume that there are no inclusion relations between the $\mathfrak{p}_i$'s and that $\mathrm{ht}(\mathfrak{p}_1) \leqq \mathrm{ht}(\mathfrak{p}_i)$ for all i.

Now we claim that

(1) $\mathrm{ht}(I) \geqq n$,
(2) $\mathfrak{p}_i \in \mathrm{S}_n(A)^c$ for all i,
(3) $\mathrm{S}_n(A)^c = \mathrm{V}(I)$.

It follows from (3) that $\mathrm{S}_n(A) = \mathrm{D}(I)$, which shows that $\mathrm{S}_n(A)$ is open in $\mathrm{Spec}(A)$, proving the theorem. We prove these in turn.

(1) It suffices to prove that $\mathrm{ht}(\mathfrak{p}_1) \geqq n$. To prove this by contradiction, suppose that $l := \mathrm{ht}(\mathfrak{p}_1) \leqq n-1$. By (∗) we get $\mathrm{depth}(A_{\mathfrak{p}_1}) \geqq \inf\ (n-1,\ \mathrm{ht}(\mathfrak{p}_1)) = \mathrm{ht}(\mathfrak{p}_1) = l$, hence there exist $c_i \in \mathfrak{p}_1$ and $f \in A - \mathfrak{p}_1$ such that $c_1, \cdots, c_l$ is an A_f-sequence in $\mathfrak{p}_1 A_f$ and that $(c_1, \cdots, c_l)A_f$ is $\mathfrak{p}_1 A_f$-primary. Now we can take $g \in \bigcap_{i=2}^{t} \mathfrak{p}_i - \mathfrak{p}_1$ such that $IA_g = \mathfrak{p}_1 A_g$ because $\mathfrak{p}_i \not\subseteqq \mathfrak{p}_1$ for all $i \geqq 2$. Moreover, by the assumption in (NC), there

exists $h \in A - \mathfrak{p}_1$ such that $\mathrm{D}(h) \cap \mathrm{V}(\mathfrak{p}_1) \subseteqq \mathrm{S}_n(A/\mathfrak{p}_1)$, hence the condition (S_n) holds for $A_h/\mathfrak{p}_1 A_h$. Put $x = fgh$ $(\in A - \mathfrak{p}_1)$. Replacing A by A_x, we may assume that

$$\begin{cases} c_1, \cdots, c_l \text{ is an } A\text{-sequence in } \mathfrak{p}_1, \\ (c_1, \cdots, c_l) \text{ is } \mathfrak{p}_1\text{-primary (hence } \mathfrak{p}_1^r \subseteqq (\underline{c}) \text{ for some } r \in \mathbf{N}), \\ I = \mathfrak{p}_1 \text{ (hence } \overline{\mathrm{S}_n(A)^c} = \mathrm{V}(\mathfrak{p}_1)), \\ (\mathrm{S}_n) \text{ holds for } A/\mathfrak{p}_1. \end{cases}$$

Moreover, by Lemma 2.1, replacing A by A_y with some $y \in A - \mathfrak{p}_1$, we may assume that

$$\mathfrak{p}_1^i/\mathfrak{p}_1^{i+1} + (\underline{c}) \cap \mathfrak{p}_1^i \text{ is } A/\mathfrak{p}_1\text{-free } (1 \leqq i < r).$$

Now note that $\mathrm{S}_n(A)^c \neq \varnothing$. In fact, if $\mathrm{S}_n(A)^c = \varnothing$ then $\mathrm{V}(\mathfrak{p}_1) = \overline{\mathrm{S}_n(A)^c} = \varnothing$ hence $\mathfrak{p}_1 = A$, a contradiction. Therefore we have $\mathrm{S}_n(A)^c \neq \varnothing$. We would like to prove that $A_\mathfrak{p}$ satisfies the condition (S_n) for any $\mathfrak{p} \in \mathrm{S}_n(A)^c$. If this is true, then we have a contradiction since $\mathfrak{p} \notin \mathrm{S}_n(A)$. Therefore, we will have $\mathrm{ht}(\mathfrak{p}_1) \geqq n$ as desired. To prove that (S_n) holds for $A_\mathfrak{p}$, take $\mathfrak{p}' \in \mathrm{Spec}(A)$ with $\mathfrak{p}' \subseteqq \mathfrak{p}$, and $\mathfrak{p}'' \in \mathrm{V}(\mathfrak{p}' + \mathfrak{p}_1)$ such that $\mathrm{ht}(\mathfrak{p}' + \mathfrak{p}_1/\mathfrak{p}_1) = \mathrm{ht}(\mathfrak{p}''/\mathfrak{p}_1)$. (Since $\mathfrak{p}', \mathfrak{p}_1 \subseteqq \mathfrak{p}$, we have $\mathrm{V}(\mathfrak{p}' + \mathfrak{p}_1) \neq \varnothing$.) We should divide the proof into two cases.

i) The case when $\mathrm{ht}(\mathfrak{p}' + \mathfrak{p}_1/\mathfrak{p}_1) \leqq n$:
Since $\mathrm{ht}(\mathfrak{p}''/\mathfrak{p}_1) \leqq n$, $A_{\mathfrak{p}''}/\mathfrak{p}_1 A_{\mathfrak{p}''} = (A/\mathfrak{p}_1)_{\mathfrak{p}''/\mathfrak{p}_1}$ is CM. Replacing A by $A/(\underline{c})$, we may assume that $\mathfrak{p}_1^r = (0)$ and that $\mathfrak{p}_1^i/\mathfrak{p}_1^{i+1}$ is $A/\mathfrak{p}_1$-free. Therefore, $\mathrm{depth}(A_{\mathfrak{p}''}) = \mathrm{depth}(A_{\mathfrak{p}''}/\mathfrak{p}_1^r A_{\mathfrak{p}''}) = \mathrm{depth}(A_{\mathfrak{p}''}/\mathfrak{p}_1 A_{\mathfrak{p}''}) = \mathrm{ht}(\mathfrak{p}''/\mathfrak{p}_1) = \mathrm{ht}(\mathfrak{p}'')$, hence $A_{\mathfrak{p}''}$ is CM. It follows that $A_{\mathfrak{p}'} = (A_{\mathfrak{p}''})_{\mathfrak{p}' A_{\mathfrak{p}''}}$ is CM.

ii) The case when $\mathrm{ht}(\mathfrak{p}' + \mathfrak{p}_1/\mathfrak{p}_1) \geqq n$:
Let $\mathfrak{q}/\mathfrak{p}_1 \in \mathrm{V}(\mathfrak{p}' + \mathfrak{p}_1/\mathfrak{p}_1)$.Then $\mathrm{ht}(\mathfrak{q}/\mathfrak{p}_1) \geqq n$, hence $\mathrm{depth}((A/\mathfrak{p}_1)_{\mathfrak{q}/\mathfrak{p}_1}) \geqq n$. Thus, $\mathrm{depth}_{\mathfrak{p}'+\mathfrak{p}_1/\mathfrak{p}_1}(A/\mathfrak{p}_1) \geqq n$. Therefore there exist $c_i' \in \mathfrak{p}'$ such that

$$c_1', \cdots, c_n' \text{ is an } A/\mathfrak{p}_1\text{-sequence in } \mathfrak{p}' + \mathfrak{p}_1/\mathfrak{p}_1.$$

Since $\mathfrak{p}_1^i/\mathfrak{p}_1^{i+1} + (\underline{c}) \cap \mathfrak{p}_1^i$ is $A/\mathfrak{p}_1$-free, one can show that

$$c_1', \cdots, c_n' \text{ is an } A/(\underline{c})\text{-sequence in } \mathfrak{p}' + \mathfrak{p}_1/\mathfrak{p}_1.$$

Hence $c_1, \cdots, c_l, c_1', \cdots, c_n'$ is an A-sequence in $\mathfrak{p}''$, so an $A_{\mathfrak{p}''}$-sequence in $\mathfrak{p}'' A_{\mathfrak{p}''}$. Therefore,

$$c_1', \cdots, c_n', c_1, \cdots, c_l \text{ is an } A_{\mathfrak{p}''}\text{-sequence in } \mathfrak{p}'' A_{\mathfrak{p}''}.$$

Hence $c_1', \cdots, c_n'$ is an $A_{\mathfrak{p}''}$-sequence in $\mathfrak{p}' A_{\mathfrak{p}''}$, so an $A_{\mathfrak{p}'} = (A_{\mathfrak{p}''})_{\mathfrak{p}' A_{\mathfrak{p}''}}$-sequence in $\mathfrak{p}' A_{\mathfrak{p}'} = \mathfrak{p}'(A_{\mathfrak{p}''})_{\mathfrak{p}' A_{\mathfrak{p}''}}$. It follows that $\mathrm{depth}(A_{\mathfrak{p}'}) \geqq n$.

As we have remarked above, it follows from *i*), *ii*) that $\mathrm{ht}(\mathfrak{p}_1) \geqq n$.

(2) To prove it by contradiction, suppose that $\mathfrak{p}_k \in \mathrm{S}_n(A)$ for some k. Since $I \subseteqq \mathfrak{p}_k$, we have $\mathrm{ht}(\mathfrak{p}_k) \geqq n$, hence $\mathrm{depth}(A_{\mathfrak{p}_k}) \geqq \inf(n, \mathrm{ht}(\mathfrak{p}_k)) = n$. Therefore, there exist $c_i \in \mathfrak{p}_k$ and $f \in A - \mathfrak{p}_k$ such that $c_1, \cdots, c_n$ is an A_f-sequence in $\mathfrak{p}_k A_f$ and that $IA_f = \mathfrak{p}_k A_f$. Since $\mathfrak{p}_k \in \mathrm{V}(I) = \overline{\mathrm{S}_n(A)^c}$, we have $\mathrm{D}(f) \cap \mathrm{S}_n(A)^c \neq \varnothing$. Let $\mathfrak{p}$ be a minimal element of this set. Since $\mathfrak{p} \in \mathrm{S}_n(A)^c \subseteqq \mathrm{V}(I)$, we have $I \subseteqq \mathfrak{p}$, hence $\mathfrak{p} A_f \supseteqq IA_f = \mathfrak{p}_k A_f$. Therefore $c_1, \cdots, c_n$ is an A_f-sequence in $\mathfrak{p} A_f$, hence is an $A_\mathfrak{p} = (A_f)_{\mathfrak{p} A_f}$-sequence in $\mathfrak{p} A_\mathfrak{p}$. It

follows that $\operatorname{depth}(A_{\mathfrak{p}}) \geqq n = \inf\,(n, \operatorname{ht}(\mathfrak{p}))$. On the other hand, if $\mathfrak{p}' \in \operatorname{Spec}(A)$ such that $\mathfrak{p}' \subsetneqq \mathfrak{p}$, then we have $\mathfrak{p}' \notin \mathrm{D}(f) \cap \mathrm{S}_n(A)^c$ by the minimality of $\mathfrak{p}$. Since $\mathfrak{p} \in \mathrm{D}(f)$, we have $\mathfrak{p}' \in \mathrm{D}(f)$. Therefore we have $\mathfrak{p}' \notin \mathrm{S}_n(A)^c$, hence (S_n) holds for $A_{\mathfrak{p}'}$. Thus, we see that (S_n) holds for $A_{\mathfrak{p}}$, contrary to the choice of $\mathfrak{p}$.

(3) We have $\mathrm{S}_n(A)^c \subseteqq \overline{\mathrm{S}_n(A)^c} = \mathrm{V}(I)$. Suppose that $\mathrm{S}_n(A)^c \subsetneqq \mathrm{V}(I)$. Then there exists $\mathfrak{p} \in \mathrm{V}(I)$ such that $\mathfrak{p} \notin \mathrm{S}_n(A)^c$. Hence we have $\mathfrak{p}_k \subseteqq \mathfrak{p}$ for some k and $\mathfrak{p} \in \mathrm{S}_n(A)$. Therefore (S_n) holds for $(A_{\mathfrak{p}})_{\mathfrak{p}_k A_{\mathfrak{p}}} = A_{\mathfrak{p}_k}$. It follows that $\mathfrak{p}_k \in \mathrm{S}_n(A)$, contrary to (2). ∎

3. (NC) for (R_n)-condition

Consider the following condition. Let n be an integer and let A be a local ring.

(R'_n) : If $\mathfrak{p} \in \operatorname{Spec}(A)$ and $\operatorname{codim}(\mathfrak{p}) \leqq n$, then $A_{\mathfrak{p}}$ is regular.

Here the codimension of an ideal I of A is defined as follows.

$$\operatorname{codim}(I) = \dim(A) - \dim(A/I).$$

Lemma 3.1. *Let A be a local ring. Then (R_n) holds for A if and only if (R'_n) holds for $A_{\mathfrak{p}}$ for every $\mathfrak{p} \in \operatorname{Spec}(A)$.*

Proof. Suppose that (R_n) holds for A. Let $\mathfrak{p} \in \operatorname{Spec}(A)$, and let $\mathfrak{p}' \in \operatorname{Spec}(A)$ such that $\mathfrak{p}' \subseteqq \mathfrak{p}$ and that $\operatorname{codim}(\mathfrak{p}' A_{\mathfrak{p}}) \leqq n$. Then we have $\operatorname{ht}(\mathfrak{p}' A_{\mathfrak{p}}) \leqq \operatorname{codim}(\mathfrak{p}' A_{\mathfrak{p}}) \leqq n$, hence $(A_{\mathfrak{p}})_{\mathfrak{p}' A_{\mathfrak{p}}}$ is regular since (R_n) holds for $A_{\mathfrak{p}}$. It follows that (R'_n) holds for $A_{\mathfrak{p}}$. Conversely, suppose that (R'_n) holds for $A_{\mathfrak{p}}$ for every $\mathfrak{p} \in \operatorname{Spec}(A)$. Let $\mathfrak{p} \in \operatorname{Spec}(A)$, and let $\mathfrak{p}' \in \operatorname{Spec}(A)$ such that $\mathfrak{p}' \subseteqq \mathfrak{p}$ and that $\operatorname{ht}(\mathfrak{p}' A_{\mathfrak{p}}) \leqq n$. Then we have $\operatorname{codim}(\mathfrak{p}' A_{\mathfrak{p}'}) = \operatorname{ht}(\mathfrak{p}') = \operatorname{ht}(\mathfrak{p}' A_{\mathfrak{p}}) \leqq n$, hence $(A_{\mathfrak{p}})_{\mathfrak{p}' A_{\mathfrak{p}}} = A_{\mathfrak{p}'} = (A_{\mathfrak{p}'})_{\mathfrak{p}' A_{\mathfrak{p}'}}$ is regular. It follows that (R_n) holds for $A_{\mathfrak{p}}$. Therefore, (R_n) holds for A. ∎

The following theorem is the main result of this section.

Theorem 3.2. (NC) *holds for* $\mathbb{P} = (\mathrm{R}_n)$.

Proof. We prove this theorem by induction on n. It is easy to see that (NC) holds for $\mathbb{P} = (\mathrm{R}_0)$, hence we assume $n \geqq 1$ in the rest. We discuss in the same way as the proof of Theorem 2.2. Suppose that a ring A satisfies the assumption in (NC). Let I be one of the maximal elements of the set $\{I \mid I \text{ is an ideal of } A \text{ and } \mathrm{R}_n(A)^c \subseteqq \mathrm{V}(I)\}$. We may assume that

$$\begin{cases} (\mathrm{R}_{n-1}) \text{ holds for } A \ \cdots \ (*), \\ I \subsetneqq A, \\ \sqrt{I} = I, \\ \overline{\mathrm{R}_n(A)^c} = \mathrm{V}(I), \\ I = \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_t \text{ (with some } \mathfrak{p}_i \in \operatorname{Spec}(A)), \\ \text{there are no inclusion relations between the } \mathfrak{p}_i\text{'s}, \\ \operatorname{ht}(\mathfrak{p}_1) \leqq \operatorname{ht}(\mathfrak{p}_i) \text{ for all } i. \end{cases}$$

Now we prove that $\mathrm{ht}(\mathfrak{p}_1) \geqq n$. To prove this by contradiction, suppose that $l := \mathrm{ht}(\mathfrak{p}_1) \leqq n-1$. By $(*)$ we see that $A_{\mathfrak{p}_1}$ is regular. Hence replacing A by A_x for some $x \in A - \mathfrak{p}_1$, we may assume that

$$\begin{cases} c_1, \cdots, c_l \text{ is an } A\text{-sequence in } \mathfrak{p}_1 \text{ (with some } c_i \in \mathfrak{p}_1), \\ (c_1, \cdots, c_l) = \mathfrak{p}_1, \\ I = \mathfrak{p}_1 \text{ (hence } \overline{\mathrm{R}_n(A)^c} = \mathrm{V}(\mathfrak{p}_1)), \\ (\mathrm{R}_n) \text{ holds for } A/\mathfrak{p}_1 \ \cdots\ (**). \end{cases}$$

Since $\mathrm{R}_n(A)^c \neq \varnothing$, one can take $\mathfrak{p} \in \mathrm{R}_n(A)^c$. Then we have $\mathfrak{p}_1 \subseteqq \mathfrak{p}$. To show that $A_\mathfrak{p}$ satisfies (R'_n), we take $\mathfrak{p}' \in \mathrm{Spec}(A)$ such that $\mathfrak{p}' \subseteqq \mathfrak{p}$ and that $\mathrm{codim}(\mathfrak{p}'A_\mathfrak{p}) \leqq n$. There exists $\mathfrak{p}'' \in \mathrm{V}(\mathfrak{p}' + \mathfrak{p}_1)$ such that $\mathrm{codim}((\mathfrak{p}' + \mathfrak{p}_1/\mathfrak{p}_1)A_\mathfrak{p}) = \mathrm{codim}((\mathfrak{p}''/\mathfrak{p}_1)A_\mathfrak{p})$ $(= \mathrm{codim}(\mathfrak{p}''A_\mathfrak{p}/\mathfrak{p}_1A_\mathfrak{p}))$. We have

$$\begin{cases} \mathrm{codim}((\mathfrak{p}' + \mathfrak{p}_1/\mathfrak{p}_1)A_\mathfrak{p}) = \mathrm{ht}(\mathfrak{p}/\mathfrak{p}_1) - \mathrm{ht}(\mathfrak{p}/\mathfrak{p}' + \mathfrak{p}_1) = \mathrm{ht}(\mathfrak{p}) - l - \mathrm{ht}(\mathfrak{p}/\mathfrak{p}' + \mathfrak{p}_1), \\ \mathrm{codim}((\mathfrak{p}' + \mathfrak{p}_1/\mathfrak{p}')A_\mathfrak{p}) = \mathrm{codim}((c_1, \cdots, c_l)(A/\mathfrak{p}')_{\mathfrak{p}/\mathfrak{p}'}) \leqq l, \\ \mathrm{codim}((\mathfrak{p}' + \mathfrak{p}_1/\mathfrak{p}')A_\mathfrak{p}) = \mathrm{ht}(\mathfrak{p}/\mathfrak{p}') - \mathrm{ht}(\mathfrak{p}/\mathfrak{p}' + \mathfrak{p}_1). \end{cases}$$

It follows that

$$\mathrm{codim}(\mathfrak{p}''A_\mathfrak{p}/\mathfrak{p}_1A_\mathfrak{p}) = \mathrm{codim}((\mathfrak{p}' + \mathfrak{p}_1/\mathfrak{p}_1)A_\mathfrak{p}) \leqq \mathrm{ht}(\mathfrak{p}) - \mathrm{ht}(\mathfrak{p}/\mathfrak{p}') = \mathrm{codim}(\mathfrak{p}'A_\mathfrak{p}) \leqq n.$$

By $(**)$ we see that (R_n) holds for $(A/\mathfrak{p}_1)_{\mathfrak{p}/\mathfrak{p}_1} = A_\mathfrak{p}/\mathfrak{p}_1A_\mathfrak{p}$. By Lemma 3.1, we see that $A_{\mathfrak{p}''}/\mathfrak{p}_1A_{\mathfrak{p}''} = (A_\mathfrak{p}/\mathfrak{p}_1A_\mathfrak{p})_{\mathfrak{p}''A_\mathfrak{p}/\mathfrak{p}_1A_\mathfrak{p}}$ is regular, which shows that $A_{\mathfrak{p}''}$ is regular. It follows that $(A_\mathfrak{p})_{\mathfrak{p}'A_\mathfrak{p}} = (A_{\mathfrak{p}''})_{\mathfrak{p}'A_{\mathfrak{p}''}}$ is regular. Therefore we see that $A_\mathfrak{p}$ satisfies (R'_n). Let $\mathfrak{q} \in \mathrm{Spec}(A)$ such that $\mathfrak{q} \subseteqq \mathfrak{p}$. If $\mathfrak{q} \in \mathrm{R}_n(A)$, then (R_n) holds for $A_\mathfrak{q}$, hence (R'_n) holds for $A_\mathfrak{q}$. If $\mathfrak{q} \in \mathrm{R}_n(A)^c$, then we see that (R'_n) holds for $A_\mathfrak{q}$, discussing in the same way as above. Thus, it follows from Lemma 3.1 that (R_n) holds for $A_\mathfrak{p}$. Since $\mathfrak{p} \in \mathrm{R}_n(A)^c$, we have a contradiction. Thus we have shown that $\mathrm{ht}(\mathfrak{p}_1) \geqq n$, hence $\mathrm{ht}(I) \geqq n$.

Therefore we can arrange the order of $\mathfrak{p}_1, \cdots, \mathfrak{p}_t$ to satisfy the following conditions.

$$\mathrm{ht}(\mathfrak{p}_i) \begin{cases} = n \ (1 \leqq i \leqq s), \\ > n \ (s < i \leqq t), \end{cases}$$

$$A_{\mathfrak{p}_i} \text{ is } \begin{cases} \text{non-regular } (1 \leqq i \leqq r), \\ \text{regular } (r < i \leqq s). \end{cases}$$

Put $J = \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_r$. Let $\mathfrak{p} \in \mathrm{R}_n(A)^c$. Then there exists $\mathfrak{p}' \in \mathrm{Spec}(A)$ such that $\mathfrak{p}' \subseteqq \mathfrak{p}$, $\mathrm{ht}(\mathfrak{p}') \leqq n$, and that $A_{\mathfrak{p}'}$ is non-regular. By $(*)$ we get $\mathrm{ht}(\mathfrak{p}') = n$. Replacing $\mathfrak{p}$ by $\mathfrak{p}'$, we may assume that $\mathrm{ht}(\mathfrak{p}) = n$. Since $\mathrm{R}_n(A)^c \subseteqq \mathrm{V}(I)$, we have $I \subseteqq \mathfrak{p}$, hence $\mathfrak{p}_k \subseteqq \mathfrak{p}$ for some k. Since $\mathrm{ht}(\mathfrak{p}) = n$, we have $\mathfrak{p}_k = \mathfrak{p}$ and $1 \leqq k \leqq s$, and since $A_\mathfrak{p}$ is non-regular, we have $1 \leqq k \leqq r$. It follows that $J \subseteqq \mathfrak{p}_k = \mathfrak{p}$, i.e. $\mathfrak{p} \in \mathrm{V}(J)$. Therefore, we have $\mathrm{R}_n(A)^c \subseteqq \mathrm{V}(J)$. Since the opposite inclusion is obvious by the choice of $\mathfrak{p}_i$,we have $\mathrm{R}_n(A)^c = \mathrm{V}(J)$. Thus, we get $\mathrm{R}_n(A) = \mathrm{D}(J)$, which shows that $\mathrm{R}_n(A)$ is open in $\mathrm{Spec}(A)$. ■

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