The Brown-Peterson spectrum BP was originally constructed as a spectrum that "kills Bockstein" in the Steenrod algebra $\mathcal{A} = HZ/p^*(HZ/p)$, that is, we have $H^*(BP) \cong \mathcal{A}/\mathcal{A}\beta\mathcal{A}$. Quillen then showed that it splits off the complex cobordism spectrum MU using the properties of formal group laws. Priddy showed that it can be characterized by the cellular construction successively killing off stasble homotopy groups in odd degrees starting from the sphere. Since its introduction, it has played a central role in stable homotopy theory, for example Adams-Novikov spectral sequence became a subject on its own. However, an explicit cellular construction has been lacking.

Mitchell and Priddy constructed a sequence of spectra $\{D(n)\}$ realizing the Cartan-Serre length filtration of the Steenrod algebra. They fit in cofibration sequences $D(n) \to D(n+1) \to \Sigma^{n+1}M(n)$. Thus $M(n)$ realizes the filtration n part of the Steenrod algebra.

They also constructed a sequence of spectra $BP(n)$ realing the "length n" part of $H^*(BP) \cong \mathcal{A}/\mathcal{A}\beta\mathcal{A}$ where BP denotes the Brown-Peterson spectrum.

This leads naturally to our first question: is there a sequence of spectra realizing the length filtration on $H^*(BP)$? A positive answer to this question also gives a nice cellular construction of BP.

Mitchell and Priddy also showed the splitting of $BP(n)$ off $BU(1)^n$ at all primes (after completion at p) as well as that of $M(n)$ off $BO(1)^n$ at prime 2 using the Steienberg idempotent, and deduced their splitting off $BU(n)$ and $BO(n)$ using the Becker-Gottlieb transfer. On the other hand, more recently Galatius, Tillmann, Madsen and Weiss showed that there are spectra $MTO(n)$ and $MTU(n)$ such that there are cofibration sequences $\Sigma^n MTO(n) \to \Sigma^{n+1}MTO(n+1) \to \Sigma^\infty\Sigma^{n+1}BO(n+1)_+$, $\Sigma^{2n}MTU(n) \to \Sigma^{2n+2}MTU(n+1) \to \Sigma^\infty\Sigma^{n+1}BU(n+1)_+$, and that $\Sigma^n MTO(n)$ and $\Sigma^{2n}MTU(n)$ filter MO and MU respectively. Furthermore, classical results of Thom and Quillen show that MO split as wedge of suspensions of $HZ/2$ and MU (after localization at p) wedge of suspensions of BP.

All these lead to more questions: does $D(n)$ split off $\Sigma^n MTO(n)$? And do the spectra realizing the length filtration on $H^*(BP)$, if they exist, split off $\Sigma^{2n}MTU(n)$'s at p?

In our previous work we showed the splitting of $D(n)$ off $\Sigma^n MTO(n)$ using the Whitehead conjecture. In this talk we show the existence of a sequence of spectra $\{\Sigma^{2n}BTP(n)\}$ realizing the Cartan-Serre filtration, with maps $BTP(n) \to MTU(n)$. Furthermore they split off $MTU(n)$. When $p = n = 2$, we can describe the remaining summand concretely, and we get $MTU(2) \cong BTP(2) \vee BSU(3)_+$ at prime 2.

The Iwahori-Hecke relation among the Steinberg idempotents is one of the key ingredients of the proof. This is joint work with Hadi Zare (University of Tehran, IPM).