Fourier transform of a function appears in the Bochner-Riesz means

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This is a note for my study. The facts in here are already known. If you find some mistakes, I am very grateful if you inform me of them.

1 The purpose and notations

Let $\delta > 0$, and $m_\delta(t) := |t|^\delta \chi_{(-\infty,0)}(t) = (-t)_+^\delta$, $(t \in \mathbb{R})$. Here $a_+ := \max(1, 0)$. This function appears in the Bochner-Riesz measn, with $\delta > 0$ and $R > 0$,

$$\begin{aligned} S_R^\delta[f](x) &:= \int_{\mathbb{R}^n} e^{ix\xi} \left(1 - |\xi/R|^2\right)_+^\delta \hat{f}(\xi) d\xi \\ &= R^{-2\delta} \int e^{ix\xi} \left(R^2 - |\xi|^2\right)_+^\delta \hat{f}(\xi) d\xi \\ &= R^{-2\delta} \int e^{ix\xi} m_\delta \left(|\xi|^2 - R^2\right) \hat{f}(\xi) d\xi. \end{aligned}$$

The aim of this note is show that

$$\hat{m_\delta}(s) = c_\delta (s + i0)^{-(1+\delta)} \quad \text{in } \mathcal{S}'(\mathbb{R}), \tag{1}$$

where $c_\delta := i^{1+\delta}\Gamma(\delta + 1)$, and for $\varphi \in \mathcal{S}(\mathbb{R})$

$$\langle (s + i0)^{-(1+\delta)}, \varphi \rangle := \lim_{\varepsilon \downarrow 0} \int_{\mathbb{R}} \frac{\varphi(t)}{(t + i\varepsilon)^{1+\delta}} dt. \tag{2}$$

For the existence of this limit, see Section 3 below.

In our calculus, then the Gamma function: $\Gamma(\delta) := \displaystyle\int_0^\infty e^{-t} t^{\delta-1} dt$ for $\delta > 0$ is involved. Fourier transform and its inverse that we use in this note are following:

$$\hat{f}(\xi) := \int_{\mathbb{R}^n} e^{-ix\xi} f(x) dx, \quad \check{f}(\xi) := (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\xi} f(x) dx.$$

! After finished writing this note, I found similar calculus in [1, page 170, 49] and [4, page 356].

Using (1), we can obtain a representation of the Bochner-Riesz means with Schrödinger evolution operator $e^{-it\Delta}$:

$$\begin{aligned} S_R^\delta[f](x) &= c_\delta R^{-2\delta} \int_{\mathbb{R}^n} e^{ix\xi} \int_{\mathbb{R}} e^{i(|\xi|^2 - R^2)s} \hat{m_\delta}(s) ds \hat{f}(\xi) d\xi \\ &= (2\pi)^n c_\delta R^{-2\delta} \int_{\mathbb{R}} e^{-isR^2} \frac{1}{(s + i0)^{1+\delta}} e^{-is\Delta} f(x) ds. \end{aligned}$$

On the other hand, the same argument give us a representation of the Riesz means T_R^δ with wave evolution operator $e^{it\sqrt{-\Delta}}$:

$$\begin{aligned} T_R^\delta[f](x) &:= \int_{\mathbb{R}^n} e^{ix\xi} \left(1 - |\xi/R|\right)_+^\delta \hat{f}(\xi) d\xi \\ &= (2\pi)^n c_\delta R^{-\delta} \int_{\mathbb{R}} e^{-isR} \frac{1}{(s + i0)^{1+\delta}} e^{is\sqrt{-\Delta}} f(x) ds. \end{aligned}$$

Thses formulation were used in [2] for showing that in 2D case, local smoothing conjecture for wave equations implies the Bochner-Riesz means conjecture.

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If we consider the function $M_\delta(\xi) := \left(1 - |\xi|^2\right)_+^\delta$ on $\mathbb{R}^n$ instead of m_δ on $\mathbb{R}$ above, we already know that

$$\hat{M}_\delta(x) = C_\delta \frac{J_{n/2+\delta}(|x|)}{|x|^{n/2+\delta}},$$

where $C_\delta := 2^{n/2-1}(2\pi)^{n/2}\Gamma(\delta+1)$ and J_m is the Bessel function: for $m > -1/2$ and $t \in \mathbb{R}$

$$J_m(t) := \frac{(t/2)^m}{\Gamma(m+1/2)\Gamma(1/2)} \int_{-1}^{1} e^{its}(1-s^2)^{m-1/2} ds.$$

It is well-known that the Bessel function has the following estimate:

$$|J_m(t)| \lesssim \min(|t|^m, |t|^{-1/2})$$

and the asymptotic behavior:

$$J_m(t) = ct^{-1/2}\cos(t - \theta_m) + R_m(t) = t^{-1/2}\left(c_1 e^{it} + c_2 e^{-it}\right) + R_m(t),$$

where $\theta_m := \pi(m+1/2)/2$ and $|R_m(t)| \lesssim \min(|t|^m, |t|^{-3/2})$. Therefore, we have

$$\hat{M}_\delta(x) = c\left(\frac{c_1 e^{i|x|} + c_2 e^{-i|x|}}{|x|^{(n+1)/2+\delta}} + c\frac{R_{n/2+\delta}(|x|)}{|x|^{n/2+\delta}}.\right)$$

2 The proof of (1)

For $\varphi \in \mathcal{S}(\mathbb{R})$, Lebesgue convergence theorem ensures that

$$\begin{aligned}
\langle \hat{m_\delta}, \varphi \rangle &= \int_{-\infty}^{0} |t|^\delta \hat{\varphi}(t) dt \\
&= \int_0^\infty t^\delta \int_{\mathbb{R}^n} e^{its}\varphi(s) ds dt \\
&= \lim_{\varepsilon \downarrow 0} \int_0^\infty t^\delta \int_{\mathbb{R}} e^{it(s+i\varepsilon)}\varphi(s) ds dt \\
&= \lim_{\varepsilon \downarrow 0} \int_{\mathbb{R}} \varphi(s) \int_0^\infty e^{it(s+i\varepsilon)} t^\delta dt ds.
\end{aligned}$$

The desired equality is proved if we show that for $s \neq 0$ and $\varepsilon > 0$

$$\int_0^\infty e^{it(s+i\varepsilon)} t^\delta dt = i^{1+\delta} \frac{\Gamma(\delta+1)}{(s+i\varepsilon)^{1+\delta}}. \tag{3}$$

2.1 The case $s > 0$

In this case, we divide the proof into two steps: the first step is to show

$$\int_0^\infty e^{it(s+i\varepsilon)} t^\delta dt = i^{1+\delta} \int_0^\infty e^{-t(s+i\varepsilon)} t^\delta dt,$$

and in the second one, we prove that the integral in the right hand side above coincides with

$$\frac{\Gamma(\delta+1)}{(s+i\varepsilon)^{\delta+1}}.$$

Fix $R \in (0,\infty)$ and denote that

$$\begin{cases} \gamma_1(t) := t, \ (0 < t < R) \\ \gamma_2(t) := Re^{i\theta}, \ (0 < \theta < \pi/2) \\ \gamma_3(t) := it, \ (R < t < 0), \end{cases}$$

and $\gamma := \gamma_1 + \gamma_2 + \gamma_3$. From Cauchy's integral theorem, we have

$$\begin{aligned}
0 &= \int_\gamma e^{i(s+i\varepsilon)z} z^\delta dz \\
&= \int_0^R e^{it(s+i\varepsilon)} t^\delta dt \\
&+ iR^{1+\delta} \int_0^{\pi/2} \left[e^{isR\cos\theta} e^{-i\varepsilon R \sin\theta} e^{i(1+\delta)\theta} \right] \times \left[e^{-sR\sin\theta} e^{-\varepsilon R\cos\theta} \right] d\theta \\
&+ i^{1+\delta} \int_R^0 e^{-t(s+i\varepsilon)} t^\delta dt.
\end{aligned}$$

Therefore, it holds that

$$\int_0^\infty e^{it(s+i\varepsilon)} t^\delta dt = \lim_{R\to\infty} \int_0^R e^{it(s+i\varepsilon)} t^\delta dt = i^{1+\delta} \int_0^\infty e^{-t(s+i\varepsilon)} t^\delta dt.$$

Next, we see that the right hand side is independent of $\varepsilon > 0$. More precisely, we show that for all $\varepsilon > 0$

$$(s+i\varepsilon)^{1+\delta} \int_0^\infty e^{-t(s+i\varepsilon)} t^\delta dt = s^{1+\delta} \int_0^\infty e^{-ts} t^\delta dt = \Gamma(\delta+1).$$

To do so, fix $R > 0$ and define

$$\begin{cases} \tilde\gamma_1(t) := ts, \ (0 < t < R) \\ \tilde\gamma_2(t) := Rs + it, \ (0 < t < R\varepsilon) \\ \tilde\gamma_3(t) := t(s+i\varepsilon), \ (R < t < 0), \end{cases}$$

and then $\tilde\gamma := \tilde\gamma_1 + \tilde\gamma_2 + \tilde\gamma_3$. From Cauchy's integral theorem again, one has

$$\begin{aligned}
0 &= \int_{\tilde\gamma} e^{-z} z^\delta dz \\
&= \int_0^R e^{-ts} (ts)^\delta s dt \\
&+ \int_0^{R\varepsilon} e^{-sR} e^{-it} \left[Rs + it \right]^\delta i dt \\
&+ \int_R^0 e^{-t(s+i\varepsilon)} (t(s+i\varepsilon))^\delta (s+i\varepsilon) dt.
\end{aligned}$$

Therefore,

$$\int_0^R e^{-t(s+i\varepsilon)} (t(s+i\varepsilon))^\delta (s+i\varepsilon) dt \to \int_0^\infty e^{-ts} (ts)^\delta s dt = \int_0^\infty e^{-t} t^\delta dt = \Gamma(\delta+1),$$

thus,

$$\int_0^\infty e^{-t(s+i\varepsilon)} t^\delta dt = \frac{\Gamma(\delta+1)}{(s+i\varepsilon)^\delta}.$$

As a result, we obtain that for $s > 0$ and $\varepsilon > 0$

$$\int_0^\infty e^{it(s+i\varepsilon)} t^\delta dt = i^{1+\delta} \int_0^\infty e^{-t(s+i\varepsilon)} t^\delta dt = i^{1+\delta} \frac{\Gamma(\delta+1)}{(s+i\varepsilon)^{1+\delta}} = \frac{c_\delta}{(s+i\varepsilon)^{(1+\delta)}}.$$

2.2 The case $s < 0$

With the same argument as above, we show that for $s < 0$ and $\varepsilon > 0$

$$\int_0^\infty e^{it(s+i\varepsilon)} t^\delta dt = \frac{i^{1+\delta}\Gamma(\delta+1)}{(s+i\varepsilon)^{1+\delta}}.$$

Fix $R > 0$ and define

$$\begin{cases} \eta_1(t) := t, \ (R < t < 0) \\ \eta_2(t) := Re^{i\theta}, \ (-\pi/2 < \theta < 0) \\ \eta_3(t) := -it, \ (0 < t < R). \end{cases}$$

From Cauchy's integral theorem with $\eta := \eta_1 + \eta_2 + \eta_3$, we see

$$\begin{aligned}
0 &= \int_\eta e^{iz(s+i\varepsilon)} z^\delta dz \\
&= \int_R^0 e^{it(s+i\varepsilon)} t^\delta dt \\
&+ iR^{1+\delta} \int_{-\pi/2}^0 \left[e^{isR\cos\theta} e^{-i\varepsilon R \sin\theta} e^{i(1+\delta)\theta} \right] \times \left[e^{-sR\sin\theta} e^{-\varepsilon R\cos\theta} \right] d\theta \\
&+ (-i)^{1+\delta} \int_0^R e^{t(s+i\varepsilon)} t^{1+\delta} dt,
\end{aligned}$$

then it follows

$$\int_0^\infty e^{it(s+i\varepsilon)} t^\delta dt = \lim_{R\to\infty} \int_0^R e^{it(s+i\varepsilon)} t^\delta dt = (-i)^{1+\delta} \int_0^\infty e^{t(s+i\varepsilon)} t^\delta dt.$$

To see this right hand side coincides with $(-1)^{1+\delta} \dfrac{\Gamma(\delta+1)}{(s+i\varepsilon)^{1+\delta}}$, we consider a path integral along

$$\begin{cases} \tilde{\eta}_1(t) := t(s+i\varepsilon), \ (0 < t < R) \\ \tilde{\eta}_2(t) := (1-t)R(s+i\varepsilon) + tRs, \ (0 < t < 1) \\ \tilde{\eta}_3(t) := ts, \ (R < t < 0). \end{cases}$$

From Cauchy's integral theorem with $\tilde{\eta} := \tilde{\eta}_1 + \tilde{\eta}_2 + \tilde{\eta}_3$,

$$\begin{aligned}
0 &= \int_{\tilde{\eta}} e^z z^\delta dz \\
&= (s+i\varepsilon)^{1+\delta} \int_0^R e^{t(s+i\varepsilon)} t^\delta dt \\
&+ (-i\varepsilon R) \int_0^1 e^{sR} e^{i\varepsilon R(1-t)} \left(Rs + i\varepsilon R(1-t) \right)^\delta dt \\
&+ s^{1+\delta} \int_R^0 e^{ts} t^\delta dt.
\end{aligned}$$

Hence, we have that for all $\varepsilon > 0$

$$(s+i\varepsilon)^{1+\delta} \int_0^\infty e^{t(s+i\varepsilon)} t^\delta dt = s^{1+\delta} \int_0^\infty e^{ts} t^\delta dt = \left(\frac{s}{|s|} \right)^{1+\delta} \Gamma(\delta+1) = (-1)^{1+\delta} \Gamma(\delta+1),$$

thus for $s < 0$ and $\varepsilon > 0$

$$\int_0^\infty e^{t(s+i\varepsilon)} t^\delta dt = \frac{(-1)^{1+\delta}\Gamma(\delta+1)}{(s+i\varepsilon)^{1+\delta}}.$$

As a consequence, we have with the same s and ε

$$\int_0^\infty e^{it(s+i\varepsilon)} t^\delta dt = i^{1+\delta} \frac{\Gamma(\delta+1)}{(s+i\varepsilon)^{1+\delta}}.$$

The proof of (1) is completed.

3 Existence of the limit in (2)

In this section, we show that for $\varphi \in \mathcal{S}$ and $\delta > 0$, there exists the limit

$$\lim_{\varepsilon \downarrow 0} \int_{\mathbb{R}} \frac{\varphi(s)}{(s+i\varepsilon)^{1+\delta}} ds.$$

To see this, let us define for $m \in \mathbb{N} \cup \{0\}$

$$R_m(s) := \sum_{j=0}^m \frac{\partial^j \varphi(0)}{j!} s^j.$$

Using this, we decompose the integral into three parts: $\int_{\mathbb{R}} \frac{\varphi(s)}{(s+i\varepsilon)^{1+\delta}} ds = I + II + III$, where

$$\begin{cases} I = \int_{|s|\leq 1} \frac{\varphi(s) - R_{[\delta]}(s)}{(s+i\varepsilon)^{1+\delta}} ds \\ II = \int_{|s|\leq 1} \frac{R_{[\delta]}(s)}{(s+i\varepsilon)^{1+\delta}} ds \\ III = \int_{|s|>1} \frac{\varphi(s)}{(s+i\varepsilon)^{1+\delta}} ds \end{cases}$$

It is easy to see that

$$\lim_{\varepsilon \downarrow 0} I = \int_{|s|\leq 1} \frac{\varphi(s) - R_{[\delta]}(s)}{s^{1+\delta}} ds \quad \text{and} \quad \lim_{\varepsilon \downarrow 0} III = \int_{|s|>1} \frac{\varphi(s)}{s^{1+\delta}} ds.$$

For the second part, we observe that $\lim_{\varepsilon \downarrow 0} \int_{|s|\leq 1} \frac{ds}{(s+i\varepsilon)^{1+\delta}} ds = \delta^{-1} \left(e^{-i\pi\delta} - 1 \right)$. By using the similar calculus and integration by parts, we can verify the existence of the limit $\lim_{\varepsilon \downarrow 0} II$. For instance, in the case $\delta < 1$, it follows that

$$\lim_{\varepsilon \downarrow 0} \int_{\mathbb{R}} \frac{\varphi(s)}{(s+i\varepsilon)^{1+\delta}} ds = \int_{|s|\leq 1} \frac{\varphi(s) - \varphi(0)}{s^{1+\delta}} ds + \int_{|s|>1} \frac{\varphi(s)}{s^{1+\delta}} ds + \frac{\varphi(0)}{\delta} \left(e^{-i\pi\delta} - 1 \right).$$

References

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[4] E.M. Stein, *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals*, Princeton Mathematical Series **43**, Princeton University Press, 1993.