\( A_\infty \) constants between \( BMO \) and weighted \( BMO \)

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Abstract

In this short article, we consider estimates of the ratio

\[ \|f\|_{BMO(w)}/\|f\|_{BMO} \]

from above and below, where \( w \) belongs to Muckenhoupt class \( A_\infty \). The upper bound of the ratio was proved by Hytönen and Pérez in [6] with the optimal power. We establish the lower bound of the ratio and give two other proofs of the upper bound.

Keywords BMO, Muckenhoupt classes

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1 Introduction

In this paper, we are interested in estimates of the ratio

\[ \|f\|_{BMO(w)}/\|f\|_{BMO} \]

with respect to the weight \( w \) belonging to Muckenhoupt class \( A_\infty \). Our purposes are to establish the lower bound of the ratio and to give two other proofs of the upper bound due to Hytönen and Pérez in [6]. In [9], Muckenhoupt and Wheeden proved that for any \( w \in A_\infty \), it holds \( BMO(w) = BMO \). Recently, Hytönen and Pérez [6] gave the upper bound;

\[ \|f\|_{BMO(w)} \leq c_n \|w\|_{A_\infty} \|f\|_{BMO}. \]  \hfill (1)

where \( \|w\|_{A_\infty} \) is Wilson’s \( A_\infty \) constant, see Definition 2.4. Moreover, they [6] proved that the power 1 of \( \|w\|_{A_\infty} \) cannot be replaced by any smaller quantity. Main result in this paper is the following lower bound of the ratio.

Theorem 1.1. There exists \( c_n > 0 \) such that for any \( w \in A_\infty \),

\[ \|f\|_{BMO} \leq c_n \log(2[n]_A) \|f\|_{BMO(w)}. \]  \hfill (2)

Remark 1.1.  1. We do not know whether the order \( \log(2[n]_A) \) is optimal or not.

2. If the inequality

\[ \|f\|_{BMO} \leq c_n \|f\|_{BMO(w)} \]

is true, the exponent 0 of \( \{w\}_{A_\infty} \) is optimal. In fact, for \( w(x) = t \chi_E(x) + \chi_{E^c}(x) \in A_1 \) with a compact set \( E \subset \mathbb{R}^n \) and large \( t \), it follows

\[ \|\log w\|_{BMO} = \|\log w\|_{BMO(w)} = \frac{1}{2} \log t. \]

We will give two other proofs of the upper bound (1). To verify (1) in [6], they used the reverse Hölder inequality;

\[ \{w^r\}_{Q}^{1/r_w} \leq 2(w)_{Q}, \]

for a cube \( Q \subset \mathbb{R}^n \) and \( r_w = 1 + (c_n \|w\|_{A_\infty})^{-1} \). Our proofs of (1) are not based on this type inequality. Our main tools are a dual inequality with the sharp maximal operator \( M^*_f \) due to Lerner [7] and another representation of \( \|w\|_{A_\infty} \).

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These estimates are related to the sharp weighted inequalities for Calderón-Zygmund operators. The sharp weighted inequality for an operator \( T \) means the inequality
\[
\| T f \|_{L^p(w)} \leq c_{n,p,T} \Phi([w]_{A_p}) \| f \|_{L^p(w)}
\]
with the optimal growth function \( \Phi \) on \([1, \infty)\) in the sense that \( \Phi \) cannot be replaced by any smaller function. Recently, Hytönen [5] solved so-called \( A_2 \) conjecture i.e. for any Calderón-Zygmund operator \( T \) (3) holds with \( \Phi(t) = t \). By combining this with the extrapolation theorem in [1], we can see that for \( p \in (1, \infty) \) (3) with \( \Phi(t) = t^{\max(1, 1/(p-1))} \) holds and the exponent \( \max(1, 1/(p-1)) \) is optimal. From the upper bound (1), it immediately follows
\[
\| T \|_{BMO(w)} \leq c_n \| T \|_{L^\infty} \| w \|_{A_\infty} \log(2 \| w \|_{A_\infty}).
\]
which corresponds to (3) with \( p = \infty \). Further, they [6] showed the optimality of the exponent 1 of \( \| w \|_{A_\infty} \). On the other hand, our lower bound (2) yields that
\[
\| T \|_{BMO(w)\to BMO(w)} \leq c_n \| T \|_{BMO\to BMO} \| w \|_{A_\infty} \log(2 \| w \|_{A_\infty}).
\]

## 2 Preliminaries

We say \( w \) a weight if \( w \) is a non-negative and locally integrable function. For a subset \( E \subset \mathbb{R}^n \), \( \chi_E \) means the characteristic function of \( E \) and \( |E| \) denotes the volume of \( E \). By a "cube" \( Q \) we mean a cube in \( \mathbb{R}^n \) with sides parallel to the coordinate axes. Throughout this article we use the following notations; \( w(Q) = \int_Q w \, dx \), \( f_Q = \frac{1}{|Q|} \int_Q f \, dx \) and \( (f)_Q = \frac{1}{w(Q)} \int_Q f \, dx \).

Firstly, we recall definitions of Muckenhoupt classes \( A_p \) and \( BMO \) spaces.

**Definition 2.1.** A weight \( w \) is said to be in the Muckenhoupt class if the following \( A_p \) constant \([w]_{A_p}\) is finite;
\[
[w]_{A_p} := \sup_Q (w(Q) | Q|)^{-p} |Q|^p_{L^\infty(Q)},
\]
and
\[
[w]_{A_\infty} := \sup_Q (w(Q) \log(\log w^{-1}(Q))).
\]

**Remark 2.1.**
1. \([w]_{A_p} \leq 1 \) and \( p < q \Rightarrow A_p \subset A_q \).
2. Because \( \lim_{r \to 0} |f|_{L^r(Q)}^{1/r} \exp(\log |f| Q) \) it follows \( \lim_{p \to \infty} [w]_{A_p} = [w]_{A_\infty} \).

**Definition 2.2.** With a weight \( w \), one defines \( BMO(w) \) as the space of locally integrable functions \( f \) with respect to \( w \) such that
\[
\| f \|_{BMO(w)} = \sup_Q (|f - (f)_Q|_Q) < \infty.
\]

**Remark 2.2.** There is another weighted \( BMO \), \( BMO_w \), which is defined by
\[
\| f \|_{BMO_w} = \inf_Q \frac{1}{w(Q)} \int_Q |f - c| \, dx < \infty.
\]

It is known that for \( w \in A_\infty \), this space is the dual space of the weighted Hardy space \( H^1(w) \), i.e. \( BMO_w = (H^1(w))^* \), see [3].

The definition of Wilson’s constant \( \| w \|_{A_\infty} \) uses the restricted Hardy-Littlewood maximal operator.

**Definition 2.3.** For any measurable subset \( E \subset \mathbb{R}^n \), Hardy-Littlewood maximal operator \( M_E \) restricted to \( E \) is defined by
\[
M_E f(x) = \sup_{R \subset E} (|f|_R),
\]
where the supremum is taken over all cubes \( R \) containing \( x \) and included in \( E \). When \( E = \mathbb{R}^n \), we write \( M = M_E \).
Definition 2.4.
\[ \| w \|_{A_\infty} = \sup_Q \frac{1}{w(Q)} \int_Q M_Q w \, dx. \]

Remark 2.3. 1. \( w \in A_\infty \iff \| w \|_{A_\infty} < \infty \), and \( \| w \|_{A_\infty} \leq c_n[w]_{A_\infty} \).

2. There are several equivalent quantities to \( \| w \|_{A_\infty} \):
\[ \| w \|_{A_\infty} \approx \sup_Q \frac{1}{w(Q)} \int_Q w \log \left( e + \frac{1}{w(Q)} \right) \, dx \]
\[ \approx \sup_Q \frac{1}{w(Q)} \| w \|_{L \log L(Q)} \]
\[ \approx \sup_Q \frac{1}{w(Q)} \int_Q M(\chi_Q w) \, dx \]
\[ \approx \sup_Q \frac{1}{w(Q)} \int_Q |R_j(\chi_Q w)| \, dx, \]
where \( j = 1, \ldots, n \), \( \| f \|_{L \log L(Q)} \) is defined by
\[ \inf \left\{ \lambda > 0; \frac{|f|}{\lambda} \log \left( e + \frac{|f|}{\lambda} \right) \right\}_Q \leq 1 \]
and \( R_j \) is the \( j \)-th Riesz transformation. The first and second equivalences are proved by \( L \log L \) theory due to Stein [10]. The third and fourth ones were proved by Fujii [2]. From the third representation, we obtain an inequality
\[ M(\chi_Q w)(2Q) \leq c_n[w]_{A_\infty} w(Q), \]
which should be compared with the doubling inequality with \([w]_{A_\infty}\):
\[ w(2Q) \leq 2^{2^n}[w]_{A_\infty}^2 w(Q), \]
see for example [4].

3 Lower bound

Owing to a version of John-Nirenberg inequality in the context of non-doubling measures in [8], one obtains a variant of the equivalence
\[ \| f \|_{BMO} \approx \sup_Q \| f - (f)_Q \|_{\exp L(Q)} \]  
(4)
with constants independent of weights.

Lemma 3.1. There exist constants \( c_1, c_2 > 0 \) such that for any \( w \in A_\infty \), it follows
\[ c_1 \sup_Q \| f - (f)_Q \|_{\exp L(Q,w)} \leq \| f \|_{BMO(w)} \]
\[ \leq c_2 \sup_Q \| f - (f)_Q \|_{\exp L(Q,w)}, \]
where \( \| f \|_{\exp L(Q,w)} \) is defined by
\[ \inf \left\{ \lambda > 0; \exp \left( \frac{|f|}{\lambda} \right) - 1 \right\}_{Q,w} \leq 1 \].

With this lemma, we give a proof of our lower bound, Theorem 1.1.

Proof of Theorem 1.1. From the definition of \( \| f \|_{\exp L(Q,w)} \) above, it follows
\[ \left( \exp \left( \frac{|f|}{\| f \|_{\exp L(Q,w)}} \right) \right)_{Q,w} \leq 2. \]
By using the version of Jensen’s inequality
\[ \exp(g) \leq [w] A_{\infty}(\exp(g)) Q, \]  
one obtains
\[ \langle |f| \rangle Q \leq \log(2[w] A_{\infty}) \|f\|_{\exp L(Q;w)}. \]  
The proof is completed by this inequality and Lemma 3.1 as follows:
\[ \langle |f - \langle f \rangle Q| \rangle Q \leq 2 \log(2[w] A_{\infty}) \|f - \langle f \rangle Q\|_{\exp L(Q;w)} \]
\[ \leq c_n \log(2[w] A_{\infty}) \|f\|_{BMO(w)}. \]

\[ \text{Remark 3.1.} \quad \text{The inequality (5) is equivalent to} \]
\[ \exp(\log|f|) \leq [w] A_{\infty} \langle |f| \rangle Q, \]  
which should be compared with (7). (6) can be verified by taking \( p \to \infty \) in
\[ \langle |f|^{1/p} \rangle_Q \leq [w] A_{p} \langle |f| \rangle_Q; \]
see 2 in Remark 2.1.

4 Two other proofs of the upper bound

Here, we give two other proofs of the upper bound without reverse Hölder inequality.

4.1 Method based on a dual inequality

The key inequality in this method is the following dual inequality with local sharp maximal operator due to Lerner [7]:

**Proposition 4.1.** There exists \( c_n > 0 \) so that for any \( \lambda < c_n \)
\[ \frac{1}{|Q|} \int_Q |f - \langle f \rangle_Q| g dx \leq c_n \int_Q M^*_f M g dx, \]
where \( M^*_f(x) = \sup_{Q \ni x} \inf_{c \in C} (\chi_{Q}(f - c))^*(\lambda|Q|), \) \( 0 < \lambda < 1 \) and \( g^* \) means the non-increasing rearrangement of \( g. \)

Using this proposition, we can immediately show the optimal upper bound (1) as follows:

**Proof of (1).**
\[ \langle |f - \langle f \rangle Q;w| \rangle_{Q,w} \leq 2 \langle |f - \langle f \rangle Q| \rangle_{Q,w} \]
\[ \leq c_n \frac{1}{w(Q)} \int_Q M^*_f M g dx \]
\[ \leq c_n \|f\|_{BMO} \|w\|_{A_{\infty}}. \]
4.2 Method based on another representation of $\|w\|_{A_\infty}$

Next, we give a proof of (1) by using another representation of $\|w\|_{A_\infty}$.

**Proposition 4.2.**

$$\|w\|_{A_\infty} \approx \sup_{Q,f} \frac{\langle |f| \rangle_{Q,w}}{\|f\|_{\exp L(Q)}},$$

where $\|f\|_{\exp L(Q)}$ is defined by

$$\inf \left\{ \lambda > 0 : \left( \exp \left( \frac{|f|}{\lambda} \right) - 1 \right) \leq 1 \right\}.$$

**Remark 4.1.** This form should be compared with

$$[w]_{A_\infty} = \sup_{Q,f} \frac{\exp(\log |f|)_{Q}}{\langle |f| \rangle_{Q,w}},$$

see for example [3].

We show this proposition and then give a proof of (1).

**Proof.** By H"older inequality in the context of Orlicz spaces, we have

$$\langle |f| \rangle_{Q,w} \leq c_n \frac{|Q|}{w(Q)} \|f\|_{\exp L(Q)} \|w\|_{L,\log L(Q)} \leq c_n \|w\|_{A_\infty} \|f\|_{\exp L(Q)}.$$

On the other hand, for a cube $Q$, from the duality, we can find a function $g \in \exp L(Q)$ such that

$$\|w\|_{L,\log L(Q)} \|g\|_{\exp L(Q)} \leq c_n \langle |g| \rangle_{Q,w},$$

and then, by using the representation of $\|w\|_{A_\infty}$ in Remark 2.3, one obtains

$$\|w\|_{A_\infty} \leq c_n \sup_{Q} \frac{1}{\langle w \rangle_{Q}} \|w\|_{L,\log L(Q)} \leq c_n \sup_{Q} \frac{\langle |g| \rangle_{Q,w}}{\|g\|_{\exp L(Q)}} \leq c_n \sup_{Q,f} \frac{\langle |f| \rangle_{Q,w}}{\|f\|_{\exp L(Q)}}.$$

Proof of (1). From Proposition 4.2, it holds

$$\langle |f| \rangle_{Q,w} \leq c_n \|w\|_{A_\infty} \|f\|_{\exp L(Q)}.$$  \hspace{1cm} (7)

Therefore,

$$\langle |f - (f)_{Q,w}| \rangle_{Q,w} \leq 2 \langle |f - (f)_{Q}| \rangle_{Q,w} \leq c_n \|w\|_{A_\infty} \|f - (f)_{Q}\|_{\exp L(Q)} \leq c_n \|w\|_{A_\infty} \|f\|_{BMO}.$$

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