An application of weighted Hardy spaces to the Navier-Stokes equations

Yohei Tsutsui *

Abstract

In this article, we consider the mapping properties of convolution operators with smooth functions on weighted Hardy spaces $H^p(w)$ with w belonging to Muckenhoupt class A_∞. As a corollary, one obtains decay estimates of heat semigroup on weighted Hardy spaces.

After a weighted version of the div-curl lemma is established, these estimates on weighted Hardy spaces are applied to the investigation of the decay property of global mild solutions to Navier-Stokes equations with the initial data belonging to weighted Hardy spaces.

Keywords Weighted Hardy spaces, convolution operators, div-curl lemma, Navier-Stokes equations
2010 Mathematics Subject Classification 42B30, 35Q30

1 Introduction

Aims of this article are to establish estimates for the heat semigroup and div-curl estimates on weighted Hardy spaces and to investigate time decay of solutions to Navier-Stokes equations with the initial data belonging to weighted Hardy spaces. Weights, we treat in this paper, belong to Muckenhoupt class A_∞.

The first aim is to find a sufficient condition on weights that ensures the boundedness of convolution operators with smooth functions on weighted Hardy spaces $H^p(w)$, see Definition 1.2 below, of the form;

$$\|f * \varphi\|_{H^q(\sigma)} \leq c\|f\|_{H^p(w)}. \tag{1}$$

The same inequalities on Lebesgue spaces with power weights were treated in [20] where the author assumed $w \in A_p$ and $\sigma \in A_q$. Meanwhile, Theorem 1.1 below does not need such assumption on weights, also see Lemma 1.1. As a corollary of Theorem 1.1, $H^p(|\cdot|^{\alpha p}) - H^q(|\cdot|^{\beta q})$ estimates for the heat semigroup $e^{t\Delta}$ are given. Their decay order of t can be large as possible, see Corollary 1.1. This is one of advantages of the usage of weighted Hardy spaces instead of weighted Lebesgue spaces. In the proof of Theorem 1.1, atomic decompositions by García-Cuerva [5] and Strömberg and Torchinsky [18], and the molecular characterization in Taibleson and Weiss [19] and Lee and Lin [11] are applied.

Next aim is to establish the so-called "div-curl lemma" on weighted Hardy spaces. Div-curl lemma was proved by Coifman, Lions, Meyer and Semmes [3]: for divergence free vector fields u

$$\|(u \cdot \nabla)v\|_{H^r} \lesssim \|u\|_{H^p}\|\nabla v\|_{H^q}$$

where $n/(n+1) < p, q < \infty$ and $1/r = 1/p + 1/q < 1 + 1/n$. At the case $p = \infty$, Auscher, Russ and Tchamitchian [1] verified that the inequalities still hold. The proof of our div-curl lemma relies on a pointwise estimate of the grand maximal function of the bilinear form $(u \cdot \nabla)v$ due to Miyachi [15] in the non-endpoint cases $p < \infty$ and the approach of Auscher, Russ and Tchamitchian [1] in the endpoint case $p = \infty$, see Theorem 1.2 and Theorem 1.3.

Finally we investigate the time decay property of global solutions in Kato [10] to the incompressible homogeneous Navier-Stokes equations

$$\text{(N-S)} \quad \begin{cases} \partial_t u - \Delta u + (u \cdot \nabla)u + \nabla p = 0, \\ \text{div } u = 0, \\ u(0) = a \end{cases}$$

when the initial data a belongs to weighted Hardy spaces. Here $u = (u_1, \cdots, u_n)$ is the unknown velocity vector field, p is the unknown pressure scalar field and $a = (a_1, \cdots, a_n)$ is the given initial velocity with div $a = \nabla \cdot a = 0$. In this research, Theorem 1.1 is applied to the linear estimate and Theorems 1.2 and 1.3 are

*Department of Mathematics, Faculty of Science and Engineering, Waseda University, Shinjuku, Tokyo 169-8555, Japan.
e-mail: `y.tsutsui@kurenai.waseda.jp`

applied to control the nonlinear term $(u\cdot\nabla)u$. Kato [10] and Giga and Miyakawa [7] showed if $\|a\|_{L^n}$ is small and $a\in L^p$ for some $p\in(1,2)$, then the mild solution u has that $\|u(t)\|_{L^2}\lesssim t^{-\gamma}$ with $\gamma=n(1/p-1/2)/2$. Wiegner [21] proved that for $\theta\ge 0$, if $a\in L^2$ and $\|e^{t\Delta}a\|_{L^2}\lesssim t^{-\theta}$, (i.e. $a\in\dot{B}^{-2\theta}_{2,\infty}$), then the weak solution u has that $\|u(t)\|_{L^2}\lesssim t^{-\gamma_W}$ with $\gamma_W=\min(\theta,(n+2)/4)$. Observe that $\gamma<(n+2)/4 \iff n/(n+1)<p$. We make γ be close to the critical order $(n+2)/4$ of Wiegner with the aid of weighted Hardy spaces. But our analysis can not reach to the critical order $(n+2)/4$, see 2 of Remark 1.6.

The real variable theory of Hardy spaces was initiated by Fefferman and Stein [4], and then its weighted version by García-Cuerva [5]. The fundamental properties of weighted Hardy spaces: density, duality, boundedness of Fourier multipliers, etc..., were studied by Strömberg and Torchinsky [18]. Two atomic decompositions with different notations of atom were given in [5] and [18], and will be applied to the proof of Theorem 1.1. Lee and Lin [11] gave a weighted version of the molecular characterization due to Taibleson and Weiss [19]. Our sufficient conditions for the boundedness of convolution operators are similar to that for the fractional integral operators in Gatto, Gutiérrez and Wheeden [6].

For the application, we need a weighted version of so-called "div-curl lemma" due to Coifman, Lions, Meyer and Semmes [3]. Our "div-curl lemma" in the non-endpoint case Theorem 1.2 follows from the pointwise estimates of bilinear forms with the cancellation property by Miyachi [15]. Since the proof uses the boundedness of the Riesz transforms, this method does not work on the endpoint case. To get the div-curl lemma in the case Theorem 1.3, we apply an approach by Auscher, Russ and Tchamitchian [1] which does not need such boundedness.

There are papers which studied the Navier-Stokes equations with Hardy spaces, for example Miyakawa [16] and [17]. Applying the theory of Hardy spaces seems to be natural, because the nonlinear term $(u\cdot\nabla)u$ has the cancellation property: $\int(u\cdot\nabla)u dx=0$ and then belongs to H^1 from "div-curl lemma" under the suitable assumption on the velocity u.

To state our results, we begin with definitions of Muckenhoupt class and weighted Hardy spaces.

We say w is a "weight" if w is a non-negative and locally integrable function. For a subset $E\subset\mathbb{R}^n$, χ_E means the characteristic function of E and $|E|$ the volume of E. Throughout this article we use the following notations;

$$w(E)=\int_E w dx,\ \langle f\rangle_E=\frac{1}{|E|}\int_E f dx,\ \langle f\rangle_{E;w}=\frac{1}{w(E)}\int_E f w dx.$$

By a " cube " Q we mean a cube in $\mathbb{R}^n$ with sides parallel to the coordinate axes. $B(x,r)$ means a open ball centered at x with radius r. We fix a smooth function Φ satisfying supp $\Phi\subset B(0,1)$, $0\le\Phi\le 1$, $\Phi\equiv 1$ on $B(0,1/2)$. We also use the notation $M_{r;w}f(x)=\sup_{Q\ni x}\langle|f|^r\rangle^{1/r}_{Q;w}$, where the supremum is taken over all cubes Q containing x, and M denotes the Hardy-Littlewood maximal operator. $A\lesssim B$ and $A\approx B$ mean $A\le c_0B$ and $c_1B\le Ac_2B$ with positive constants c_0,c_1 and c_2. In what follows, c denotes a constant that is independent of the functions involved, which may differ from line to line.

Definition 1.1. *A weight w is said to be in the Muckenhoupt class A_p, $(1\le p\le\infty)$, if the A_p constant $[w]_{A_p}$ is finite:*

$$[w]_{A_1}:=\sup_Q\langle w\rangle_Q\|w^{-1}\|_{L^\infty(Q)},$$

$$[w]_{A_p}:=\sup_Q\langle w\rangle_Q\langle w^{1-p'}\rangle_Q^{p-1},\quad(1<p<\infty),$$

and

$$[w]_{A_\infty}:=\sup_Q\langle w\rangle_Q\exp(\langle\log w^{-1}\rangle_Q),$$

where the suprema are taken over all cubes Q. Also, we define $q_w:=\inf\{q\in[1,\infty);w\in A_q\}$.

It is well-known that $w(x)=|x|^\alpha\in A_p$ if and only if $-n<\alpha\le 0$ when $p=1$ and $-n<\alpha<n(p-1)$ when $p>1$.

Remark 1.1.

1. *$[w]_{A_p}\ge 1$ and $1\le p<q\le\infty\Rightarrow A_p\subsetneq A_q$.*
2. *A_p classes have the openness property: if $p\in(1,\infty]$ and $w\in A_p$, there exists $q\in(1,p)$ so that $w\in A_q$.*
3. *$w\notin A_{q_w}$*, if $q_w>1$.

It is well known that all A_∞ weights satisfy the reverse Hölder inequality. In a recent study of the sharp weighted inequalities for Calderón-Zygmund operators, the optimal orders of the reverse Hölder inequality were found by Lerner, Ombrosi and Pérez [13] for A_p weights with $p<\infty$ and Hytönen and Pérez [9] for A_∞ weights.

Proposition A ([9])**.** *Every $w \in A_\infty$ satisfy the "reverse Hölder inequality";*

$$\langle w^{r_w} \rangle_Q^{1/r_w} \le 2\langle w \rangle_Q,$$

with $r_w := 1 + \dfrac{1}{2^{n+11}\|w\|_{A_\infty}}$, where $\|w\|_{A_\infty}$ is another A_∞ constant of w, see [9] for example.

The weighted Hardy spaces $H^p(w)$ with $w \in A_\infty$ are defined as follows.

Definition 1.2. *Let $0 < p \le \infty$ and $w \in A_\infty$. Define $H^p(w)$ as a space of all tempered distributions f whose the maximal function $M_\Phi f(x) = \sup_{t>0} |f * \Phi_t(x)|$ belongs to $L^p(w)$, and*

$$\|f\|_{H^p(w)} := \|M_\Phi f\|_{L^p(w)},$$

where $L^\infty(w)$ denotes L^∞.

Remark 1.2. *It is well-known that when $1 < p < \infty$ and $w \in A_p$, it holds $H^p(w) = L^p(w)$. On the other hand, if $L^p(w) = H^p(w)$, then w has to belong to A_p. This fact also is true for open subsets in $\mathbb{R}^n$, see [14]. Furthermore, if $1 \le p < q$, then there is a $w \in A_q$ so that the Dirac mass belongs to $H^p(w)$, see pp.86 in [18].*

In [18], several characterizations of $H^p(w)$ by maximal functions, for example the grand maximal function f^*_m, were established. This maximal function is defined as follows; for $m \in \mathbb{N} \cup \{0\}$, $x \in \mathbb{R}^n$ and $t \in (0, \infty)$, $\mathcal{I}_m(x,t)$ denotes a space of all function $\psi \in C^\infty(B(x,t))$ with

$$\|\partial^\alpha \psi\|_{L^\infty} \le t^{-(n+|\alpha|)} \quad \text{for} \quad |\alpha| \le m.$$

The grand maximal function f^*_m is then defined by

$$f^*_m(x) = \sup \left\{ |f(\psi)|\, ; \psi \in \bigcup_{t \in (0,\infty)} \mathcal{I}_m(x,t) \right\}.$$

We denote by $\hat{\mathcal{D}}_0$ by the set of all $f \in \mathcal{S}$ with $\hat{f}$ belonging to $\mathcal{D}$ and vanishing in a neighbourhood of $\xi = 0$, where $\hat{f}$ means the Fourier transform of f. Strömberg and Torchinsky [18] proved that $\hat{\mathcal{D}}_0$ is a dense subspace of $H^p(w)$ for $p \in (0, \infty)$ and doubling measures w.

Our first result of the paper reads as follows.

Theorem 1.1. *Let $0 < p \le q < \infty$ and $w, \sigma \in A_\infty$. If there exists $K > 0$ such that*

$$[w,\sigma]_{X^K_{p,q}} = \sup_B \min\left(1, |B|^K\right) \frac{\sigma(B)^{1/q}}{w(B)^{1/p}} < \infty,$$

where the supremum is taken over all balls B, then for any $\varphi \in \mathcal{S}$ we have

$$\|f * \varphi\|_{H^q(\sigma)} \le c[w,\sigma]_{X^K_{p,q}} \|f\|_{H^p(w)}$$

where the constant c depends on $p, q, n, \varphi, [w]_{A_\infty}$ and $[\sigma]_{A_\infty}$.

Remark 1.3. *1. Gatto, Gutiérrez and Wheeden [6] showed that for $0 < p < q < \infty$, $0 < m \in \mathbb{N}$ and doubling measures w and σ, $\|I_m f\|_{H^q(\sigma)} \lesssim \|f\|_{H^p(w)}$ with $f \in \mathcal{S}_m$ if and only if*

$$\sup_Q |Q|^{m/n} \frac{\sigma(Q)^{1/q}}{w(Q)^{1/p}} < \infty,$$

where $I_m f(x) = \mathcal{F}^{-1}[|\cdot|^{-m}\hat{f}](x)$ and $\mathcal{S}_m = \{\varphi \in \mathcal{S}; \partial^\alpha \hat{\varphi}(0) = 0,\ |\alpha| \le m\}$.

2. The constant c can be written by $c = \tilde{c}|\varphi|_{\mathcal{S}}$ where the new constant $\tilde{c}$ is independent of φ and $|\cdot|_{\mathcal{S}}$ denotes a semi-norm of $\mathcal{S}$.

Especially, it is not hard to check the finiteness of $[w,\sigma]_{X^K_{p,q}}$ for power weights w and σ. For example , see pp. 285-286 in [8].

Lemma 1.1. *Let $0 < p \le q < \infty$ and $-n/q < \beta \le \alpha < \infty$. For $w(x) = |x|^{\alpha p}$ and $\sigma(x) = |x|^{\beta q}$, we can find $K > 0$ such that $[w, \sigma]_{X^K_{p,q}} < \infty$.*

Combining Theorem 1.1 and Lemma 1.1 with the homogeneity of $H^p(w)$ with power weight w, we get decay estimates of heat semigroup on weighted Hardy spaces with such weights.

Corollary 1.1. *Let $0 < p \le q < \infty$, $-n/q < \beta \le \alpha < \infty$, $w(x) = |x|^{\alpha p}$ and $\sigma(x) = |x|^{\beta q}$. Then, it holds*

$$\left\| e^{t\Delta} f \right\|_{H^q(\sigma)} \lesssim t^{-\gamma} \|f\|_{H^p(w)}$$

with $\gamma = n(1/p - 1/q)/2 + (\alpha - \beta)/2$. Also, it follows that for $p < \infty$ and $0 \le \alpha < \infty$

$$\|e^{t\Delta} f\|_{L^\infty} \lesssim t^{-\gamma} \|f\|_{H^p(w)},$$

with $\gamma = n/(2p) + \alpha/2$.

Remark 1.4.

1. *Because it follows that*

$$\|g\|_{L^q(\sigma)} \le c\|g\|_{H^q(\sigma)} \tag{2}$$

holds for $q \in (0, \infty)$, a doubling measure σ and $g \in H^p(w) \cap L^1_{loc}$, the inequalities in Theorem 1.1 and Corollary 1.1 hold for $L^q(\sigma)$ replaced by $H^q(\sigma)$ in the left hand side. The inequality (2) follows from atomic decomposition for weighted Hardy spaces due to Strömberg and Torchinsky [18], see Theorem C below.

2. *For $0 < p \le q < \infty$, $w(x) = |x|^{\alpha p}$, $\sigma(x) = |x|^{\beta q}$ and $0 \le \varphi \in \mathscr{S}$, if*

$$\|f * \varphi\|_{L^q(\sigma)} \le c\|f\|_{L^p(w)}$$

holds, then exponents have to fulfill

$$-n/q < \beta \le \alpha \le n(1 - 1/p),$$

which should be compared with the condition on exponents in Lemma 1.1.

3. *In [20], the author proved the same inequality with $L^p(|\cdot|^{\alpha p})$ replaced by $H^p(|\cdot|^{\alpha p})$. In order to show that, we needed the restriction on exponents*

$$-n/q < \beta \le \alpha < n(1 - 1/p),$$

which implies $w(x) = |x|^{\alpha p} \in A_p$ and $\sigma(x) = |x|^{\beta q} \in A_q$.

4. *The second inequality in Corollary 1.1 is verified by $H^p - L^\infty$ estimate for the heat semigroup from Miyakawa [16] and the first one.*

Our second result is a generalization of div-curl lemma in [3] which plays an important role in our application. Except for the case $p = \infty$, our weighted div-curl lemma reads as follows.

Theorem 1.2. *Let $n/(n+1) < p, q < \infty$ and $1/r = 1/p + 1/q < 1 + 1/n$. Suppose that there exist $\tau \in (1, p(1 + 1/n))$ and $\rho \in (1, q(1 + 1/n))$ such that $\tau/p + \rho/q < 1 + 1/n$, $w \in A_\tau$ and $\sigma \in A_\rho$. Then, we have*

$$\|(u \cdot \nabla)v\|_{H^r(\mu)} \le c\|u\|_{H^p(w)} \|\nabla v\|_{H^q(\sigma)},$$

with div $u = 0$ and $\mu = w^{r/p} \sigma^{r/q}$.

Remark 1.5. *The weight μ belongs to $A_{r(\tau/p+\rho/q)}$, $1 < r(\tau/p + \rho/q) < r(1 + 1/n)$, and it holds*

$$[\mu]_{A_{r(\tau/p+\rho/q)}} \le [w]^{r/p}_{A_\tau} [\sigma]^{r/q}_{A_\rho}.$$

Because we use Theorem 1.2 with $w(x) = |x|^{\alpha p}$ and $\sigma(x) = 1$ in Theorem 1.4 below, it is convenience to rewrite Theorem 1.2 as follows.

Theorem 1.2′. *Let $n/(n+1) < p, q < \infty$ and $-n/p < \alpha < n(1 - 1/p) + 1$. If $1/r = 1/p + 1/q < \min(1 + 1/n, 1 + 1/n - \alpha/n)$, then one has*

$$\|(u \cdot \nabla)v\|_{H^r(\mu)} \le c\|u\|_{H^p(w)} \|\nabla v\|_{H^q},$$

where $w(x) = |x|^{\alpha p}$ and $\mu(x) = |x|^{\alpha r}$.

For the endpoint case $p = \infty$, we can get rid of the assumption on the weight σ in the non-endpoint case above. To give $(u \cdot \nabla)v$ a definition as a tempered distribution, we define Y by a space of all locally integral functions f satisfying that there exist $c_f > 0$ and a seminorm $|\cdot|_{\mathcal{S}}$ of $\mathcal{S}$ so that $\int |f(x)\varphi(x)|dx \le c_f |\varphi|_{\mathcal{S}}$, for all $\varphi \in \mathcal{S}$. Obviously, $L^p(w) \subset Y$ when $1 \le p \le \infty$ and $w \in A_p$.

Theorem 1.3. *Let $n/(n+1) < q < \infty$ and $\sigma \in A_\infty$. Then, it holds*

$$\|(u \cdot \nabla)v\|_{H^q(\sigma)} \lesssim \|u\|_{L^\infty} \|\nabla v\|_{H^q(\sigma)},$$

for $u \in L^\infty$ with div $u = 0$ and $v \in Y$ with each $\partial_j v_k \in L^1_{loc}$, provided that $\sigma \in A_{q(1+1/n)}$ in the case $0 < q \le 1$.

By using Theorems 1.1, 1.2 and 1.3, we consider the time decay property of solutions to the Navier-Stokes equations. In particular, we treat with time-global solutions with the small initial data $a \in L^n$ due to Kato [10]. Solving the Cauchy problem (N-S) can be reduced to finding a divergence free solution u of the integral equation

$$\text{(I.E.)} \quad u(t) = e^{t\Delta}u_0 - B(u,u)(t)$$

where $e^{t\Delta}$ is the heat semigroup,

$$B(u,v)(t) = \int_0^t e^{(t-s)\Delta}\mathbb{P}(u \cdot \nabla)v(s)ds$$

and $\mathbb{P} = \{\delta_{i,j} + R_iR_j\}_{1 \le i,j \le n}$ denotes the Leray-Hopf operator or the Weyl-Helmholtz projection which is the orthogonal projection on solenoidal vector field. Of course, the operator $e^{t\Delta}$ is defined by the convolution

$$e^{t\Delta}f(x) := f * G_{\sqrt{t}}(x),$$

where G is the Gaussian $G(x) := \frac{1}{(4\pi)^{n/2}} e^{-|x|^2/4}$ and $G_t(x) := t^{-n}G(x/t)$.

Our third result in this paper reads as follows.

Theorem 1.4. *Let $1 \le p < \infty$, $-n/p < \alpha < n(1-1/p)+1$ and $w(x) = |x|^{\alpha p} \in A_{p(1+1/n)}$. Then, there exists $\delta > 0$ such that for any $a \in L^n \cap H^p(w)$ with $\|a\|_{L^n} + \|a\|_{H^p(w)} \le \delta$ and div $a = 0$, we can construct a solution $u \in L^\infty(0,\infty; L^n \cap H^p(w)) \cap C([0,\infty); L^n \cap H^p(w)) \cap C^\infty((0,\infty) \times \mathbb{R}^n)$ of (I.E.) satisfying*

$$\lim_{t \searrow 0} \|u(t) - a\|_{L^n} = \lim_{t \searrow 0} \|u(t) - a\|_{H^p(w)} = 0$$
$$\sup_{t>0} t^{1/2} \|\nabla u(t)\|_{H^p(w)} < \infty.$$

Moreover, for $q \in [p,\infty)$ and $\beta \in (-n/q, n(1-1/q)+1)$ with $\beta \le \alpha$, the solution u satisfies the following decay property;

$$\|u(t)\|_{H^q(\sigma)} \lesssim t^{-n(1/p-1/q)/2-(\alpha-\beta)/2}\delta, \tag{3}$$

with $\sigma(x) = |x|^{\beta q} \in A_{q(1+1/n)}$. In particular, in the case $p < q$ or $\beta < \alpha$, it holds that

$$\|u(t)\|_{H^q(\sigma)} = o(t^{-\gamma}) \quad \text{as } t \searrow 0, \tag{4}$$

where $\gamma = n(1/p-1/q)/2 + (\alpha-\beta)/2$.

Remark 1.6. *1. The decay order " $n(1/p-1/q)/2 + (\alpha-\beta)/2$ " in (3) is dominated by $(n+1)/2$.*

2. In particular, if $1 \le p \le 2$ and $0 \le \alpha < n(1-1/p)+1$, it holds that

$$\|e^{t\Delta}a\|_{L^2} = o(t^{-\gamma}) \quad \text{and} \quad \|u(t)\|_{L^2} = O(t^{-\gamma}) \quad \text{as } t \nearrow \infty, \tag{5}$$

where $\gamma = n(1/p-1/2)/2 + \alpha/2$. Observe that $\gamma < (n+2)/4$. As we mentioned above, $(n+2)/4$ is a critical order of Wiegner [21]. The following equivalence should be remarked;

$$\alpha = n\left(1 - \frac{1}{p}\right) + 1 \iff \gamma = \frac{n+2}{4}.$$

The more α is close to our critical value $n(1-1/p)+1$, the more γ is close to Wiegner's critical one $(n+4)/2$. The restriction on α from above stems from div-curl estimates (Theorem 1.2), that is, the influence of the non-linear term $(u \cdot \nabla)u$.

3. *Owing to the density of $\hat{\mathcal{D}}_0$ in $H^p(w)$ with $0 < p < \infty$, see [18], the former statement in (5) still holds with other exponents. More precisely, for $0 < p \le q < \infty$, $-n/q < \beta \le \alpha < \infty$ and $b \in H^p(w)$, it follows*

$$\|e^{t\Delta} b\|_{H^q(\sigma)} = o(t^{-\gamma}) \quad \text{as } t \nearrow \infty$$

with $\gamma = n(1/p - 1/q)/2 + (\alpha - \beta)/2$.

This article is organized as follows. In next section, we prepare several estimates for atoms and maximal functions of them that are used in the proof of Theorem 1.1. In Section 3, Theorem 1.1 is proved by such estimates, atomic decomposition and molecular characterization of weighted Hardy spaces. The proof is divided into three parts, the cases $0 < p \le q \le 1$, $0 < p \le 1 < q < \infty$ and $1 < p \le q < \infty$. The first and second cases rely on the molecular characterization. The third one uses the atomic decomposition due to [18] and the duality argument. In Section 4, we show the weighted version of "div-curl lemma" by using the pointwise estimates for some bilinear forms due to Miyachi [15] and an argument in [1]. Finally, in Section 5, we apply results in previous sections to get the time decay estimates of solutions to Navier-Stokes equations with the small initial data $a \in L^n \cap H^p(w)$.

2 Basic estimates for atoms

In this section, we prepare several estimates for an atom $a \in L^\infty$ satisfying

$$\operatorname{supp} a \subset B_0 = B(x_0, r_0) \text{ and } \int x^\alpha a(x) dx = 0, \ (|\alpha| \le N),$$

with some $N \in \mathbb{N} \cup \{0\}$.

We begin with two estimates for this atom in terms of the size of B_0.

Lemma 2.1. *Let $\varphi \in \mathcal{S}$.*

(i)

$$\|a * \varphi\|_{L^\infty} \lesssim \min\left(1, |B_0|^{1+(N+1)/n}\right) \|a\|_{L^\infty}. \tag{6}$$

(ii) *For $x \notin 2B_0$ and $M \ge 0$,*

$$|a * \varphi(x)| \lesssim \min\left(1, |B_0|^{1+(N+1)/n}, |B_0||x - x_0|^{-M}\right) \|a\|_{L^\infty}. \tag{7}$$

Proof. (i): It is easy to see that $\|a * \varphi\|_{L^\infty} \lesssim \|a\|_{L^\infty}$. Using the moment condition, we also have

$$\begin{aligned} |a * \varphi(x)| &= |\int_{B_0} a(y)\Big(\varphi(x-y) - \sum_{|\gamma| \le N} \frac{(-1)^\gamma}{\gamma!} \partial^{|\gamma|}\phi(x - x_0)(y - x_0)^\gamma\Big) dy| \\ &\lesssim \|a\|_{L^\infty} |B_0|^{1+(N+1)/n} \|\nabla^{N+1}\varphi\|_{L^\infty} \\ &\lesssim |B_0|^{1+(N+1)/n} \|a\|_{L^\infty}. \end{aligned}$$

(ii): Because $x \notin 2B_0$, the third bound in (7) follows;

$$\begin{aligned} |a * \varphi(x)| &\le c\|a\|_{L^\infty} \int_{B_0} \frac{1}{|x-y|^M} dy \\ &\le c\|a\|_{L^\infty} |B_0||x - x_0|^{-M}. \end{aligned}$$

□

Next, we consider estimates for the maximal function $M_\Phi(a * \varphi)$ with the previous estimates Lemma 2.1.

Lemma 2.2. (i)

$$\|M_\Phi(a * \varphi)\|_{L^\infty} \lesssim \min\left(1, |B_0|^{1+(N+1)/n}\right). \tag{8}$$

(ii) *For $x \notin 4B_0$,*

$$M_\Phi(a * \varphi)(x) \lesssim \max\left(1, |B_0|^{-(N+1)/n}\right) \left(\frac{r_0}{|x - x_0|}\right)^{n+N+1}. \tag{9}$$

Proof. (8) immediately follows from (6). To verify (9), fix $x \notin 4B_0$.

Since $B(x,t) \cap 2B_0 = \emptyset$ for $t \leq \frac{|x-x_0|}{2}$, we have from (7)

$$\begin{aligned}
|a * \varphi * \Phi_t(x)| &= \left| \int_{B(x,t)} (a * \varphi(y)) \Phi_t(x-y) dy \right| \\
&\lesssim \int_{B(x,t)} |B_0||y-x_0|^{-(n+N+1)} |\Phi_t(x-y)| dy \\
&\lesssim |B_0||x-x_0|^{-(n+N+1)} \\
&\lesssim |B_0|^{-(N+1)/n} \left(\frac{r_0}{|x-x_0|} \right)^{n+N+1}.
\end{aligned}$$

Thus, it suffices to show that for $t > \frac{|x-x_0|}{2}$,

$$|a * \varphi * \Phi_t(x)| \lesssim \max\left(1, |B_0|^{-(N+2)/n}\right) \left(\frac{r_0}{|x-x_0|} \right)^{n+N+1}.$$

For the sake of simplicity, let $b(x) = a * \varphi(x)$ and $\Psi(y) = \Phi\left(\frac{x}{t} - y\right)$. The moment condition yields the following bound.

$$\begin{aligned}
|a * \varphi * \Phi_t(x)| &= \left(b_{1/t} * \Phi\right)_t (x) \\
&= t^{-n} \int b_{1/t}(y) \Psi(y) dy \\
&\lesssim t^{-n} \int |b_{1/t}(y)| \left| y - \frac{x_0}{t} \right|^{N+1} dy \\
&= t^{-(n+N+1)} \int |b(y)||y-x_0|^{N+1} dy.
\end{aligned}$$

On one hand, it holds that

$$\begin{aligned}
\int_{2B_0} |b(y)||y-x_0|^{N+1} dy &\lesssim \int_{2B_0} |y-x_0|^{N+1} dy \\
&\lesssim r_0^{n+N+1}.
\end{aligned}$$

On the other hand, we have that

$$\begin{aligned}
\int_{(2B_0)^c} |b(y)||y-x_0|^{N+1} dy &\lesssim \int_{(2B_0)^c} |B_0||y-x_0|^{-(n+N+2)} |y-x_0|^{N+1} dy \\
&\leq |B_0| \int_{|y-x_0| \geq 2r_0} |y-x_0|^{-(n+1)} dy \\
&\lesssim |B_0| r_0^{-1} \\
&\lesssim r_0^{n-1}.
\end{aligned}$$

Therefore, the desired estimate is obtained;

$$\begin{aligned}
|a * \varphi * \Phi_t(x)| &\lesssim |x-x_0|^{-(n+N+1)} \left(r_0^{n+N+1} + r_0^{n-1}\right) \\
&= \left(1 + r_0^{-(N+2)}\right) \left(\frac{r_0}{|x-x_0|} \right)^{n+N+1} \\
&\lesssim \max\left(1, |B_0|^{-(N+2)/n}\right) \left(\frac{r_0}{|x-x_0|} \right)^{n+N+1}.
\end{aligned}$$

□

3 Proof of Theorem 1.1

We divide the proof of Theorem 1.1 into three parts. For definiteness, we specify the class of w and σ; $w \in A_\tau$ and $\sigma \in A_\rho$ with $1 < \tau, \rho < \infty$.

3.1 The case $0 < p \le q \le 1$

To prove Theorem 1.1 in this case, we use the atomic decomposition in [5] and the theory of molecular characterization in [11] for weighted Hardy spaces.

Definition 3.1 ([5]). *Let $0 < p < \infty$ with $w \in A_\infty$ and $s \ge \left[n\left(\frac{q_w}{p}-1\right)\right]$. We say a function a is a (p, q, s) atom w.r.t. w if a satisfies the following conditions;*
(i) supp $a \subset Q$
(ii) $\|a\|_{L^\infty(w)} \le w(Q)^{-1/p}$
(iii) $\int x^\alpha a(x)dx = 0, \ (|\alpha| \le s)$.

Atomic decomposition for weighted Hardy spaces with atoms above was established by García-Cuerva [5].

Theorem A ([5]). *Let $0 < p \le 1$ and $w \in A_\infty$. For every $f \in H^p(w)$ and $s \in \mathbb{N} \cup \{0\}$ there exist (p, ∞, s) atoms w.r.t. w $\{a_j\}_j$ and $\{\lambda_j\} \in l^p$ such that $f = \sum_j \lambda_j a_j$ in $\mathcal{S}' \cap H^p(w)$ and $\|\{\lambda_j\}_j\|_{l^p} \lesssim \|f\|_{H^p(w)}$.*

The concept of molecule was introduced by Taibleson and Weiss [19], in which the characterization with molecules was given. The weighted version of them were studied by Lee and Lin [11].

Definition 3.2 ([19] and [11]). *Let $0 < p \le 1, \ w \in A_\infty$. Suppose that*

$$s \ge \left[n\left(\frac{q_w}{p}-1\right)\right], \ \varepsilon > \max\left(\frac{sr_w}{n(r_w-1)} + \frac{1}{r_w-1}, \frac{1}{p}-1\right),$$

$a = 1 - 1/p + \varepsilon$ and $b = 1 + \varepsilon, \ (a, b > 0)$. Then, a function M is said to be a $(p, \infty, s, \varepsilon)$-molecule w.r.t. w centered at x_0 if M satisfies the following conditions;
(i) $M(\cdot)w(B(x_0, |\cdot - x_0|))^b \in L^\infty$,
(ii) $\mathcal{N}_w(M) := \|M\|_{L^\infty(w)}^{a/b} \|M(\cdot)w(B(x_0, |\cdot - x_0|))^b\|_{L^\infty}^{1-a/b} < \infty$,
(iii) $\int x^\alpha M(x)dx = 0, \ (|\alpha| \le s)$.

The condition on ε above is used for the next Theorem B only.

To investigate the mapping property for several linear operators, the following theorem is useful.

Theorem B ([11]). *Let $0 < p \le 1$ and $s \ge \left[n\left(\frac{q_w}{p}-1\right)\right]$. Assume that*

$$\varepsilon > \max\left(\frac{sr_w}{n(r_w-1)} + \frac{1}{r_w-1}, \frac{1}{p}-1\right),$$

$a = 1 - 1/p + \varepsilon$ and $b = 1 + \varepsilon$. For any M, $(p, \infty, s, \varepsilon)$-molecule w.r.t. w, $\|M\|_{H^p(w)} \lesssim \mathcal{N}_w(M)$.

Proof of Theorem 1.1 in the case $0 < p \le q \le 1$. From Theorem A, $f \in H^p(w)$ can be decomposed as $f = \sum_j \lambda_j a_j$ with supp $a_j \subset B_j = B(x_j, r_j)$ and $\int x^\alpha a_j(x)dx = 0, \ (|\alpha| \le N)$ with a sufficiently large N. Since

$$\|f * \varphi\|_{H^q(\sigma)} \le \left(\sum_{j=1}^\infty |\lambda_j|^q \|a_j * \varphi\|_{H^q(\sigma)}^q\right)^{1/q},$$

it is sufficient to prove that

$$\text{(I)} \ \{a_j * \varphi\}_j \text{ are } (q, \infty, \tilde{N}, \varepsilon) - \text{molecules w.r.t. } \sigma, \text{where}$$
$$\tilde{N} = \left[n\left(\frac{q_\sigma}{q}-1\right)\right] \text{ and } \varepsilon > \max\left(\frac{\tilde{N}r_\sigma}{n(r_\sigma-1)} + \frac{1}{r_\sigma-1}, \ \frac{1}{q}-1\right),$$
$$\text{(II)} \ \sup_j \mathcal{N}_\sigma(a_j * \varphi) \lesssim [w, \sigma]_{X^K_{p,q}}.$$

The moment condition in (I) is easily checked.

Let $a = 1 - 1/q + \varepsilon$ and $b = 1 + \varepsilon$. Because $\sigma(B(x_j, |x - x_j|))^b \le c_\sigma \sigma(B_j)^b$ for $x \in 2B_j$ from the doubling property of σ, an estimate

$$\sup_{x \in 2B_j} \left| a_j * \varphi(x)\ \sigma(B(x_j, |x - x_j|))^b \right| \lesssim \min\left(1, |B_j|^{1+(N+1)/n}\right) \frac{\sigma(B_j)^b}{w(B_j)^{1/p}} \tag{10}$$

follows from (6).

On the other hand, for $m \ge n\rho b$, it holds

$$\sup_{x \notin 2B_j} \left| a_j * \varphi(x)\ \sigma(B(x_j, |x - x_j|))^b \right| \lesssim |B_j|^{1-m/n} \frac{\sigma(B_j)^b}{w(B_j)^{1/p}}. \tag{11}$$

To verify (11), we take an integer $l > 1$ so that $x \in B(x_j, 2lr_j) \backslash B(x_j, 2(l-1)r_j)$. Since $x \in B(x_j, 2lr_j) \iff \frac{|x - x_j|}{2l} \le r_j$, one obtains, for $x \in B(x_j, 2lr_j)$,

$$\sigma(B(x_j, |x - x_j|))^b \le \sigma(B(x_j, 2lr_j))^b \lesssim l^{n\rho b} \sigma(B_j)^b,$$

where we have used that $\sigma(\lambda B) \le c\lambda^{n\rho}\sigma(B)$ for $\lambda > 1$. Also,

$$\begin{aligned}
|a_j * \varphi(x)| &\lesssim \|a_j\|_{L^\infty} \int_{B_j} \frac{1}{|x - y|^m} dy \\
&\le \|a_j\|_{L^\infty} |B_j| \frac{1}{(2l-1)^m r_j^m} \\
&\lesssim l^{-m} |B_j|^{1-m/n} w(B_j)^{-1/p}.
\end{aligned}$$

Hence, taking $m \in [n\rho b, \infty)$ ensures (11).

From (10) and (11), we obtain

$$\begin{aligned}
&\|a_j * \varphi(\cdot) \sigma(B(x_j, |\cdot - x_j|))^b\|_{L^\infty} \\
&\qquad \lesssim \max\left(\min\left(1, |B_j|^{1+(N+1)/n}\right), |B_j|^{1-m/n}\right) \frac{\sigma(B_j)^b}{w(B_j)^{1/p}}.
\end{aligned}$$

Hence, $\mathcal{N}_\sigma(a_j * \varphi)$ is dominated by a constant multiple of

$$\sup_B \mathcal{N}(B) \frac{\sigma(B)^{1/q}}{w(B)^{1/p}},$$

where the supremum is taken over all balls B and

$$\begin{aligned}
\mathcal{N}(B) = &\min\left(1, |B_j|^{1+(N+1)/n}\right)^{a/b} \\
&\times \max\left(\min\left(1, |B_j|^{1+(N+1)/n}\right), |B_j|^{1-m/n}\right)^{1-a/b}.
\end{aligned}$$

If N satisfies $\frac{a}{b} \frac{1 + m + N - n}{n} + 1 \ge K$, then for $|B| \le 1$

$$\begin{aligned}
\mathcal{N}(B) &\le |B|^{(1+(N+1)/n)a/b} \max\left(|B|^{1+(N+1)/n}, |B|^{1-m/n}\right)^{1-a/b} \\
&\le |B|^{(1+(N+1)/n)a/b + (1-m/n)(1-a/b)} \\
&\le |B|^K \\
&\lesssim \min(1, |B|^K).
\end{aligned}$$

On the other hand because we may assume $n < m$, for $|B| \ge 1$

$$\mathcal{N}(B) \le 1 \times \max\left(1, |B|^{1-m/n}\right)^{1-a/b} \lesssim 1 \lesssim \min(1, |B|^K).$$

As a consequence, we get the uniform bound;

$$\mathcal{N}_\sigma(a_j * \varphi) \lesssim \sup_B \min\left(1, |B|^K\right) \frac{\sigma(B)^{1/q}}{w(B)^{1/p}},$$

for all $j \in \mathbb{N}$, and the proof is completed.

□

3.2 The case $0 < p \le 1 < q < \infty$

Proof of Theorem 1.1 in the case $0 < p \le 1 < q < \infty$. Let $f \in H^p(w)$ have the same decomposition as that in the previous subsection. Now, since $q > 1$, we have

$$\begin{aligned} \|f * \varphi\|_{H^q(\sigma)} &= \left\| \sum_j \lambda_j a_j * \varphi \right\|_{H^q(\sigma)} \\ &\le \sup_j \|a_j * \varphi\|_{H^q(\sigma)} \sum_j |\lambda_j| \\ &\lesssim \sup_j \|a_j * \varphi\|_{H^q(\sigma)} \|f\|_{H^p(w)}. \end{aligned}$$

Here, we remark that the argument in [11] ensures that even if $q > 1$, for a $(q, \infty, \tilde{N}, \varepsilon)$-molecule M, it holds

$$\|M\|_{H^q(\sigma)} \lesssim \mathcal{N}_\sigma(M).$$

Therefore, the proof is completed, provided that it follows

$$\sup_j \mathcal{N}_\sigma(a_j * \varphi) \lesssim [w, \sigma]_{X^K_{p,q}}.$$

The inequality can be verified by the same argument as that in the previous subsection.

□

3.3 The case $1 < p \le q < \infty$

Finally, we prove Theorem 1.1 with $1 < p \le q < \infty$. To do so, we make use of an atomic decomposition in [18] for weighted Hardy spaces. Another concept of atom was introduced by Strömberg and Torchinsky [18]. The definition is independent of weight.

Definition 3.3 ([18])**.** *Let $N \in \mathbb{N} \cup \{0\}$. A function a is said to be an $\langle \infty, N \rangle$-atom if a satisfies the following conditions;*
(i) supp $a \subset B$
(ii) $\|a\|_{L^\infty} \le 1$
(iii) $\displaystyle\int x^\alpha a(x) dx = 0, \ (|\alpha| \le N)$.

The following atomic decomposition in [18] can work in the case $p \in (0, \infty)$. In fact, they proved the following theorem with doubling measures w.

Theorem C (Atomic decomposition by using $\langle \infty, s \rangle$-atoms, [18])**.** *Let $0 < p < \infty$ and $w \in A_\infty$. For every $f \in H^p(w)$, we can find $N(p, w) \in \mathbb{N}$ so that for any $N \ge N(p, w)$ there exist $\lambda_j > 0$ and $\langle \infty, N \rangle$-atom a_j supported in B_j such that $f = \sum_j \lambda_j a_j$ in $\mathcal{S}' \cap H^p(w)$ and for all $s \in (0, \infty)$*

$$\left\| \left(\sum_j \lambda_j^s \chi_{B_j} \right)^{1/s} \right\|_{L^p(w)} \lesssim \|f\|_{H^p(w)}.$$

Proof of Theorem 1.1 in the case $1 < p \le q < \infty$. Let $1/s = 1 + 1/p - 1/q$. Let N^* and N be integers satisfying

$$\begin{aligned} &2\,(n\rho/q + n\tau/s) - (n+1) < N^* \text{ and} \\ &K \le \min\left(1 + \frac{N+1}{n}, \ \frac{1}{2} + \frac{N - N^*}{2n}\right) = \frac{1}{2} + \frac{N - N^*}{2n}. \end{aligned} \tag{12}$$

Then, we decompose f with $\langle \infty, N \rangle$-atoms $\{a_j\}_j$; $f = \sum_j \lambda_j a_j$ in $\mathcal{S}' \cap H^p(w)$ with

$$\left\| \left(\sum_j \lambda_j^s \chi_{B_j} \right)^{1/s} \right\|_{L^p(w)} \lesssim \|f\|_{H^p(w)}.$$

We divide $\|f*\varphi\|_{H^q(\sigma)}$ into two parts as follows; $\|f*\varphi\|_{H^q(\sigma)} \le \mathrm{I}+\mathrm{II}$, where

$$\mathrm{I} = \left\| \sum_{j=1}^{\infty} \lambda_j M_\Phi(a_j*\varphi)\chi_{4B_j} \right\|_{L^q(\sigma)} \quad \text{and}$$

$$\mathrm{II} = \left\| \sum_{j=1}^{\infty} \lambda_j M_\Phi(a_j*\varphi)\chi_{(4B_j)^c} \right\|_{L^q(\sigma)}.$$

3.3.1 Estimate of I

We begin the estimate of I with the duality;

$$\mathrm{I} \le \sup_g \sum_{j=1}^{\infty} \lambda_j \|M_\Phi(a_j*\varphi)\|_{L^\infty} \int_{4B_j} |g|\sigma dx,$$

where the supremum is taken over all g with $\|g\|_{L^{q'}(\sigma)} \le 1$. From (12), we have

$$\|M_\Phi(a_j*\varphi)\|_{L^\infty} \lesssim \min\left(1, |B_j|^K\right).$$

Let $1/r_1 = 1+\alpha-1/q,\ 1/r_2 = \alpha(\tau-1),\ 1/r_3 = 1/q-\alpha\tau$ and $0<\alpha\le \dfrac{1}{qr'_\sigma\tau}\left(\le \dfrac{1}{2q}\right)$. It holds

$$1<r_i<\infty,\ 1/r_1+1/r_2+1/r_3=1,\ \alpha r_2 = \tau'-1 \text{ and } r_1<q'.$$

Moreover,

$$r_3/q \le r_\sigma,$$

in fact,

$$\begin{aligned} r_3/q \le r_\sigma &\iff \frac{1}{qr_\sigma} \le \frac{1}{r_3} = \frac{1}{q}-\alpha\tau \\ &\iff \alpha\tau \le \frac{1}{q}-\frac{1}{qr_\sigma} = \frac{1}{qr'_\sigma} \\ &\iff \alpha \le \frac{1}{qr'_\sigma\tau}. \end{aligned}$$

From Hölder inequality, one has

$$\begin{aligned} \int_{4B_j} |g|\sigma dx &= \int_{4B_j} |g| w^\alpha \sigma^{1/q'} w^{-\alpha}\sigma^{1/q} dx \\ &\le \|gw^\alpha\sigma^{1/q'}\|_{L^{r_1}(4B_j)}\ \|w^{-\alpha}\|_{L^{r_2}(4B_j)}\ \|\sigma^{1/q}\|_{L^{r_3}(4B_j)}. \end{aligned}$$

The first term can be estimated by the maximal function;

$$\begin{aligned} \cdot\|gw^\alpha\sigma^{1/q'}\|_{L^{r_1}(4B_j)} &= w(4B_j)^{1/r_1}\left(\frac{1}{w(4B_j)}\int_{4B_j}\left(|g|w^{\alpha-1/r_1}\sigma^{1/q'}\right)^{r_1} wdx\right)^{1/r_1} \\ &\le w(B_j)^{1/r_1-1/s}\left(\int_{4B_j} M_{r_1;w}(|g|w^{\alpha-1/r_1}\sigma^{1/q'})(y)^s wdx\right)^{1/s}. \end{aligned}$$

Since $\alpha r_2 = \tau'-1 = \dfrac{1}{\tau-1}$, $\alpha\ge 0$ and $w\in A_\tau$, the second term has the following bound;

$$\begin{aligned} \cdot\|w^{-\alpha}\|_{L^{r_2}(4B_j)} &= |4B_j|^{1/r_2}\left(\langle w^{-\alpha r_2}\rangle_{4B_j}^{1/\alpha r_2}\right)^\alpha \\ &\le |4B_j|^{1/r_2}\left([w]_{A_\tau}\langle w\rangle_{4B_j}^{-1}\right)^\alpha \\ &\lesssim |B_j|^{\alpha+1/r_2} w(B_j)^{-\alpha}. \end{aligned}$$

The relation $r_3/q \le r_\sigma$ and the reverse Hölder inequality yield the bound of the third term;

$$
\begin{aligned}
\cdot\|\sigma^{1/q}\|_{L^{r_3}(4B_j)} &= |4B_j|^{1/r_3}\left(\langle \sigma^{r_3/q}\rangle_{4B_j}^{q/r_3}\right)^{1/q}\\
&\le |4B_j|^{1/r_3}\left(2\langle\sigma\rangle_{4B_j}\right)^{1/q}\\
&\lesssim |B_j|^{1/r_3-1/q}\sigma(B_j)^{1/q}.
\end{aligned}
$$

Therefore, combining three estimates above, we get

$$
\begin{aligned}
\int_{4B_j}|g|\sigma dx &\lesssim |B_j|^{1/r_2+1/r_3+\alpha-1/q}w(B_j)^{1/r_1-1/s-\alpha}\sigma(B_j)^{1/q}\\
&\quad\times\left(\int_{4B_j}M_{r_1;w}(|g|w^{\alpha-1/r_1}\sigma^{1/q'})^s w dx\right)^{1/s} \qquad (13)\\
&= \frac{\sigma(B_j)^{1/q}}{w(B_j)^{1/p}}\left(\int_{4B_j}M_{r_1;w}(|g|w^{\alpha-1/r_1}\sigma^{1/q'})^s w dx\right)^{1/s}.
\end{aligned}
$$

Thus, one has

$$
\begin{aligned}
\mathrm{I} &\lesssim \sup_g\sum_{j=1}^{\infty}\lambda_j\min(1,|B_j|^K)\frac{\sigma(B_j)^{1/q}}{w(B_j)^{1/p}}\left(\int_{4B_j}M_{r_1;w}(|g|w^{\alpha-1/r_1}\sigma^{1/q'})^s w dx\right)^{1/s}\\
&\le [w,\sigma]_{X^K_{p,q}}\sup_g\left(\int\sum_{j=1}^{\infty}\lambda_j^s\chi_{4B_j}w^{s/p}M_{r_1;w}(|g|w^{\alpha-1/r_1}\sigma^{1/q'})^s w^{1-s/p}dx\right)^{1/s}\\
&= [w,\sigma]_{X^K_{p,q}}\sup_g\left\|\left(\sum_j\lambda_j^s\chi_{4B_j}\right)^{1/s}\right\|_{L^p(w)}\left\|M_{r_1;w}(|g|w^{\alpha-1/r_1}\sigma^{1/q'})^s\right\|^{1/s}_{L^{(p/s)'}(w)}.
\end{aligned}
$$

Here we have used that $(1-s/p)(p/s)'=1$. Since M_w is a bounded operator on $L^r(w)$, $(1<r<\infty)$ with the uniform operator norm for w, we see that

$$
\begin{aligned}
\|M_{r_1;w}(|g|w^{\alpha-1/r_1}\sigma^{1/q'})^s\|^{1/s}_{L^{(p/s)'}(w)} &= \|M_{r_1;w}(|g|w^{\alpha-1/r_1}\sigma^{1/q'})\|_{L^{q'}(w)}\\
&\lesssim \|g\|_{L^{q'}(\sigma)}.
\end{aligned}
$$

Here, the relations $s(p/s)'=q' \iff 1/s-1/p=1-1/q$, and $1+q'(\alpha-1/r_1)=0$ have been used. Therefore, thanks to Lemma 4 in Chapter VIII in [18], we can complete the estimate of I;

$$
\begin{aligned}
\mathrm{I} &\lesssim [w,\sigma]_{X^K_{p,q}}\left\|\left(\sum_j\lambda_j^s\chi_{4B_j}\right)^{1/s}\right\|_{L^p(w)}\\
&\lesssim [w,\sigma]_{X^K_{p,q}}\|f\|_{H^p(w)}.
\end{aligned}
$$

3.3.2 Estimate of II

Next, we verify that

$$
\mathrm{II} = \left\|\sum_{j=1}^{\infty}\lambda_j M_\Phi(a_j*\varphi)\chi_{(4B_j)^c}\right\|_{L^q(\sigma)}
$$

has the same bound as that of I. Here, we use the same exponents r_1,α and s as these in the previous subsection. From Lemma 2.2 and (12), it follows that

$$
M_\Phi(a_j*\varphi)(x)\chi_{(4B_j)^c}(x) \lesssim \min\left(1,|B_j|^K\right)\left(\frac{r_j}{|x-x_j|}\right)^M\chi_{(4B_j)^c}(x),
$$

where $M = (n + N^* + 1)/2$. In fact,

$$M_\Phi(a_j * \varphi)(x)\chi_{(4B_j)^c}(x) \le \|M_\Phi(a_j * \varphi)\|_{L^\infty}^{1/2} M_\Phi(a_j * \varphi)(x)^{1/2}\chi_{(4B_j)^c}(x)$$
$$\lesssim \min\left(1, |B_j|^K\right)\left(\frac{r_j}{|x - x_j|}\right)^{(n+N^*+1)/2} \chi_{(4B_j)^c}(x).$$

Hence, we obtain

$$\begin{aligned} \mathrm{II} &\lesssim \left\| \sum_{j=1}^{\infty} \lambda_j \min\left(1, |B_j|^K\right)\left(\frac{r_j}{|x - x_j|}\right)^M \chi_{(4B_j)^c} \right\|_{L^q(\sigma)} \\ &= \sup_g \left| \int \sum_j \lambda_j \min\left(1, |B_j|^K\right)\left(\frac{r_j}{|x - x_j|}\right)^M \chi_{(4B_j)^c} g\sigma dx \right| \\ &\le \sup_g \sum_j \lambda_j \min\left(1, |B_j|^K\right) \int_{(2B_j)^c} \left(\frac{r_j}{|x - x_j|}\right)^M |g|\sigma dx \end{aligned}$$

with the same supremum above. From (13), the last integral has the similar bound as that of I;

$$\begin{aligned} \int_{(2B_j)^c} \left(\frac{r_j}{|x - x_j|}\right)^M |g|\sigma dx &\lesssim \sum_{k \ge 1} 2^{-kM} \int_{2^{k+1}B_j} |g|\sigma dx \\ &\lesssim \sum_k 2^{-kM} \frac{\sigma(2^{k+1}B_j)^{1/q}}{w(2^{k+1}B_j)^{1/p}} \left(\int_{2^{k+3}B_j} M_{r_1;w}(|g|w^{\alpha - 1/r_1}\sigma^{1/q'})^s w dx \right)^{1/s} \\ &\le \sum_k \left(\frac{2^{n\rho/q}}{2^M}\right)^k \frac{\sigma(B_j)^{1/q}}{w(B_j)^{1/p}} \left(\int_{2^{k+3}B_j} M_{r_1;w}(|g|w^{\alpha - 1/r_1}\sigma^{1/q'})^s w dx \right)^{1/s}. \end{aligned}$$

Thus, we get the desired estimate of (II) as follows;

$$\begin{aligned} \mathrm{II} &\lesssim [w, \sigma]_{X_{p,q}^K} \sup \sum_j \lambda_j \sum_k 2^{k(n\rho/q - M)} \\ &\qquad \times \left(\int_{2^{k+3}B_j} M_{r_1;w}(|g|w^{\alpha - 1/r_1}\sigma^{1/q'})^s w dx \right)^{1/s} \\ &\le [w, \sigma]_{X_{p,q}^K} \sup_g \sum_k 2^{k(n\rho/q - M)} \\ &\qquad \times \left(\int \sum_j \lambda_j^s \chi_{2^{k+3}B_j} w^{s/p} M_{r_1;w}\left(|g|w^{\alpha - 1/r_1}\sigma^{1/q'}\right)^s w^{1 - s/p} dx \right)^{1/s} \\ &\lesssim [w, \sigma]_{X_{p,q}^K} \sum_k 2^{k(n\rho/q - M)} \left\| \left(\sum_j \lambda_j^s \chi_{2^{k+3}B_j} \right)^{1/s} \right\|_{L^p(w)} \\ &\lesssim [w, \sigma]_{X_{p,q}^K} \sum_k 2^{k(n\rho/q + n\tau/s - M)} \left\| \left(\sum_j \lambda_j^s \chi_{B_j} \right)^{1/s} \right\|_{L^p(w)} \\ &\lesssim [w, \sigma]_{X_{p,q}^K} \|f\|_{H^p(w)}. \end{aligned}$$

Here we have used Lemma 4 in Chapter VIII in [18] and the relation $n\rho/q + n\tau/s - M < 0$. As a consequence, the proof is completed.

□

4 Div-curl estimate for weighted Hardy spaces

In this section, we establish the weighted version of the so-called "div-curl lemma" due to Coifman, Lions, Meyer and Semmes [3].

4.1 Non-endpoint cases $p < \infty$

The purpose in this case is achieved by means of the pointwise estimate of bilinear form with the cancellation property by Miyachi [15].

Proof. It is enough to show the inequality with $u, \nabla v \in \hat{\mathcal{D}}_0$. We denote the j-th Riesz transform by $R_j = -\partial_j |\nabla|^{-1} = \mathcal{F}^{-1}\left[-i\frac{\xi_j}{|\xi|}\mathcal{F}\right]$. Since the divergence free condition of u yields the cancellation

$$\sum_{k=1}^{n} R_k u_k = 0,$$

one obtains

$$\|(u\cdot\nabla)v\|_{H^r(\mu)} \lesssim \sum_{j,k=1}^{n} \|u_k R_k(|\nabla| v_j) - (R_k u_k)|\nabla| v_j\|_{H^r(\mu)}.$$

It is sufficient to prove

$$\|\Lambda(f,g)\|_{H^r(\mu)} \lesssim \|f\|_{H^p(w)}\|g\|_{H^q(\sigma)} \tag{14}$$

with $f, g \in \hat{\mathcal{D}}_0$, where $\Lambda(f,g)(x) = \sum_{\lambda=1}^{2}\left(T_1^\lambda f\right)(x)\left(T_2^\lambda g\right)(x),\ T_1^1 h = h,\ T_1^2 h = -R_j h,\ T_2^1 h = R_j h$ and $T_2^2 h = h$. Note that the bilinear form Λ has the moment condition

$$\int \Lambda(f,g)dx = 0,$$

for all $f, g \in \hat{\mathcal{D}}_0$. To prove (14), we make use of the pointwise estimate for Λ due to Miyachi [15]. More precisely, he showed the following; for every $K \in \mathbb{N}\cup\{0\}$,

$$\Lambda(f,g)_1^*(x) \lesssim \sum_{\lambda=1}^{2} G_K(f, T_1^\lambda, m_1)(x) G_K(g, T_2^\lambda, m_2)(x) \tag{15}$$

where $G_L(h,S,m)(x) = M_m(h_L^*)(x) + (Sh)_L^*(x)$, provided that $1/m_j = 1/\theta_j + s_j/n$ and, θ_j and s_j satisfy the following conditions; $0 < 1/\theta_j < 1,\ 1/\theta_1 + 1/\theta_2 \le 1,\ 0 \le s_j < 1,\ s_1 + s_2 \le 1,\ 1/p < 1/\theta_1 + s_1/n$ and $1/q < 1/\theta_2 + s_2/n$.
We check that our assumption ensures the existence of exponents (m_j, θ_j, s_j) satisfying these conditions.

Let $m_1 = p/\tau \in (n/(n+1), p)$ and $m_2 = q/\rho \in (n/(n+1), q)$. Obviously, $1/p < 1/m_1 < 1 + 1/n, 1/q < 1/m_2 < 1 + 1/n$ and $1/r < 1/m_1 + 1/m_2 < 1 + 1/n$. It is possible to take $\theta_j, (j = 1,2)$, fulfilling the following conditions;

- $0 < 1/\theta_1 < 1,\ 1/\theta_1 \le 1/m_1$ and $1/m_1 - 1/n \le 1/\theta_1 \le 1 + 1/n - 1/m_2$,
- $0 < 1/\theta_2 \le \min(1 - 1/\theta_1, 1/m_2),\ 1/m_2 - 1/n \le 1/\theta_2$ and $1/m_1 + 1/m_2 - 1/\theta_1 - 1/n \le 1/\theta_2$.

Then, we define $s_j = n(1/m_j - 1/\theta_j), (j = 1,2)$. From the conditions on θ_j above, it follows that $0 \le s_j < 1$ and $s_1 + s_2 \le 1$. Therefore, from (15) we get

$$\|\Lambda(f,g)\|_{H^r(\mu)} \lesssim \sum_{\lambda=1}^{2} \|G_K(f, T_1^\lambda, m_1)\|_{L^p(w)} \|G_K(g, T_2^\lambda, m_2)\|_{L^q(\sigma)}.$$

Because $m_1 < p$ and $w \in A_{p/m_1}$, it follows $\|M_{m_1}(f_K^*)\|_{L^p(w)} \lesssim \|f\|_{H^p(w)}$. By using Theorem 14 in Chapter XI in [18], we have the boundedness of the Riesz transforms on $H^p(w)$, which implies

$$\|(T_j^\lambda f)_K^*\|_{L^p(w)} \lesssim \|f\|_{H^p(w)}.$$

Hence, we get the desired estimate (14). □

4.2 Endpoint case $p=\infty$

The argument above is false in the case $p=\infty$, because the Riesz transform R_j is a unbounded operator on L^∞. By following the argument by Auscher, Russ and Tchamitchian [1], we can show the endpoint case. For the proof of the theorem, we recall notations that were used in [1]. For $x\in\mathbb{R}^n$ and $1\le m\le\infty$, let $F_m(x)$ be a set of all vector-valued functions $\Psi=(\psi_1,\cdots,\psi_n)$ with $\psi_j\in L^m$ supported in a ball $B=B(x_B,r_B)$ containing x so that there exists a function $g_\Psi\in L^m$ such that $\mathrm{div}\Psi=g_\Psi$ in $\mathcal{S}'$, supp $g_\Psi\subset B$ and $\|\Psi\|_{L^m}+r_B\|g_\Psi\|_{L^m}\le|B|^{-1/m'}$. The similar one $\tilde{F}_m(x)$ is defined by a set of all vector-valued functions $\tilde\Psi=(\tilde\psi_1,\cdots,\tilde\psi_n)$ so that $\tilde\psi_j\in W^{1,m}$ supported in a ball $B=B(x_B,r_B)$ satisfying $\|\tilde\Psi\|_{L^m}+r_B\|D\tilde\Psi\|_{L^m}\le|B|^{-1/m'}$ where $D\tilde\Psi$ is the $n\times n$ matrix whose jth column is $\nabla\tilde\psi_j$. The maximal operators N_m and $\tilde{N}_m$ are defined by for locally integrable functions f and $\{f_j\}_{j=1}^n$

$$N_mf(x)=\sup_{\Psi\in F_m(x)}\int f(y)g_\Psi(y)dy\quad\text{and}\quad\tilde{N}_m\mathbb{F}(x)=\sup_{\tilde\Psi\in\tilde{F}_m(x)}\int\mathbb{F}(y)\cdot\tilde\Psi(y)dy,$$

where $\mathbb{F}=(f_1,\cdots,f_n)$.

In the proof, we use the following basic fact;

Lemma 4.1. *Let $1<p<\infty$ and $w\in A_p$. Then*

$$\int_{|x|>M}\frac{w(x)}{|x|^{np}}dx\lesssim M^{-np}w(B(0,M)).$$

Proof. It is sufficient to prove

$$\left\|\sum_{j=1}^n u_j\partial_jv\right\|_{H^q(\sigma)}\lesssim\|u\|_{L^\infty}\|\nabla v\|_{H^q(\sigma)},$$

for every vector-valued functions $u\in L^\infty$ with div $u=0$ and every functions $v\in Y$ with $\partial_jv\in H^q(\sigma)$. Firstly, we give a definition of $\sum_{j=1}^n u_j\partial_jv$ as a tempered distribution as follows; for $\varphi\in\mathcal{S}$

$$\left\langle\sum_{j=1}^n u_j\partial_jv,\varphi\right\rangle=-\sum_{j=1}^n\int u_j(y)v(y)\partial_j\varphi(y)dy.$$

Our assumption ensures that $\sum_{j=1}^n u_jv\partial_j\varphi\in\mathcal{S}$. Then, it follows

$$\sum_{j=1}^n u_j\partial_jv*\Phi_t(x)=-C_\Phi\|u\|_{L^\infty}\int v(y)\left[\sum_{j=1}^n\tilde{u}_j(y)\partial_{y_j}\Phi_t(x-y)\right]dy,$$

where C_Φ is a large constant depending on n and Φ, and $\tilde{u}_j(y)=\dfrac{u_j(y)}{C_\Phi\|u\|_{L^\infty}}$. Owing to the divergence free condition on u, we see that for every $x\in\mathbb{R}^n$

$$\sum_{j=1}^n\tilde{u}_j(y)\partial_{y_j}\Phi_t(x-y)=\sum_{j=1}^n\partial_{y_j}\left(\tilde{u}_j(y)\Phi_t(x-y)\right)\quad\text{in}\quad\mathcal{S}'({\mathbb{R}^n}_y).$$

Hence, we obtain the pointwise estimate

$$M_\Phi\left(\sum_{j=1}^n u_j\partial_jv\right)(x)\le C_\Phi\|u\|_{L^\infty}N_mv(x),$$

for all $m\in[1,\infty]$. The maximal function N_mv is pointwisely controlled by another one $\tilde{N}_m\nabla v$. To check this, let $\Psi\in F_m(x)$ be supported in B and $m\in(1,\infty)$. Because the compactness of the support of $g_\Psi\in L^m$ implies $\int g_\Psi dx=0$, from Theorem 10 in [1] there exists $\tilde\Psi=(\tilde\psi_1,\cdots,\tilde\psi_n)$ with $\tilde\psi_j\in W^{1,m}(B)$ such that $g_\Psi=\mathrm{div}\tilde\Psi$ a.e. on $\mathbb{R}^n$ and $\|D\tilde\Psi\|_{L^m}\le c\|g_\Psi\|_{L^m}$. The Poincaré inequality then ensures that $\frac{1}{C^*}\tilde\Psi\in\tilde{F}_m(x)$ with some constant $C^*>0$. Therefore, we can see that

$$\int v(y)g_\Psi(y)dy=-\int\nabla v(y)\cdot\tilde\Psi(y)dy,$$

which implies that $N_m v(x) \le c\tilde{N}_m \nabla v(x)$ for all $x \in \mathbb{R}^n$. As a consequence, one obtain the pointwise estimate

$$M_\Phi\left(\sum_{j=1}^n u_j \partial_j v\right)(x) \lesssim \|u\|_{L^\infty} \tilde{N}_m \nabla v(x),$$

with $1 < m < \infty$. Therefore, it suffices to prove that

$$\|N_m^* f\|_{L^q(\sigma)} \lesssim \|f\|_{H^p(w)} \tag{16}$$

with some $m \in (1,\infty)$, where $N_m^* f(x) = \sup_{\psi \in \Lambda_m(x)} |\int f(y)\psi(y)dy|$ and $\Lambda_m(x)$ is a space of all functions $\psi \in W^{1,m}$ supported in a ball $B = B(x_B, r_B)$ satisfying $\|\psi\|_{L^m} + r_B\|\nabla\psi\|_{L^m} \le |B|^{-1/m'}$.

Let $f = \sum_{j=1}^\infty \lambda_j a_j$ be a atomic decomposition from Theorem A when $q \le 1$ or Theorem C when $q > 1$.

We shall show the inequality (16) in the case $0 < q \le 1$. From the openness property of Muckenhoupt classes, there exists $m > n$ so that $\sigma \in A_{q(1/m'+1/n)}$. Because

$$\|N_m^* f\|_{L^q(\sigma)} \lesssim \left(\sum_{j=1}^\infty |\lambda_j|^q \|N_m a_j\|_{L^q(\sigma)}^q\right)^{1/q},$$

it is sufficient to show that $\sup_j \|N_m^* a_j\|_{L^q(\sigma)} \lesssim 1$. To do so, let a be an atom, i.e. supp $a \subset B_0 = B(x_0, r_0)$, $\|a\|_{L^\infty} \le \sigma(B_0)^{-1/q}$ and $\int x^\alpha a(x)dx = 0$ for all $|\alpha| \le 1$. Fix $x \in \mathbb{R}^n$ and take $\psi \in \Lambda_m(x)$ supported in $\tilde{B} = B(\tilde{x}, \tilde{r})$. Since $|\int a(y)\psi(y)dy| \le \|a\|_{L^\infty}\|\psi\|_{L^1} \le \sigma(B_0)^{-1/q}$, we see that

$$\|(N_m^* a)\chi_{4B_0}\|_{L^q(\sigma)} \lesssim 1.$$

Consider the case $x \notin 4B_0$. In this case, if $c\tilde{r} < |x - x_0|$ with $c > 12/7$, then $|\int a\psi dy| = 0$. On the other hand, in the case $|x - x_0| \le c\tilde{r}$, the same calculus as that in pp.74-75 in [1] yields

$$|\int a(y)\psi(y)dy| \lesssim \left(\frac{r_0}{|x - x_0|}\right)^{1+n/m'} \sigma(B_0)^{-1/q}.$$

Here we have used that $m > n$. Hence, by using Lemma 4.1, one obtains $\|(N_m^* a)\chi_{(4B_0)^c}\|_{L^q(\sigma)} \lesssim 1$, which completes the proof in this case.

To give a proof in the case $q \in (1,\infty)$, let a be an atom, i.e. supp $a \subset B_0$, $\|a\|_{L^\infty} \le 1$ and $\int x^\alpha a(x)dx = 0$ for all $|\alpha| \le 1$. Obviously, $\|N_m^* a\|_{L^\infty} \le 1$. From the argument in Subsection 3.5, this estimate implies (16) with all $m \ge 0$. Remark that this discussion can work under the assumption $\sigma \in A_\infty$. The proof is completed. □

5 Application to the decay property of solutions to Navier-Stokes equations

In this final section, we prove Theorem 1.4 by using theorems that are established in the previous sections.

Proof. The openness property of Muckenhoupt classes, see Remark 1.1, ensures the existence of $N \in (1, q(1 + 1/n))$ such that $\sigma \in A_N$. From our assumption, we can take $\tau, \theta \in (n, \infty)$ fulfilling

$$\alpha < n(1 + 1/n - 1/p - 1/\tau),\ N < q(1 + 1/n - 1/\tau) \text{ and } 1/n - 1/\tau < 1/\theta < 1/n.$$

Let $1/r = 1/p + 1/\tau$. Then, we define

$$\begin{aligned}\|u\|_X &= \sup_{t>0}\|u(t)\|_{L^n} + \sup_{t>0} t^{1/2}\|u(t)\|_{L^\infty} + \sup_{t>0} t^{1-n/(2\tau)}\|\nabla u(t)\|_{L^\tau} \\ &= \|u\|_{X_1} + \|u\|_{X_2} + \|u\|_{X_3},\end{aligned}$$

$$\begin{aligned}\|u\|_Y &= \sup_{t>0}\|u(t)\|_{H^p(w)} + \sup_{t>0} t^{n(1/p-1/q)/2+(\alpha-\beta)/2}\|u(t)\|_{H^q(\sigma)} \\ &\quad + \sup_{t>0} t^{1/2}\|\nabla u(t)\|_{H^p(w)} \\ &= \|u\|_{Y_1} + \|u\|_{Y_2} + \|u\|_{Y_3},\end{aligned}$$

and let X be a Banach space of all divergence free vector functions u satisfying $\|u\|_X < \infty$ and $\lim_{t\searrow 0}\Big(t^{1/2}\|u(t)\|_{L^\infty} + t^{1-n/(2\tau)}\|\nabla$ 0, and $Y = \{u; \|u\|_Y < \infty,\ \mathrm{div} u = 0\}$. Let

$$\|h\|_{\bar{X}_\theta} = \sup_{t>0} t^{n(1/n-1/\theta)/2}\|h(t)\|_{L^\theta} \le \|h\|_{X_1}^{n/\theta}\|h\|_{X_2}^{1-n/\theta}.$$

We construct solutions in $X \cap Y$ through Picard's iteration scheme.

Step 1. It is not hard to see that $\|u_0\|_X \le c\delta$ and

$$\lim_{t\searrow 0} t^{1/2}\|e^{t\Delta}a\|_{L^\infty} = \lim_{t\searrow 0} t^{1-n/(2\tau)}\|\nabla e^{t\Delta}a\|_{L^\tau} = 0$$

by density. Moreover, Theorem 1.1 gives $\|u_0\|_{Y_1} \le c\delta$.

Step 2. $\|B(f,g)\|_X \le c\|f\|_X\|g\|_X$ can be seen from the following estimates:

$$\begin{aligned}\cdot\|B(f,g)(t)\|_{L^n} &\lesssim \int_0^t (t-s)^{-(n/\theta-1)/2}\|f(s)\|_{L^n}\|g\|_{L^\theta}ds \\ &\lesssim \|f\|_{X_1}\|g\|_{\bar{X}_\theta}\end{aligned}$$

$$\begin{aligned}\cdot\|B(f,g)(t)\|_{L^\infty} &\lesssim \int_0^t (t-s)^{-n/(2\theta)}\|f(s)\|_{L^\theta}\|\nabla g(s)\|_{L^\tau}ds \\ &\lesssim t^{-1/2}\|f\|_{\bar{X}_\theta}\|g\|_{X_3}\end{aligned}$$

$$\begin{aligned}\cdot\|\nabla B(f,g)(t)\|_{L^\tau} &\lesssim \int_0^t (t-s)^{-(n/\theta+1)/2}\|f(s)\|_{L^\theta}\|\nabla g(s)\|_{L^\tau}ds \\ &\lesssim t^{-1+n/(2\tau)}\|f\|_{\bar{X}_\theta}\|g\|_{X_3}.\end{aligned}$$

These estimates also imply that $B(f,g) \in X$ and $\lim_{t\searrow 0}\|B(f,g)(t)\|_{L^n} = 0$ whenever f and g belong to X. By using Corollary 1.1 and Theorem 1.2, one obtains

$$\begin{aligned}\|B(f,g)(t)\|_{H^p(w)} &\le c\int_0^t (t-s)^{-n(1/p+1/\tau-1/p)/2}\|(f\cdot\nabla)g(s)\|_{H^r(|\cdot|^{\alpha r})}ds \\ &\le c\int_0^t (t-s)^{-n/(2\tau)}s^{-(1-n/(2\tau))}ds\|f\|_{Y_1}\|g\|_{X_3} \\ &\le c\|f\|_{Y_1}\|g\|_{X_3}.\end{aligned} \tag{17}$$

On one hand, we have

$$\begin{aligned}\|e^{(t-s)\Delta}\mathbb{P}(f\cdot\nabla)g(s)\|_{H^q(\sigma)} &\lesssim (t-s)^{-n(1/r-1/q)/2-(\alpha-\beta)/2}\|(f\cdot\nabla)g(s)\|_{H^r(|\cdot|^{r\alpha})} \\ &\lesssim (t-s)^{-n(1/p+1/\tau-1/q)/2-(\alpha-\beta)/2}\|f(s)\|_{H^p(w)}\|\nabla g(s)\|_{L^\tau} \\ &\lesssim (t-s)^{-n(1/p+1/\tau-1/q)/2-(\alpha-\beta)/2}s^{-(1-n/(2\tau))}\|f\|_{Y_1}\|g\|_{X_3}.\end{aligned}$$

On the other hand, it follows that with $1/\tilde{r} = 1/q + 1/\tau$

$$\begin{aligned}\|e^{(t-s)\Delta}\mathbb{P}(f\cdot\nabla)g(s)\|_{H^q(\sigma)} &\lesssim (t-s)^{-n(1/\tilde{r}-1/q)/2}\|(f\cdot\nabla)g(s)\|_{H^{\tilde{r}}(|\cdot|^{\beta\tilde{r}})} \\ &\lesssim (t-s)^{-n/(2\tau)}\|f(s)\|_{H^q(\sigma)}\|\nabla g(s)\|_{L^\tau} \\ &\lesssim (t-s)^{-n/(2\tau)}s^{-n(1/p-1/q)/2-(\alpha-\beta)/2-(1-n/(2\tau))}\|f\|_{Y_2}\|g\|_{X_3}.\end{aligned}$$

Combining these estimates, it holds that

$$\|B(f,g)\|_{Y_2} \le c(\|f\|_{Y_1} + \|f\|_{Y_2})\|g\|_{X_3}. \tag{18}$$

Further, $\|B(f,g)\|_{Y_3} \le c\|f\|_{Y_1}\|g\|_{X^3}$ can be found from

$$\|\nabla e^{(t-s)\Delta}\mathbb{P}(f\cdot\nabla)g(s)\|_{H^p(w)} \lesssim (t-s)^{-1/2-n/(2\tau)}s^{-(1-n/(2\tau))}\|f\|_{Y_1}\|g\|_{X_3}.$$

As a consequence, we can find a global solution $u \in X \cap Y$ with $\|u\|_X + \|u\|_Y \le c\delta$ and $u(t) \to a$ in L^n as t tends to 0. It is not hard to see that $u \in C([0,\infty); L^n)$. Furthermore, from Proposition 15.1 in [12], we see $u \in C^\infty((0,\infty)\times\mathbb{R}^n)$.

Step 3. We shall show that $u \in C([0,\infty); H^p(w))$. From (ii) of Remark 4.5 in [2], we can see that $\|e^{t\Delta}a - a\|_{H^p(w)} \to 0$ as t tends to 0. (17) implies the convergence $\|B(u,u)(t)\|_{H^p(w)} \to 0$ as t tends to 0. The continuity of the linear part $e^{t\Delta}a$ in $H^p(w)$ can be proved by Theorem 1.1. From 2 of Remark 1.3, we have

$$\|e^{t\Delta}a - e^{s\Delta}a\|_{H^p(w)} \lesssim |G_{\sqrt{t}} - G_{\sqrt{s}}|_{\mathcal{S}}\|a\|_{H^p(w)} \to 0 \quad \text{as } |t-s| \to 0.$$

To check the continuity of the non-linear part $B(u,u)$, we consider that for $t_+ > t_-$

$$\begin{aligned} B(u,u)(t_+) - & B(u,u)(t_-) \\ &= \int_0^{t_-} e^{(t_+-s)\Delta}\mathbb{P}(u\cdot\nabla)u(s) - e^{(t_--s)\Delta}\mathbb{P}(u\cdot\nabla)u(s)ds \\ &+ \int_{t_-}^{t_+} e^{(t_+-s)\Delta}\mathbb{P}(u\cdot\nabla)u(s)ds. \end{aligned}$$

From Theorem 1.3, we have the uniform bound for t

$$\|e^{(t-s)\Delta}\mathbb{P}(u\cdot\nabla)u(s)\|_{H^p(w)} \lesssim \|u(s)\|_{L^\infty}\|\nabla u(s)\|_{H^p(w)}. \tag{19}$$

The continuity of the heat semigroup on weighted Hardy spaces, proved by Bui [2], ensures that the first term goes to 0 as $|t_+ - t_-| \searrow 0$. Remark that $\mathbb{P}(u\cdot\nabla)u \in H^r(|\cdot|^{\alpha r})$. The uniform estimate (19) implies that the second term tends to 0 as $|t_+ - t_-| \searrow 0$.

Step 4. Finally, we check (4). In the case $p < q$ or $\beta < \alpha$, from the density of $\hat{\mathcal{D}}_0$ in $H^p(w)$, it holds that

$$\lim_{t\searrow 0} t^{n(1/p-1/q)/2+(\alpha-\beta)/2}\|e^{t\Delta}a\|_{H^q(\sigma)} = 0.$$

This and the bilinear estimate (18) yield (4). □

Acknowledgements

The author would like to thank Professor Hideo Kozono for suggesting the problem and valuable comments. The author is grateful to Professor Akihiko Miyachi for fruitful suggestions and Professors Yoshihiro Sawano and Hitoshi Tanaka and Doctor Xiao Lian for valuable discussion and pointing out some typos. The author would like to thank the referee for carefully reading. This work was supported by Grant-in-Aid for JSPS Fellows.

References

[1] P. Auscher, E. Russ and P. Tchamitchian, *Hardy Sobolev spaces on strongly Lipschitz domains of* $\mathbb{R}^n$, J. Funct. Anal. **218** (2005), no. 1, 54-109.

[2] H.Q. Bui, *Weighted Besov and Triebel spaces: interpolation by the real method*, Hiroshima Math. J. **12** (1982), no. 3, 581-605.

[3] R. Coifman, P.L. Lions, Y. Meyer and S. Semmes, *Compensated compactness and Hardy spaces*, J. Math. Pures Appl. **72** (1993), 247-286.

[4] C. Fefferman and E. Stein, H^p *spaces of several variables*, Acta Math. **129**, (1972), 137-193.

[5] J. García-Cuerva, *Weighted* H^p*-spaces*, Disertationes Math. **162** (1979).

[6] A.E. Gatto, C.E. Gutiérrez and R.L. Wheeden, *Fractional integrals on weighted* H^p *spaces*, Trans. Amer. Math. Soc. **289** (1985), no. 2, 575-589.

[7] Y. Giga and T. Miyakawa, *Solutions in* L^r *of the Navier-Stokes initial value problem*, Arch. Rational Mech. Anal. **89** (1985), no. 3, 267-281.

[8] L. Grafakos, *Modern Fourier Analysis*, Springer-Verlag, Graduate Texts in Mathematics. **250**, Second Edition, (2008).

[9] T. Hytönen and C. Pérez, *Sharp weighted bounds involving* A_∞, Anal. PDE **6** (2013), no. 4, 777-818.

[10] T. Kato, *Strong* L^p*-solutions of the Navier-Stokes equation in* $\mathbb{R}^n$*, with applications to weak solutions*, Math. Z. **187** (1984), 471-480.

[11] M.-Y. Lee and C.-C. Lin, *The molecule characterization of weighted Hardy spaces*, J. Funct. Anal. **188** (2002), 442-460.

[12] P. G. Lemarié-Rieusset, *Recent development in the Navier-Stokes problem*, Chapman & Hall/CRC Research Notes in Mathematics. **431**, Chapman & Hall/CRC, Boca Raton, FL. (2002).

[13] A.K. Lerner, S. Ombrosi and C. Pérez, *Sharp* A_1 *bounds for Calderón-Zygmund operators and the relationship with a problem of Muckenhoupt-Wheeden*, Int. Math. Res. Not. (2008), 11pp.

[14] A. Miyachi, *Weighted Hardy spaces on a domain*, Proceedings of the Second ISAAC Congress, Vol. 1 (Fukuoka, 1999), 59-64, Int. Soc. Anal. Appl. Comput., 7, Kluwer Acad. Publ., Dordrecht, 2000.

[15] A. Miyachi, *Hardy space estimate for the product of singular integrals*, Canad. J. Math. **52** (2000), no.2, 381-411.

[16] T. Miyakawa, *Hardy spaces of solenoidal vector fields, with applications to the Navier-Stokes equations*, Kyushu J. Math. **50** (1996), no. 1, 1-64.

[17] T. Miyakawa, *Application of Hardy space techniques to the time-decay problem for incompressible Navier-Stokes flows in* $\mathbb{R}^n$, Funkcial. Ekvac. **41** (1998), no. 3, 383-434.

[18] J.-O. Strömberg and A. Torchinsky, *Weighted Hardy spaces*, Lecture Notes in Mathematics, **1381**. Springer-Verlag, Berlin, (1989).

[19] M.H. Taibleson and G. Weiss, *The molecular characterization of certain Hardy spaces. Representation theorems for Hardy spaces*, Astérisque, **77**, Soc. Math. France, Paris, (1980), 67-149.

[20] Y. Tsutsui, *The Navier-Stokes equations and weak Herz spaces*, Adv. Differential Equations **16** (2011), no. 11-12, 1049-1085.

[21] M. Wiegner, *Decay results for weak solutions of the Navier-Stokes equations on* $\mathbb{R}^n$, J. London Math. Soc. (2) **35** (1987), no. 2, 303-313.