Div - curl lemma with critical power weights in dimension three

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Abstract

In $\mathbf{R}^3$, a div - curl lemma with critical exponents in terms of Hardy spaces associated to Herz spaces is given.

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1 Introduction

Div-curl lemma means an inequality of the form: for two vector-valued functions F and G

$$\|F \cdot G\|_Z \le c\|F\|_X \|G\|_Y$$

under the assumption div$F$ = curl $G = 0$ with some quasi-Banach spaces X, Y and Z. Coifman, Lions, Meyer and Semmes [5] investigated the type of inequalities and gave several applications. Their study was motivated from the theory of compensated compactness due to Murat and Tataru [15].

One of examples of the form above is $(u \cdot \nabla)v$ with div$u = 0$:

$$\begin{aligned}(u \cdot \nabla)v &= \left(\sum_{j=1}^{3} u_j \partial_j v_1, \sum_{j=1}^{3} u_j \partial_j v_2, \sum_{j=1}^{3} u_j \partial_j v_3\right)\\ &= \left(\sum_{j=1}^{3} \partial_j (u_j v_1), \sum_{j=1}^{3} \partial_j (u_j v_2), \sum_{j=1}^{3} \partial_j (u_j v_3)\right) = \nabla \cdot (u \otimes v)\end{aligned}$$

where $u = (u_1, u_2, u_3)$, $v = (v_1, v_2, v_3)$ and $u \otimes v$ is a 3×3 matrix, whose (i, j) component is $u_i v_j$. This term appears in the incompressible viscous Navier-Stokes equation with $v = u$:

$$\text{(N-S)} \quad \begin{cases} \partial_t u - \Delta u + (u \cdot \nabla)u + \nabla \pi = 0, \\ \text{div } u = 0, \\ u(0) = a. \end{cases}$$

In this article, we focus on this non linear, but bilinear term. From Hölder inequality, if $u \in L^p(\mathbb{R}^3)^3$ and $\nabla v \in L^{p'}(\mathbb{R}^3)^{3\times 3}$ where $p \in (1, \infty)$ and $p' = p/(p-1)$, then $(u \cdot \nabla)v \in L^1(\mathbb{R}^3)^3$. With the help of the cancellation property:

$$\int_{\mathbb{R}^3} \sum_{j=1}^{3} u_j \partial_j v_k dx = 0 \quad \text{for all} \quad k \in \{1, 2, 3\},$$

the term belongs to a better function space, Hardy space $H^1(\mathbb{R}^3)^3 \subset L^1(\mathbb{R}^3)^3$. This interesting result was found by Coifman-Lions-Meyer-Semmes [5] as the following form: let $3/4 < p, q < \infty$ and $1/r = 1/p + 1/q < 4/3$. For vector fields u and v, it follows that

$$\|(u \cdot \nabla)v\|_{H^r} \le c\|u\|_{H^p} \|\nabla v\|_{H^q} \tag{1}$$

provided that div $u = 0$. Here $H^p(\mathbb{R}^3) = H^p$ is the Hardy space.

Their result has several generalizations. Because the moment of order one;

$$\int_{\mathbb{R}^3} x^\alpha (u \cdot \nabla)v(x)dx \quad (|\alpha| = 1)$$

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does not vanish in general, there is no hope that the term belongs to the Hardy space $H^{3/4}(\mathbb{R}^3)^3$. However, with a modification, the endpoint inequality holds:

$$\|(u\cdot\nabla)v\|_{H^{3/4,\infty}} \le c\|u\|_{L^p}\|\nabla v\|_{L^q}$$

for $p\in(1,\infty)$ and $q\in(1,3)$, where $H^{3/4,\infty}(\mathbb{R}^3)$ is the weak Hardy space, see [5] and also [13]. Although, (1) cannot also deal with the case $p=\infty$, Auscher-Russ-Tchamitchian [1] gave the endpoint bound:

$$\|(u\cdot\nabla)v\|_{H^1} \le c\|u\|_{L^\infty}\|\nabla v\|_{H^1}.$$

It is not allowed to replace $L^\infty(\mathbb{R}^3)$ by $BMO(\mathbb{R}^3)$, because if u is a constant vector field the left hand side is not zero in general, but $\|u\|_{BMO}=0$. Bonami-Feuto-Grellier [2] established a version of [1] as follows:

$$\|(u\cdot\nabla)v\|_{H^\Phi} \le c\|u\|_{bmo}\|\nabla v\|_{H^1}$$

where $H^\Phi(\mathbb{R}^3)$ is a Hardy-Orlicz space related to the Orlicz function $\Phi(t)=\dfrac{t}{\log(e+t)}$ and $bmo(\mathbb{R}^3)$ is the local BMO space, introduced by Goldberg [8]. (1) with power weights was established by Lu and Yang [11] and Miyachi [12] in terms of Herz spaces $\dot{K}^\alpha_{p,q}(\mathbb{R}^3)$, which is a generalization of Lebesgue spaces with weights, see Remark 1.1 below. In the previous paper [17], we proved a similar result, in which weights belong to Muckenhoupt classes $A_p(\mathbb{R}^3)$: let $3/4<p,q<\infty$, $w\in A_{4p/3}(\mathbb{R}^3)$ and $\sigma\in A_{4q/3}(\mathbb{R}^3)$.

(i): Suppose that $1/r=1/p+1/q<4/3$ and there exist $\tilde{p}\in(1,4p/3)$ and $\tilde{q}\in(1,4q/3)$ so that $w\in A_{\tilde{p}}(\mathbb{R}^3)$, $\sigma\in A_{\tilde{q}}(\mathbb{R}^3)$ and $\tilde{p}/p+\tilde{q}/q<4/3$. Then,

$$\|(u\cdot\nabla)v\|_{H^r(\mu)} \le c\|u\|_{H^p(w)}\|\nabla v\|_{H^q(\sigma)}$$

where div $u=0$ and $\mu^{1/r}=w^{1/p}\sigma^{1/q}$.
(ii): It follows

$$\|(u\cdot\nabla)v\|_{H^q(\sigma)} \le c\|u\|_{L^\infty}\|\nabla v\|_{H^q(\sigma)} \tag{2}$$

where div $u=0$.

See Remark 1.1 below for the definition of weighted Hardy spaces $H^p(w)=H^p(\mathbb{R}^3;w)$. When $\sigma(x)=|x|^{\alpha q}$, the range of α, for which (ii) can be applied, is

$$-3/q<\alpha<3(1-1/q)+1=:\alpha_q.$$

The purpose of this article is to establish the same estimate at the end-point case $\alpha=\alpha_q$ in 3-D case. $H^q(\sigma)$ norm with $\sigma(x)=|x|^{\alpha_q q}$ is related to the optimal decay of $L^2(\mathbb{R}^3)$ energy of solutions to (N.-S.). Before we see the relation, we shall recall a result by Wiegner [19] for the decay rate of $L^2(\mathbb{R}^3)$ energy of weak solutions to (N.-S.). He [19] proved that if $L^2(\mathbb{R}^3)$ initial data a satisfies

$$\|e^{t\Delta}a\|_{L^2}\le ct^{-\theta}\ \left(\text{i.e. } a\in\dot{\mathbf{B}}^{-2\theta}_{2,\infty}(\mathbb{R}^3)\right),$$

then the corresponding weak solution u fulfills $\|u(t)\|_{L^2}\le ct^{-\gamma}$ where $\gamma=\min(\theta,5/4)$. It is well known that the order $5/4$ is optimal in general. More precisely, if

$$\lim_{t\to\infty}t^{5/4}\|u(t)\|_{L^2}=0,$$

then the initial data and solution have to satisfy some symmetric conditions, see [14] for the detail. It seems that (2) with $\sigma(x)=|x|^{\alpha_q q}$ is relevant to this order $5/4$, because we have that for $q\in(0,2]$

$$\|e^{t\Delta}a\|_{L^2}\le ct^{-5/4}\|a\|_{H^q(\sigma)}\text{ where }\sigma(x)=|x|^{\alpha_q q},$$

see [17] for the proof. The present author [17] investigated the $L^2(\mathbb{R}^3)$ decay of mild solutions by Kato [10] and constructed solutions whose decay order of $L^2(\mathbb{R}^3)$ energy is $\gamma<5/4$. One of reasons why the order γ in [17] did not reach to the optimal order $5/4$ is that (ii) cannot allow us to take $\sigma(x)=|x|^{\alpha_q q}$ in (2). As mentioned in Remark 7.3 in [12], the bilinear term $(u\cdot\nabla)v$ does not belong to $H^p(w)$ with $w(x)=|x|^{\alpha_p p}$. This observation tells us that if we try to establish (2) with $\sigma(x)=|x|^{\alpha_q q}$, we has to replace $H^q(\sigma)$ in the left hand side by some larger spaces. For the purpose, we use Hardy spaces associated to Herz spaces, as in [11] and [12]. Although, the author does not know whether or not it is possible to construct global solutions having optimal $L^2(\mathbb{R}^3)$ decay from the similar argument as the previous paper [17], by using a critical div-curl lemma established in this article.

We explain notations. $\mathcal{S}(\mathbb{R}^3)$ and $\mathcal{S}'(\mathbb{R}^3)$ denote the Schwartz spaces of rapidly decreasing smooth functions and tempered distributions on $\mathbb{R}^3$, respectively. For a measurable subset $E \subset \mathbb{R}^3$, $|E|$ and χ_E are the volume and the characteristic function of E, respectively. For any integers j, A_j denotes a annulus $\{x \in \mathbb{R}^3; 2^{j-1} \le |x| < 2^j\}$, and χ_j is the characteristic function of A_j. $B(x,r)$ is a ball in $\mathbb{R}^3$, centered at x of radius r. $\langle g\rangle_B$ means the average $|B|^{-1}\int_B g(x)dx$. Also, $A \approx B$ means $c_1 B \le A \le c_2 B$ with positive constants c_1 and c_2. In what follows, c denotes a constant that is independent of the functions involved, which may differ from line to line.

Definition 1.1. *Let $p,q \in (0,\infty]$ and $\alpha \in \mathbb{R}$. Define Herz spaces $\dot{K}^{\alpha}_{p,q}(\mathbb{R}^3)$ as*

$$\dot{K}^{\alpha}_{p,q}(\mathbb{R}^3) := \left\{ f \in L^p(\mathbb{R}^3\backslash\{0\}); \|f\|_{\dot{K}^{\alpha}_{p,q}} := \left\| \left\{ 2^{j\alpha}\|f\chi_j\|_{L^p} \right\}_{j\in\mathbb{Z}} \right\|_{l^q} < \infty \right\}.$$

To define Hardy spaces, we fix a radial function $\phi \in C^\infty(\mathbb{R}^3)$ supported on $B(0,1)$ satisfying $0 \le \phi \le 1$, $\phi \equiv 1$ on $B(0,1/2)$ and $\int \phi(x)dx = 1$. For $f \in \mathcal{S}'$, we define

$$M_\phi f(x) := \sup_{t>0} |\langle f, \phi_t(x-\cdot)\rangle|, \quad \text{where} \quad \phi_t(x) = t^{-3}\phi(x/t).$$

Definition 1.2. *Let $p,q \in (0,\infty]$ and $\alpha \in \mathbb{R}$. Define Hardy spaces associated with Herz spaces $H\dot{K}^{\alpha}_{p,q}(\mathbb{R}^3)$ as*

$$H\dot{K}^{\alpha}_{p,q}(\mathbb{R}^3) := \left\{ f \in \mathcal{S}'; \|f\|_{H\dot{K}^{\alpha}_{p,q}} := \|M_\phi f\|_{\dot{K}^{\alpha}_{p,q}} < \infty \right\}.$$

Remark 1.1. *1. These spaces cover Lebesgue spaces and Hardy spaces with power weight:*

$$\dot{K}^{\alpha}_{p,p}(\mathbb{R}^3) = L^p(w) \quad \text{and} \quad H\dot{K}^{\alpha}_{p,p}(\mathbb{R}^3) = H^p(w)$$

when $w(x) = |x|^{\alpha p}$ with $0 < p < \infty$, where $\|f\|_{L^p(w)} := \|fw^{1/p}\|_{L^p}$. Here, for $w \in A_\infty(\mathbb{R}^3)$, $\|f\|_{H^p(w)} := \|M_\phi f\|_{L^p(w)}$. If $w \equiv 1$, then we use H^p instead of $H^p(1)$.

2. For $1 < p < \infty$, it is well known that

$$w(x) = |x|^{\alpha p} \in A_p(\mathbb{R}^3) \iff -3/p < \alpha < 3(1-1/p).$$

Here A_p is the Muckenhoupt class. From this, we can see that $H\dot{K}^{\alpha}_{p,q}(\mathbb{R}^3) = \dot{K}^{\alpha}_{p,q}(\mathbb{R}^3)$ with such α.

Hardy spaces are characterized in terms of the the grand maximal function f^*_m. This maximal function is defined as follows: for $m \in \mathbb{N}\cup\{0\}$, $x \in \mathbb{R}^3$ and $t \in (0,\infty)$, $\mathcal{I}_m(x,t)$ denotes a space of all smooth functions $\psi \in C^\infty(\mathbb{R}^3)$ supported in $B(x,t)$ with

$$\|\partial^\alpha \psi\|_{L^\infty} \le t^{-(3+|\alpha|)} \quad \text{for} \quad |\alpha| \le m.$$

The grand maximal function f^*_m is then defined by

$$f^*_m(x) := \sup\left\{ |\langle f,\psi\rangle|; \psi \in \bigcup_{t\in(0,\infty)} \mathcal{I}_m(x,t) \right\}.$$

Uchiyama [18] showed an inequality between $M_\phi f$ and f^*_m:

$$f^*_m(x) \le c M_{3/(3+m)}(M_\phi f)(x),$$

where $M_r f(x) := \sup_{B\ni x} \langle |f|^r\rangle_B^{1/r}$ where the supremum is taken over all balls B containing x. We also write $M_1 = M$. From this, we can see that

$$\|f\|_{H\dot{K}^{\alpha}_{p,q}} = \|M_\phi f\|_{\dot{K}^{\alpha}_{p,q}} \approx \|f^*_m\|_{\dot{K}^{\alpha}_{p,q}}$$

for $0 < p,q \le \infty$, $-3/p < \alpha < \infty$ and $m > 3(1/p-1) + \max(0,\alpha)$.

We denote by $\hat{\mathcal{D}}_0(\mathbb{R}^3)$ the set of all $f \in \mathcal{S}(\mathbb{R}^3)$ with $\hat{f}$ belonging to $\mathcal{D}(\mathbb{R}^3)$ and vanishing in a neighborhood of $\xi = 0$, where $\hat{f}$ means the Fourier transform of f. Strömberg and Torchinsky [16] proved that $\hat{\mathcal{D}}_0(\mathbb{R}^3)$ is a dense subspace of $H^p(w)$ for $p \in (0,\infty)$ and doubling measures w. Miyachi [12] showed that $\hat{\mathcal{D}}_0$ is also a dense subspace of $H\dot{K}^{\alpha}_{p,q}(\mathbb{R}^3)$ for $0 < p,q < \infty$ and $-3/p < \alpha < \infty$.

To give $(u\cdot\nabla)v$ a definition as a tempered distribution, we define Y by a space of all locally integrable functions f satisfying that there exist $c_f > 0$ and a seminorm $|\cdot|_{\mathcal{S}}$ of $\mathcal{S}$ so that $\int |f(x)\varphi(x)|dx \le c_f|\varphi|_{\mathcal{S}}$, for all $\varphi \in \mathcal{S}$. Obviously, $L^p(w) \subset Y$ when $1 \le p \le \infty$ and $w \in A_p$.

The main result reads as follows.

Theorem 1.1. *For $3/4 < p < \infty$, it holds*

$$\|(u\cdot\nabla)v\|_{H\dot{K}^{\alpha_p}_{p,\infty}} \le c\|u\|_{L^\infty}\|\nabla v\|_{H\dot{K}^{\alpha_p}_{p,3/4}},$$

for $u \in L^\infty(\mathbb{R}^3)^3$ with div $u = 0$ and $v \in \left(Y \cap W^{1,r}_{loc}(\mathbb{R}^3)\right)^3$ for some $r \in (1,\infty)$.

Remark 1.2. *Using the same argument as in Section 4, we can also show a weak type estimate:*

$$\|(u\cdot\nabla)v\|_{H^{3/4,\infty}} \le c\|u\|_{L^\infty}\|\nabla v\|_{H^{3/4}}.$$

Here, for $f \in \mathcal{S}'(\mathbb{R}^3)$, $f \in H^{3/4,\infty}(\mathbb{R}^3)$ if and only if $M_\phi f \in L^{3/4,\infty}(\mathbb{R}^3)$, where $L^{p,\infty}(\mathbb{R}^3)$ is the Lorentz space. This can be also regarded as an endpoint case of the original div-curl lemma of [5]. It is enough to show

$$\|N_\infty v\|_{L^{3/4,\infty}} \le c\|\nabla v\|_{H^{3/4}}$$

instead of (4). This is achieved from the pointwise estimate (7) and a Fefferman-Stein's vector valued inequality, (2) of Theorem 1 in [7].

Our proof of Theorem 1.1 follows the argument of Auscher, Russ and Tchamitchian [1]. We recall notations that were used in [1]. For $x \in \mathbb{R}^3$ and $1 \le m \le \infty$, let $F_m(x)$ be a set of all vector-valued functions $\Psi = (\psi_1,\psi_2,\psi_3)$ and the supports of them are included in a ball $B_\Psi = B(x, r_\Psi)$ so that there exists a function $g_\Psi \in L^m(\mathbb{R}^3)$ such that div$\Psi = g_\Psi$ in $\mathcal{S}'$, supp $g_\Psi \subset B_\Psi$ and $\|\Psi\|_{L^m} + r_\Psi\|g_\Psi\|_{L^m} \le |B_\Psi|^{-1/m'}$. The maximal operator N_m is defined by for any locally integrable function v as

$$N_m v(x) := \sup_{\Psi\in F_m(x)}\left|\int v(y)g_\Psi(y)dy\right|.$$

The reason why we can deal with the critical exponent α_p is the pointwise estimate for $N_m v$, (6) in Section 4. Let $\nabla v = \sum_{j=1}^{\infty} a_j$ be an atomic decomposition with atoms $\{a_j\}_{j=1}^\infty \subset L^\infty(\mathbb{R}^3)$ satisfying

$$\text{supp } a_j \subset B_j \quad \text{and} \quad \int x^\alpha a_j(x)dx = 0 \ (|\alpha| \le N)$$

with a large $N \in \mathbb{N}$. In [1], the following pointwise estimate was used to obtain the div - curl lemma:

$$N_m v(x) \le c\sum_{j=1}^{\infty}\|a_j\|_{L^\infty}M_s(\chi_{B_j})(x)$$

for all $x \in \mathbf{R}^n$ with $m \in (1,\infty)$ and $s = 3m'/(3+m')$. On the other hand, our main estimate (7) in Section 4, corresponds to the case $m = \infty$. The proof of the pointwise estimate above in [1] relies on the solvability for the divergence equation

$$\text{div } \Psi = g \text{ in } B,$$

see Lemma 10 in [1]. In there, the solution Ψ belongs to the class $F_m(x)$ with $m < \infty$. Bourgain and Brezis [3], [4] studied this equation in bounded domains with $g \in L^3(\mathbb{R}^3)$ fulfilling $\int g(x)dx = 0$. It is a way for finding the solution Ψ to consider the Poisson equation

$$-\Delta h = g \text{ in } B.$$

If h is a solution of this equation, $\Psi = \nabla h$ solves the divergence equation. In particular, we apply the solution h with the Neumann condition $\partial_\nu h = 0$ on the boundary $\partial B(x,r)$. Fortunately, we need to consider this problem on balls and the Green/Neumann function G is known, see [6] and [20]. It is well known the equivalence between the existence of the Helmholtz decomposition and the solvability of the Neumann problem in a weak sense. This additional argument yields the our pointwise estimate (7) in Section 4.

In next chapter, we investigate the $C^2(\mathbb{R}^3\backslash\{0\})$ regularity of the solutions to the Neumann problem by using the Green/Neumann function G. In Section 3, we establish a vector-valued inequality for the Hardy-Littlewood maximal operator on Herz spaces with the critical weights. Using the regularity property and the vector valued inequality, we give a proof of Theorem 1.1 in Section 4.

2 Neumann problem for the Poisson equation in unit ball of $\mathbb{R}^3$

Let $B_0 = B(0,1) \subset \mathbb{R}^3$. We consider

$$\text{(NP)} \begin{cases} -\Delta h = g & \text{in} \quad B_0 \\ \partial_\nu h = 0 & \text{on} \quad \partial B_0, \end{cases}$$

where $g \in C_0^\infty(B_0)$ satisfying $\int_{B_0} g dx = 0$ and $\nu(y) = (\nu_1(y), \nu_2(y), \nu_3(y))$ is the outer normal vector at $y \in \mathcal{S}^2$.

The Green/Neumann function G for the problem (NP) is already known: for example see [6] or [20],

$$G(x,y) = (4\pi)^{-1} (\Gamma(x-y) - D(x,y) + N(x,y))$$

where

$$\begin{cases} \Gamma(x-y) = \dfrac{1}{|x-y|}, \\ D(x,y) = \dfrac{1}{|x||x^*-y|}, \\ N(x,y) = \log n(x,y) \quad \text{and} \\ n(x,y) = |x^*-y| \left(1 + \dfrac{x}{|x|} \cdot \dfrac{x^*-y}{|x^*-y|}\right) \quad \text{with} \quad x^* = \dfrac{x}{|x|^2} \end{cases}$$

Remark 2.1. *The following identity is important in this section: let $x, y \neq 0$, if $x^* \neq y$ and $y^* \neq x$, then*

$$|x||x^*-y| = |y||y^*-x|,$$

which implies $D(x,y) = D(y,x)$ and $|x|n(x,y) = |y|n(y,x)$.

We define

$$\begin{aligned} h(x) &= \int_{B_0} G(x,y)g(y)dy \\ &= (4\pi)^{-1} \left[\int_{B_0} \Gamma(x-y)g(y)dy - \int_{B_0} D(x,y)g(y)dy + \int_{B_0} N(x,y)g(y)dy \right] \\ &= (4\pi)^{-1} \left[h_\Gamma(x) + h_D(x) + h_N(x) \right]. \end{aligned}$$

From Lemma 4.2 in [9], we know $h_\Gamma \in C^2(B_0)$. Further, it holds $\partial_\nu h_\Gamma = 0$ on ∂B_0, see [6]. Main purpose of this section is to show the C^2 regularity of h outside B_0. We show the following.

Proposition 2.1. (i) $h_\Gamma \in C^2(\mathbb{R}^3)$ *and* $-\Delta h_\Gamma(x) = g(x)$ for all $x \in \mathbb{R}^3$.

(ii) $h_D \in C^2(\mathbb{R}^3 \backslash \{0\})$ *and*

$$-\Delta h_D(x) = \begin{cases} 0 & \text{for} \quad 0 < |x| \leq 1 \\ g(x^*)\psi_D(x) & \text{for} \quad |x| > 1, \end{cases}$$

where $\psi_D(x) = c\left(\dfrac{1}{|x|R}\right)^2 \int_{\partial B^*} \dfrac{x^*-y}{|x^*-y|} \cdot \dfrac{y^*-x}{|y^*-x|} d\sigma(y)$ *and $B^* = B(x^*, R)$ is an arbitrary ball so that $B_0 \subset\subset B^*$.*

(iii) $h_N \in C^2(\mathbb{R}^3 \backslash \{0\})$ *and $-\Delta h_N(x) = 0$ for all $x \neq 0$.*

As a consequence, we have that $h \in C^2(\mathbb{R}^3 \backslash \{0\})$, $\partial_\nu h = 0$ on ∂B_0,

$$-\Delta h(x) = \begin{cases} g(x) & \text{for} \quad 0 < |x| \leq 1 \\ g(x^*)\psi_D(x) & \text{for} \quad |x| > 1, \end{cases}$$

and then $\|\Delta h\|_{L^\infty(\mathbb{R}^3)} \leq c\|g\|_{L^\infty}$. Moreover, $\|\nabla h\|_{L^2(B_0)} \leq c\|g\|_{L^2(B_0)}$.

We divide the proof into several steps. We fix a cut-off function $\varphi \in C_0^\infty(\mathbb{R}^3)$ satisfying

$$0 \leq \varphi \leq 1, \ \varphi \equiv 1 \text{ on } B_0 \text{ and } \varphi \equiv 0 \text{ on } B(0,2)^c,$$

and then, define for small $\varepsilon > 0$

$$\varphi_\varepsilon(x,y) = \varphi\left(\frac{x-y}{\varepsilon}\right).$$

Fix $i, j \in \{1,2,3\}$.

2.1 The proof of (i)

Let

$$u_\Gamma^\varepsilon(x) = \int_{B_0} \Gamma(x-y)\left(1-\varphi_\varepsilon(x,y)\right)g(y)dy,$$
$$v_\Gamma^\varepsilon(x) = \int_{B_0} \left(\partial_{x_i}\Gamma(x-y)\right)\left(1-\varphi_\varepsilon(x,y)\right)g(y)dy,$$
$$w_\Gamma^1(x) = \int_{B_0} \left(\partial_{x_i}\Gamma(x-y)\right)g(y)dy$$

and define $w_\Gamma^2(x)$ by

$$\int_B \left(\partial_{x_i,x_j}\Gamma(x-y)\right)(g(y)-g(x))dy + g(x)\int_{\partial B}\left(\partial_{x_i}\Gamma(x-y)\right)\nu_j(y)d\sigma(y),$$

where B is an arbitrary ball so that $B_0 \subset\subset B$. Remark that for $|x|>1$, this function equals

$$\int_{B_0}\left(\partial_{x_i,x_j}\Gamma(x-y)\right)g(y)dy.$$

Since

$$\sup_{x\in\mathbb{R}^3}|h_\Gamma(x)-u_\Gamma^\varepsilon(x)| \le c\varepsilon^2\|g\|_{L^\infty} \text{ and } \sup_{x\in\mathbb{R}^3}|w_\Gamma^1(x)-\partial_{x_i}u_\Gamma^\varepsilon(x)| \le c\varepsilon\|g\|_{L^\infty},$$

we see that $\partial_{x_i}h = w_\Gamma^1 \in C(\mathbb{R}^3)$.

2.1.1 Continuity of $\partial_{x_i,x_j}h_\Gamma$

It is not hard to see that $v_\Gamma^\varepsilon \in C^\infty(\mathbb{R}^3)$ and each integrals in the definition of w_Γ^2 absolutely converge. Observe that if $\varepsilon \le 1/2$, then for $x\in\bar{B}_0$, $\partial_{x_j}v_\Gamma^\varepsilon(x)$ equals

$$\int_B \partial_{x_j}\left\{\left(\partial_{x_i}\Gamma(x-y)\right)\left(1-\varphi\left(\frac{x-y}{\varepsilon}\right)\right)\right\}(g(y)-g(x))dy + g(x)\int_{\partial B}\left(\partial_{x_i}\Gamma(x-y)\right)\nu_j(y)d\sigma(y).$$

On the other hand, in the case $x\notin\bar{B}_0$, one can see

$$\partial_{x_j}v_\Gamma^\varepsilon(x) = \int_{B_0}\left(\partial_{x_i,x_j}\Gamma(x-y)\right)g(y)dy$$

for all $\varepsilon \le \operatorname{dist}\left(B_0^c, \operatorname{supp} g\right)/2$. From these expressions, one obtains

$$\sup_{x\in\mathbb{R}^3}|w_\Gamma^2(x)-\partial_{x_j}v_\Gamma^\varepsilon(x)| \le c\varepsilon\|\nabla g\|_{L^\infty}.$$

Because it also holds $\sup_{x\in\mathbb{R}^3}|\partial_{x_i}h_\Gamma(x)-v_\Gamma^\varepsilon(x)| \le c\varepsilon\|g\|_{L^\infty}$, we have $\partial_{x_i,x_j}h_\Gamma = w_\Gamma^2$ and $f_\Gamma \in C^2(\mathbb{R}^3)$.

2.2 The proof of (ii)

Denote for small $\varepsilon>0$

$$\varphi_\varepsilon^*(x,y) = \varphi\left(\frac{x^*-y}{\varepsilon}\right),$$

then it holds $|\nabla_x\varphi_\varepsilon^*(x,y)| \le c\varepsilon^{-1}|x|^{-2}$. Let for $x\ne 0$,

$$u_D^\varepsilon(x) = \int_{B_0} D(x,y)\left(1-\varphi_\varepsilon^*(x,y)\right)g(y)dy$$
$$v_D^\varepsilon(x) = \int_{B_0}\left(\partial_{x_i}D(x,y)\right)\left(1-\varphi_\varepsilon^*(x,y)\right)g(y)dy$$
$$w_D^1(x) = \int_{B_0}\left(\partial_{x_i}D(x,y)\right)g(y)dy,$$

and define $w_D^2(x)$ as

$$\int_{B^*} \left(\partial_{x_i,x_j} D(x,y)\right)(g(y)-g(x^*))\,dy + g(x^*)\int_{\partial B^*} \left(\partial_{x_i} D(x,y)\right)\nu_j(y)d\sigma(y).$$

Remark that for $0<|x|\le 1$,

$$w_D^2(x) = \int_{B_0} \partial_{x_i,x_j} D(x,y)g(y)dy.$$

Since it holds that

$$|h_D(x)-u_D^\varepsilon(x)| \le c\frac{\varepsilon^2}{|x|}\|g\|_{L^\infty} \text{ and } |w_D^1(x)-\partial_{x_i}u_D^\varepsilon(x)| \le c\varepsilon\left(\frac{1}{|x|^2}+\frac{1}{|x|^3}\right)\|g\|_{L^\infty},$$

we can see that $h_D \in C(\mathbb{R}^3\backslash\{0\})$ and $\partial_{x_i}h_D = w_D^1 \in C(\mathbb{R}^3\backslash\{0\})$.

2.2.1 Continuity of $\partial_{x_i,x_j}h_D$

From

$$|\partial_{x_i,x_j}D(x,y)| \le c\left(|x|^{-3}+|x|^{-5}\right)\left(|x^*-y|^{-1}+|x^*-y|^{-3}\right),$$

one can check the absolute convergences of the each integral of v_D^ε and w_D^2. $v_D^\varepsilon \in C^\infty(\mathbb{R}^3\backslash\{0\})$ and has the following expressions for small $\varepsilon>0$; in the case $x\in \bar{B}_0\backslash\{0\}$,

$$\partial_{x_j}v_D^\varepsilon(x) = \int_{B_0} \partial_{x_i,x_j}D(x,y)g(y)dy$$

for all $\varepsilon < d_g/2$ where $d_g := \inf_{x\in\bar{B}_0\backslash\{0\},y\in \mathrm{supp} g} |x^*-y| > 0$, and in the other case $x\notin \bar{B}_0$, $\partial_{x_j}v_D^\varepsilon(x)$ equals

$$\int_{B^*} \partial_{x_j}\left\{\left(\partial_{x_i}D(x,y)\right)(1-\varphi_\varepsilon^*(x,y))\right\}(g(y)-g(x^*))\,dy + g(x^*)\int_{\partial B^*}\left(\partial_{x_i}D(x,y)\right)\nu_j(y)d\sigma(y),$$

for all $\varepsilon < R/2$. Hence, we can get that for small $\varepsilon>0$,

$$|w_D^2(x)-\partial_{x_j}v_D^\varepsilon(x)| \le c\varepsilon\left(\frac{1}{|x|^2}+\frac{1}{|x|^5}\right)\|\nabla g\|_{L^\infty} \text{ for all } x\neq 0.$$

Since $|\partial_{x_i}h_D(x)-v_D^\varepsilon(x)| \le c\varepsilon|x|^{-2}\|g\|_{L^\infty}$, we see $\partial_{x_i,x_j}h_D = w_D^2$.

2.2.2 The equality for $-\Delta h_D$

For all $x\neq 0$, $\Delta_x D(x,y)=0$ a.e $y\in B_0$. Thus, $-\Delta h_D(x)=0$ when $0<|x|\le 1$. On the other hand, when $|x|>1$,

$$-\Delta h_D(x) = g(x^*)\int_{\partial B^*}\nabla_x D(x,y)\cdot\nu(y)d\sigma(y) = g(x^*)\psi_D(x)$$

and $|\psi_D(x)| \le c|x|^{-2}\le c$.

2.3 The proof of (iii)

For $x\neq 0$ and $\varepsilon>0$, define

$$E_x := \left\{y\in\mathbb{R}^3; 1+\frac{x}{|x|}\cdot\frac{x^*-y}{|x^*-y|}=0\right\} \text{ and } E_x^\varepsilon := \left\{y\in\mathbb{R}^3; \cos^{-1}\left(\frac{x}{|x|}\cdot\frac{x^*-y}{|x^*-y|}\right)^1 < \varepsilon\right\}.$$

Remark that $|E_x^\varepsilon| \le c\sin^2\varepsilon \approx \varepsilon^2$. Fix a cut-off function $\psi\in C_0^\infty(\mathbb{R})$ so that $0\le\psi\le 1$, $\psi\equiv 1$ on $(-1,1)$ and $\psi\equiv 0$ on $(-2,2)$. Then, let

$$\psi_\varepsilon^\dagger(x,y) = \psi\left(\frac{\pi-\cos^{-1}\left(\frac{x}{|x|}\cdot\frac{x^*-y}{|x^*-y|}\right)}{\varepsilon}\right).$$

[1] We regard the function $\cos^{-1}(\cdot)$ as a decreasing function from $(-1,1)$ to $(0,\pi)$.

Observe that

$$\psi_\varepsilon^\dagger(x,y) = \psi_\varepsilon^\dagger(y,x) \text{ and } |\nabla_x \psi_\varepsilon^\dagger(x,y)| \le c\frac{|y|}{\varepsilon|x||x^*-y|\sin\varepsilon}\|\psi'\|_{L^\infty} \approx \frac{|y|}{\varepsilon^2|x||x^*-y|}\|\psi'\|_{L^\infty}.$$

For $x \neq 0$, let

$$\begin{aligned}
u_N^\varepsilon(x) &= \int_{B_0} N(x,y)\,(1-\varphi_\varepsilon^*(x,y))\left(1-\psi_\varepsilon^\dagger(x,y)\right)g(y)dy,\\
v_N^\varepsilon(x) &= \int_{B_0} (\partial_{x_i}N(x,y))\,(1-\varphi_\varepsilon^*(x,y))\left(1-\psi_\varepsilon^\dagger(x,y)\right)g(y)dy,\\
w_N^1(x) &= \int_{B_0} (\partial_{x_i}N(x,y))\,g(y)dy \quad \text{and}\\
w_N^2(x) &= \int_{B_0} \left(\partial_{x_i,x_j}N(x,y)\right)g(y)dy.
\end{aligned}$$

The kernel N and its derivatives have an additional singularity on lines. Integrals around there are estimated by the following lemmas.

Lemma 2.1. *If $-\infty < \alpha < 2$ and $x = r\theta \neq 0$, then*

$$\int_{B_0}\left|1+\frac{x}{|x|}\cdot\frac{x^*-y}{|x^*-y|}\right|^{-\alpha} dy = \int_{B(x^*,1)}\left|1+\frac{x}{|x|}\cdot\frac{y}{|y|}\right|^{-\alpha} dy \le c\int_{\mathcal{S}^2}\left|1+\theta\cdot\tilde\theta\right|^{-\alpha} d\tilde\theta \le c.$$

Lemma 2.2. *For any $\theta \in \mathcal{S}^2$ and large integer j,*

$$\int_{\{\tilde\theta\in\mathcal{S}^2;0\le 1+\theta\cdot\tilde\theta\le 2^{-j}\}} d\tilde\theta \le c2^{-2j}.$$

Proof. Let $\angle(\tilde\theta)$ be an angle between $-\theta$ and $\tilde\theta$. When $1+\theta\cdot\tilde\theta_0 = 2^{-j}$,

$$\angle(\tilde\theta_0) = \cos^{-1}(1-2^{-j}) = \cos^{-1}((1-a_j)^{1/2}) \quad \text{where} \quad a_j = 2^{-j+1}-2^{-2j}.$$

Here, using $\sin\varepsilon \approx \varepsilon$ for $\varepsilon \in (0,\pi/2)$, we can find $\cos^{-1}((1-\varepsilon)^{1/2}) \approx \varepsilon$. Hence, one obtains $\angle(\tilde\theta_0) \approx a_j \approx 2^{-j}$, which implies the assertion. □

2.3.1 Continuity of h_N

For any $s_1 \in (0,2)$ and $s_2 \in (0,\infty)$, we decompose

$$\begin{aligned}
|h_N(x,y) - u_N^\varepsilon(x)| \le c&\Bigg[\int_{B_0\cap B(x^*,2\varepsilon)} \left(|n(x,y)|^{s_1} + |n(x,y)|^{-s_1}\right)dy\\
&+ \int_{B_0\cap E_x^{2\varepsilon}} \left(|n(x,y)|^{s_2} + |n(x,y)|^{-s_2}\right)dy\Bigg]\|g\|_{L^\infty} = c[I+II]\|g\|_{L^\infty}.
\end{aligned}$$

I and II are controlled by positive powers of ε;

$$I \le c\varepsilon^{3-s_1} \text{ and } II \le c\left(1+|x|^{-1}\right)^{3+s_2}\varepsilon^{4-2s_2}.$$

As a consequence, we can conclude $h_N \in C(\mathbb{R}^3\backslash\{0\})$ from the estimate; for $\tau \in (0,4)$

$$|h_N(x) - u_N^\varepsilon(x)| \le c(1+|x|^{-1})^4\varepsilon^\tau\|g\|_{L^\infty}.$$

2.3.2 Continuity of $\partial_{x_i} h_N$

An elementary equality: $2(1+\theta\cdot\tilde{\theta}) = \left|\theta+\tilde{\theta}\right|^2$ for $\theta,\ \tilde{\theta}\in\mathcal{S}^2$ yields

$$\left|\frac{y_i}{|y|}+\frac{y_i^*-x_i}{|y^*-x|}\right| \le \left(2\frac{n(x,y)}{|x^*-y|}\right)^{1/2} \tag{3}$$

for all $i\in\{1,2,3\}$. Therefore, one has

$$|\partial_{x_i}N(x,y)| = \left|-\frac{|y|}{|x|n(x,y)}\left(\frac{y_i}{|y|}+\frac{y_i^*-x_i}{|y^*-x|}\right)-\frac{x_i}{|x|^2}\right| \le c\frac{|y|}{|x||x^*-y|^{1/2}n(x,y)^{1/2}}+\frac{1}{|x|}.$$

For the purpose, we decompose with $s_1\in(1,\infty)$ and $s_2\in(0,2)$,

$$\begin{aligned}
|w_N^1(x)-\partial_{x_i}u_N^\varepsilon(x)| \le c\frac{\|g\|_{L^\infty}}{|x|}\Bigg[&\int_{B_0\cap B(x^*,2\varepsilon)}\left(\frac{1}{|x^*-y|^{1/2}n(x,y)^{1/2}}+1\right)dy\\
&+\int_{B_0\cap E_x^{2\varepsilon}}\left(\frac{1}{|x^*-y|^{1/2}n(x,y)^{1/2}}+1\right)dy\\
&+\frac{1}{\varepsilon|x|}\int_{B_0\cap\{\varepsilon\le|x^*-y|\le 2\varepsilon\}}\left(|n(x,y)|^{s_1}+|n(x,y)|^{-s_2}\right)dy\\
&+\frac{1}{\varepsilon^2}\int_{B_0\cap\left\{-\cos\varepsilon\le\frac{x}{|x|}\cdot\frac{x^*-y}{|x^*-y|}\le-\cos(2\varepsilon)\right\}}\left(|n(x,y)|^{s_1}+|n(x,y)|^{-s_2}\right)\frac{|y|}{|x^*-y|}dy\Bigg]\\
&=\frac{c}{|x|}\left[I+II+III+IV\right]\|g\|_{L^\infty}.
\end{aligned}$$

The four terms have bounds as follows: for any $\tau\in(0,2)$

$$I\le c\varepsilon^{3/2},\ II\le c\varepsilon^2,\ III\le c\varepsilon^\tau|x|^{-1} \text{ and } IV\le c\varepsilon^\tau,$$

which ensure that $\partial_{x_i}h_N = w_N^1$ and $h_N\in C^1(\mathbb{R}^3\backslash\{0\})$.

2.3.3 Continuity of $\partial_{x_i,x_j} h_N$

To see this, we need a bound of the second derivatives of N with respect to x. Observe that

- $$\begin{aligned}
\frac{\partial_{x_i,x_j}n(x,y)}{n(x,y)} &= \frac{|y|}{n(x,y)}\partial_{x_i,x_j}\left(\frac{n(y,x)}{|x|}\right)\\
&=\frac{|y|}{n(x,y)}\Bigg[\frac{1}{|x||y^*-x|}+\frac{x_i}{|x|^3}\left(\frac{y_j}{|y|}+\frac{y_j^*-x_j}{|y^*-x|}\right)\\
&\quad+\frac{x_j}{|x|^3}\left(\frac{y_i}{|y|}+\frac{y_i^*-x_i}{|y^*-x|}\right)-\frac{(y_i^*-x_i)(y_j^*-x_j)}{|x||y^*-x|^3}\Bigg]+3\frac{x_ix_j}{|x|^4}\quad\text{and}
\end{aligned}$$

- $$\begin{aligned}
\frac{\partial_{x_i}n(x,y)}{n(x,y)}\frac{\partial_{x_j}n(x,y)}{n(x,y)} &= \frac{1}{n(x,y)^2}\left[\frac{|y|^2}{|x|^2}\left(\frac{y_i}{|y|}+\frac{y_i^*-x_i}{|y^*-x|}\right)\left(\frac{y_j}{|y|}+\frac{y_j^*-x_j}{|y^*-x|}\right)\right]\\
&\quad+\frac{|y|}{|x|^3n(x,y)}\left[x_i\left(\frac{y_j}{|y|}+\frac{y_j^*-x_j}{|y^*-x|}\right)+x_j\left(\frac{y_i}{|y|}+\frac{y_i^*-x_i}{|y^*-x|}\right)\right]+\frac{x_ix_j}{|x|^4}.
\end{aligned}$$

From this and (3), we see that $\partial_{x_i,x_j}N(x,y)$ can be controlled without the violent term $|n(x,y)|^{-2}$:

$$|\partial_{x_i,x_j}N(x,y)| \le c\frac{|y|}{|x|^2|n(x,y)|}\left(1+\frac{|y|}{|x^*-y|}\right)+\frac{1}{|x|^2}.$$

Using this, we have

$$|w_N^2(x) - \partial_{x_j} v_N^\varepsilon(x)| \le c\frac{1}{|x|^2}\Bigg[\int_{B_0\cap B(x^*,2\varepsilon)} \frac{1}{n(x,y)}\left(1+\frac{1}{|x^*-y|}\right)dy$$

$$+\int_{B_0\cap E_x^{2\varepsilon}} \frac{1}{n(x,y)}\left(1+\frac{1}{|x^*-y|}\right)dy$$

$$+\frac{1}{\varepsilon|x|}\int_{B_0\cap\{\varepsilon\le|x^*-y|\le 2\varepsilon\}} \frac{1}{|x^*-y|^{1/2}n(x,y)^{1/2}}dy$$

$$+\frac{1}{\varepsilon^2}\int_{B_0\cap\left\{-\cos\varepsilon\le\frac{x}{|x|}\cdot\frac{x^*-y}{|x^*-y|}\le-\cos(2\varepsilon)\right\}} \frac{1}{|x^*-y|^{3/2}n(x,y)^{1/2}}dy\Bigg]\|g\|_{L^\infty}$$

$$= c\frac{1}{|x|^2}\left(I+II+III+IV\right)\|g\|_{L^\infty}.$$

Each term is estimated as follows:

$$\begin{cases} I \le c\int_0^{2\varepsilon} t(1+t^{-1})\int_{\mathcal{S}^2}\left|1+\theta\cdot\tilde\theta\right|^{-1}d\tilde\theta dt \le c\varepsilon \\ II \le c\int_0^{2} t(1+t^{-1})\int_{\{-1\le\theta\cdot\tilde\theta\le-\cos(2\varepsilon)\}}\left|1+\theta\cdot\tilde\theta\right|^{-1}d\tilde\theta dt \le c\varepsilon^2 \\ III \le c\varepsilon^{-1}|x|^{-1}\int_0^{2\varepsilon} t^2\int_{\mathcal{S}^2}\left|1+\theta\cdot\tilde\theta\right|^{-1/2}d\tilde\theta dt \le c\varepsilon^2|x|^{-1} \\ IV \le c\varepsilon^{-3}\int_0^{2}\int_{\{-\cos\varepsilon\le\theta\cdot\tilde\theta\le-\cos(2\varepsilon)\}}d\tilde\theta dt \le c\varepsilon. \end{cases}$$

As a consequence, we have that,

$$|w_N^2(x) - \partial_{x_j} v_N^\varepsilon(x)| \le c\frac{\varepsilon}{|x|^2}\left(1+\frac{1}{|x|}\right)\|g\|_{L^\infty}.$$

Since we have also $|\partial_{x_i} h_N(x) - v_N^\varepsilon(x)| \le c\varepsilon^{3/2}|x|^{-1}\|g\|_{L^\infty}$, we can conclude $\partial_{x_i,x_j} h_N = w_N^2$, thus $h_N \in C^2(\mathbb{R}^3\backslash\{0\})$.

2.3.4 The equality for $-\Delta h_N$

Observe that $\Delta_x N(x,y) = \Delta_x\{\log|y| - \log|x| + N(y,x)\} = \Delta_x N(y,x) - |x|^{-2}$. Since

$$\partial_{y_i}^2 N(x,y) = \frac{1}{|x^*-y|n(x,y)}\left(1-\frac{(x_i^*-y_i)^2}{|x^*-y|^2}\right) - \frac{1}{n(x,y)^2}\left(\frac{x_i}{|x|}+\frac{x_i^*-y_i}{|x^*-y|}\right)^2,$$

one has $\Delta_y N(x,y) = 0$, and then $\Delta_x N(x,y) = -|x|^{-2}$. Therefore,

$$-\Delta_x h_N(x) = -\int_{B_0}\Delta_x N(x,y)g(y)dy = \frac{1}{|x|^2}\int g(y)dy = 0.$$

2.4 The boundary condition

Next, we see that f enjoys the boundary condition; $\partial_\nu h(x) = 0$ for $x \in \partial B_0$. First, for any $x \in \mathcal{S}^2$

$$\partial_{\nu(x)}\left(\Gamma(x-y) - D(x,y)\right) = \frac{1}{|x-y|}.$$

On the other hand, we see that for the same x, $\partial_{\nu(x)} N(x,y) = -\frac{|y|}{n(x,y)}x\cdot\left(\frac{y}{|y|}+\frac{y^*-x}{|y^*-x|}\right) - 1$. Since

$$x\cdot\left(\frac{y}{|y|}+\frac{y^*-x}{|y^*-x|}\right) = -\frac{n(x,y)}{|y|} + \frac{n(x,y)}{|y||x-y|},$$

we obtain $\partial_{\nu(x)} N(x,y) = -\frac{1}{|x-y|}$, and then $\partial_\nu h(x) = 0$ for any $x \in \mathcal{S}^2$.

2.5 L^2-estimate

To complete the proof of Proposition 2.1, we check the L^2 estimate for ∇h. For simplicity, we give only a proof of $\|\nabla h_N\|_{L^2(B_0)} \le c\|g\|_{L^2(B_0)}$. From a pointwise estimate of ∇N in Section 2.3.2, it is sufficient to show

$$\int_{B_0} |y||x^* - y|^{-1} \left(1 + \frac{x}{|x|} \cdot \frac{x^* - y}{|x^* - y|}\right)^{-1/2} |g(y)|dy \le c\|g\|_{L^2(B_0)}.$$

Applying Cauchy-Schwarz inequality and changing variables, we see that the left hand side is controlled by

$$\|g\|_{L^2(B_0)} \left(\int_{x^* - B_0} |y|^{-2} \left(1 + \frac{x}{|x|} \cdot \frac{y}{|y|}\right)^{-1} dy \right)^{1/2}.$$

Because $x^* - B_0 \subset \left\{y; \frac{1}{|x|} - 1 \le |y| \le \frac{1}{|x|} + 1\right\}$, this integral is uniformly bounded for $x \in B_0$. Therefore, the desired L^2 estimate is verified, and then the proof of Proposition 2.1 is completed.

3 Vector-valued inequality

The proof of main result uses a version of vector-valued inequalities for Hardy-Littlewood maximal operator. The following is a generalization of the result of Fefferman and Stein [7]. The argument in this section can be applied to other dimensional cases.

Proposition 3.1. *For $1 < r, p < \infty$ and $\alpha = 3(1 - 1/p)$,*

$$\left\| \left(\sum_{l=1}^{\infty} (M f_l)^r \right)^{1/r} \right\|_{\dot{K}^{\alpha}_{p,\infty}} \le c \left\| \left(\sum_{l=1}^{\infty} |f_l|^r \right)^{1/r} \right\|_{\dot{K}^{\alpha}_{p,1}}.$$

Because Herz spaces have the property

$$\|f^r\|_{\dot{K}^{\alpha}_{p,q}} = \|f\|^r_{\dot{K}^{\alpha/r}_{pr,qr}},$$

Proposition 3.1 can be rewritten as follows

Corollary 3.1. *For $0 < r < 1$, $r < p < \infty$ and $\alpha = 3(1/r - 1/p)$,*

$$\left\| \sum_{l=1}^{\infty} M_r f_l \right\|_{\dot{K}^{\alpha}_{p,\infty}} \le c \left\| \sum_{l=1}^{\infty} |f_l| \right\|_{\dot{K}^{\alpha}_{p,r}}.$$

This inequality with $r = 3/4$ is applied in the proof of Theorem 1.1. Note that $\alpha_{3/4} = 0$.

We give a proof of Proposition 3.1.

Proof.

$$\begin{aligned}
\text{L.H.S.} &= \sup_{k \in \mathbb{Z}} 2^{k\alpha} \left\| \left(\sum_{l=1}^{\infty} M f_l^r \right)^{1/r} \right\|_{L^p(A_k)} \le \sup_{k \in \mathbb{Z}} 2^{k\alpha} \left\| \sum_{j \in \mathbb{Z}} \left(\sum_{l=1}^{\infty} M(f_l \chi_j)^r \right)^{1/r} \right\|_{L^p(A_k)} \\
&\le \sup_{k \in \mathbb{Z}} 2^{k\alpha} \sum_{j=-\infty}^{k-2} \left\| \left(\sum_{l=1}^{\infty} M(f_l \chi_j)^r \right)^{1/r} \right\|_{L^p(A_k)} + \sup_{k \in \mathbb{Z}} 2^{k\alpha} \sum_{j=k-1}^{k+1} \left\| \left(\sum_{l=1}^{\infty} M(f_l \chi_j)^r \right)^{1/r} \right\|_{L^p(A_k)} \\
&\quad + \sup_{k \in \mathbb{Z}} 2^{k\alpha} \sum_{j=k+2}^{\infty} \left\| \left(\sum_{l=1}^{\infty} M(f_l \chi_j)^r \right)^{1/r} \right\|_{L^p(A_k)} =: \text{I} + \text{II} + \text{III}.
\end{aligned}$$

From [7], we can see that $\text{II} \le c \left\| \left(\sum_{l=1}^{\infty} |f_l|^r \right)^{1/r} \right\|_{\dot{K}^{\alpha}_{p,\infty}}$.

Since if $x \in A_k$ and $j \le k-2$,

$$\left(\sum_{l=1}^{\infty} M(f_l\chi_j)(x)^r\right)^{1/r} \le c2^{-3k}2^{3j(1-1/p)} \left\| \left(\sum_{l=1}^{\infty} |f_l|^r\right)^{1/r} \right\|_{L^p(A_j)},$$

we can see that $\mathrm{I} \le c \left\| \left(\sum_{l=1}^{\infty} |f_l|^r\right)^{1/r} \right\|_{\dot{K}^{\alpha}_{p,1}}$. On the other hand, if $x \in A_k$ and $k+2 \le j$, it holds that

$$\left(\sum_{l=1}^{\infty} M(f_l\chi_j)(x)^r\right)^{1/r} \le c2^{-3j/p} \left\| \left(\sum_{l=1}^{\infty} |f_l|^r\right)^{1/r} \right\|_{L^p(A_j)},$$

which implies that $\mathrm{III} \le c \left\| \left(\sum_{l=1}^{\infty} |f_l|^r\right)^{1/r} \right\|_{\dot{K}^{\alpha}_{p,\infty}}$ and the proof is completed. □

4 Proof of Main theorem

Proof. Because

$$\|(u\cdot\nabla)v\|_{H\dot{K}^{\alpha_p}_{p,\infty}} = \sum_{k=1}^{3} \left\| \sum_{j=1}^{3} u_j \partial_j v_k \right\|_{H\dot{K}^{\alpha_p}_{p,\infty}} = \sum_{k=1}^{3} \left\| M_\phi \left(\sum_{j=1}^{3} u_j \partial_j v_k \right) \right\|_{\dot{K}^{\alpha_p}_{p,\infty}},$$

it is enough to show the inequality

$$\left\| M_\phi \left(\sum_{j=1}^{3} u_j \partial_j v \right) \right\|_{\dot{K}^{\alpha_p}_{p,\infty}} \le c\|u\|_{L^\infty} \|\nabla v\|_{H\dot{K}^{\alpha_p}_{p,3/4}},$$

for all divergence free vector fields u and functions $v \in Y \cap W^{1,r}_{loc}(\mathbb{R}^3)$. Firstly, we give a definition of $\sum_{j=1}^{n} u_j \partial_j v$ as a tempered distribution as follows; for $\varphi \in \mathcal{S}(\mathbb{R}^3)$

$$\left\langle \sum_{j=1}^{3} u_j \partial_j v, \varphi \right\rangle := -\sum_{j=1}^{3} \int u_j(y) v(y) \partial_j \varphi(y) dy.$$

Our assumption ensures that the integral in the right hand side absolutely converges. Then, it follows

$$\sum_{j=1}^{3} u_j \partial_j v * \phi_t(x) = -C_\phi \|u\|_{L^\infty} \int v(y) \left[\sum_{j=1}^{3} \tilde{u}_j(y) \partial_{y_j} \phi_t(x-y) \right] dy,$$

where C_ϕ is a large constant depending on ϕ, and $\tilde{u}_j(y) = \dfrac{u_j(y)}{C_\phi \|u\|_{L^\infty}}$. Owing to the divergence free condition on u, we see that for every $x \in \mathbb{R}^3$

$$\sum_{j=1}^{3} \tilde{u}_j(y) \partial_{y_j} \phi_t(x-y) = \sum_{j=1}^{3} \partial_{y_j} \left(\tilde{u}_j(y) \phi_t(x-y) \right) \quad \text{in} \quad \mathcal{S}'(\mathbb{R}^3_y).$$

Hence, we obtain the pointwise estimate

$$M_\phi \left(\sum_{j=1}^{3} u_j \partial_j v \right)(x) \le C_\phi \|u\|_{L^\infty} N_m v(x),$$

for all $m \in [1,\infty]$. In particular, we use this estimate with $m = \infty$ and get

$$\left\| M_\phi \left(\sum_{j=1}^{3} u_j \partial_j v \right) \right\|_{\dot{K}^{\alpha_p}_{p,\infty}} \le c\|u\|_{L^\infty} \|N_\infty v\|_{\dot{K}^{\alpha_p}_{p,\infty}}.$$

It is enough to prove that

$$\|N_\infty v\|_{\dot{K}^{\alpha_p}_{p,\infty}} \le c\|\nabla v\|_{H\dot{K}^{\alpha_p}_{p,3/4}}. \tag{4}$$

We derive this inequality from a pointwise estimate. To prove this, we fix $\Psi \in F_\infty(x)$. Since the support of g_Ψ is a compact subset in $B_\Psi = B(x, r_\Psi)$, there exist a small $\varepsilon_0 > 0$ and a smooth positive function η so that for all $\varepsilon \in (0, \varepsilon_0)$,

$$\begin{cases} \operatorname{supp}(g_\Psi * \phi_\varepsilon) \cup \operatorname{supp}\Psi \subset\subset \operatorname{supp}\eta \subset\subset B_\Psi \\ \eta \equiv 1 \text{ on } \operatorname{supp}(g_\Psi * \phi_\varepsilon) \cup \operatorname{supp}\Psi \\ \|\eta\|_{L^q} = c_q |B_\Psi|^{1/q} \text{ for all } q \in [1, \infty]. \end{cases}$$

Define $\alpha_\varepsilon := \|\eta\|_{L^1}^{-1} \int_{B_\Psi} g_\Psi * \phi_\varepsilon(y)dy$. Remark that $\alpha_\varepsilon \to 0$ as $\varepsilon \to 0$. In fact, for a test function $\rho \in C_0^\infty(2B_\Psi)$ with $\rho \equiv 1$ on B_Ψ, we have

$$\|\eta\|_{L^1}\alpha_\varepsilon = -\langle \Psi, \nabla\rho * \phi_\varepsilon \rangle \to -\langle \Psi, \nabla\rho \rangle = 0.$$

For simplicity, let $g_\Psi^\varepsilon := g_\Psi * \phi_\varepsilon - \alpha_\varepsilon \eta$. Since $g_0^\varepsilon(y) := g_\Psi^\varepsilon(x - r_\Psi y) \in C_0^\infty(B_0)$ and $\int_{B_0} g_0^\varepsilon dy = 0$, from Section 2, we see that for $\varepsilon < \varepsilon_0$,

$$h_0^\varepsilon(y) := \int_{B_0} G(y,z) g_0^\varepsilon(z)dz$$

is a function in $C^2(\mathbb{R}^3 \backslash \{0\})$ and solves the Neumann problem; $-\Delta h_0^\varepsilon = g_0^\varepsilon$ in $B_0 \backslash \{0\}$ with $\partial_\nu h_0^\varepsilon = 0$ on ∂B_0. Therefore,

$$h_\Psi^\varepsilon(y) := r_\Psi^2 h_0^\varepsilon \left(\frac{x-y}{r_\Psi} \right)$$

is in $C^2(\mathbb{R}^3 \backslash \{x\})$ and enjoys the Neumann problem:

$$\begin{cases} -\Delta h_\Psi^\varepsilon & = g_\Psi^\varepsilon \quad \text{in} \quad B_\Psi \backslash \{x\}, \\ \dfrac{\partial h_\Psi^\varepsilon}{\partial \nu} & = 0 \quad \text{on } \partial B_\Psi. \end{cases}$$

Further, h_Ψ^ε fulfills the following estimates: for all j,

$$\|\partial_j h_\Psi^\varepsilon\|_{L^2(B_\Psi)} \le c|B_\Psi|^{-1/2} \quad \text{and} \quad \|\Delta h_\Psi^\varepsilon\|_{L^\infty(\mathbb{R}^3)} \le c|B_\Psi|^{-4/3}. \tag{5}$$

The former follows from the L^2 estimate in Proposition 2.1. Now we can see that

$$\int v g_\Psi dy = -\lim_{\varepsilon\to 0} \int \nabla v \cdot \nabla h_\Psi^\varepsilon dy.$$

Indeed, from Theorem 7.25 in [9], we can find a sequence $\{v_m\}_{m\in\mathbb{N}} \subset C^\infty(\overline{B_\Psi})$ so that $v_m \to v$ in $W^{1,r}(B_\Psi)$ as $m \to \infty$. From divergence theorem,

$$\int v g_\Psi dy = \lim_{m\to\infty} \lim_{\varepsilon\to 0} \int v_m g_\Psi^\varepsilon dy = \lim_{\varepsilon \to 0} \int \nabla v \cdot \nabla h_\Psi^\varepsilon dy.$$

Thus we obtain

$$\left| \int v g_\Psi dy \right| \le \limsup_{0<\varepsilon<\varepsilon_0} \sum_{k=1}^3 \left| \int \partial_k v \partial_k h_\Psi^\varepsilon dy \right|.$$

Since $\partial_k v \in H\dot{K}_{p,3/4}^{\alpha_p}$, following Miyachi [12], it can be decomposed as

$$\partial_k v = \sum_{j=1}^\infty a_j^{(k)}$$

where supp $a_j^{(k)} \subset B_j = B(x_j, r_j)$, $a_j^{(k)} \in L^\infty(\mathbb{R}^3)$ and $\int x^\alpha a_j^{(k)}(x)dx = 0$ for α with $|\alpha| \le 1$, also

$$\left(\sum_{j=1}^\infty \|a_j^{(k)}\|_{L^\infty}^s \chi_{B_j}(x) \right)^{1/s} \le c_s (\partial_k v)_2^*(x) \quad \text{for all } s \in (0,\infty).$$

Therefore, we have

$$\left| \int v g_\Psi dy \right| \le \limsup_{0<\varepsilon<\varepsilon_0} \sum_{k=1}^3 \sum_{j=1}^\infty \left| \int a_j^{(k)} \partial_k h_\Psi^\varepsilon dy \right|.$$

From (5), we immediately see that

$$\left| \int a_j^{(k)} \partial_k h_\Psi^\varepsilon dy \right| \le \|a_j^{(k)}\|_{L^\infty} |B_j \cap B_\Psi|^{1/2} \|\partial_k h_\Psi^\varepsilon\|_{L^2(B_\Psi)} \le c\|a_j^{(k)}\|_{L^\infty}.$$

When $x \notin 4B_j$, if $Cr_\Psi < |x - x_j|$ with $C > 8/3$, then it holds $B_j \cap B_\Psi = \varnothing$ and $\int a_j^{(k)} \partial_k h_\Psi^\varepsilon dy = 0$. On the other hand, if $Cr_\Psi \geq |x - x_j|$, then we can derive the decay estimate

$$\limsup_{0<\varepsilon<\varepsilon_0} \left| \int a_j^{(k)} \partial_k h_\Psi^\varepsilon dy \right| \leq c \|a_j^{(k)}\|_{L^\infty} \left(\frac{r_j}{|x - x_j|} \right)^4. \tag{6}$$

We may assume $x \neq x_j$. Using the moment condition on $a_j^{(k)}$ twice, one has

$$\begin{aligned}
\int a_j^{(k)}(y) \partial_k h_\Psi^\varepsilon(y) dy &= \int a_j^{(k)}(y) \left(\partial_k h_\Psi^\varepsilon(y) - \partial_k h_\Psi^\varepsilon(x_j) \right) dy \\
&= \sum_{s=1}^{3} \int_0^1 \int a_j^{(k)}(y)(y - x_j)_s (\partial_s \partial_k h_\Psi^\varepsilon)(\theta y + (1-\theta)x_j) dy d\theta \\
&= \sum_{s=1}^{3} \int_0^1 \int a_j^{(k)}(y)(y - x_j)_s \left[(\partial_s \partial_k h_\Psi^\varepsilon)(\theta y + (1-\theta)x_j) - \langle \partial_s \partial_k h_\Psi^\varepsilon \rangle_{B(x_j, \theta r_j)} \right] dy d\theta.
\end{aligned}$$

From this, the decay estimate (6) is derived as follows;

$$\begin{aligned}
\left| \int a_j^{(k)}(y) \partial_k h_\Psi^\varepsilon(y) dy \right| &\leq c r_j \|a_j^{(k)}\|_{L^\infty} \sum_{s=1}^{3} \int_0^1 \theta^{-3} \int_{B(x_j, \theta r_j)} \left| \partial_s \partial_k h_\Psi^\varepsilon(y) - \langle \partial_s \partial_k h_\Psi^\varepsilon \rangle_{B(x_j, \theta r_j)} \right| dy d\theta \\
&\leq c r_j^4 \|a_j^{(k)}\|_{L^\infty} \sum_{s=1}^{3} \|\partial_s \partial_k h_\Psi^\varepsilon\|_{BMO(\mathbb{R}^3)} \\
&\leq c r_j^4 \|a_j^{(k)}\|_{L^\infty} \|\Delta h_\Psi^\varepsilon\|_{L^\infty(\mathbb{R}^3)} \\
&\leq c \left(\frac{r_j}{r_\Psi} \right)^4 \|a_j^{(k)}\|_{L^\infty} \\
&\leq c \left(\frac{r_j}{|x - x_j|} \right)^4 \|a_j^{(k)}\|_{L^\infty}.
\end{aligned}$$

Here, we have used the boundedness of $R_j R_k$ from $L^\infty(\mathbb{R}^3)$ to $BMO(\mathbb{R}^3)$ in the third inequality, where R_j is the jth Riesz transform, and (5) in the fourth inequality.

As mentioned in [12], because $\left(\frac{1}{1 + |x - x_j|/r_j} \right)^4 \approx M_{3/4}(\chi_{B_j})(x)$, as a consequence it follows that for all $x \in \mathbb{R}^3$,

$$N_\infty v(x) \leq c \sum_{k=1}^{3} \sum_{j=1}^{\infty} \|a_j^{(k)}\|_{L^\infty} M_{3/4}(\chi_{B_j})(x). \tag{7}$$

Now, we apply Corollary 3.1 with $r = 3/4$ and obtain

$$\|N_\infty v\|_{\dot{K}_{p,\infty}^{\alpha_p}} \leq c \sum_{k=1}^{3} \left\| \sum_{j=1}^{\infty} \|a_j^{(k)}\|_{L^\infty} \chi_{B_j} \right\|_{\dot{K}_{p,3/4}^{\alpha_p}} \leq c \sum_{k=1}^{3} \|(\partial_k v)_2^*\|_{\dot{K}_{p,3/4}^{\alpha_p}} \approx \|\nabla v\|_{H\dot{K}_{p,3/4}^{\alpha_p}}.$$

Here we have used $3(1 - 1/p) + 3(4/3 - 1) = 3(1 - 1/p) + 1 = \alpha_p$. The proof is completed. □

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