A_∞ constants between BMO and weighted BMO

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Abstract

In this short article, we consider estimates of the ratio

$$\|f\|_{BMO(w)}/\|f\|_{BMO}$$

from above and below, where w belongs to Muckenhoupt class A_∞. The upper bound of the ratio was proved by Hytönen and Pérez in [6] with the optimal power. We establish the lower bound of the ratio and give two other proofs of the upper bound.

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1 Introduction

In this paper, we are interested in estimates of the ratio

$$\|f\|_{BMO(w)}/\|f\|_{BMO}$$

with respect to the weight w belonging to Muckenhoupt class A_∞. Our purposes are to establish the lower bound of the ratio and to give two other proofs of the upper bound due to Hytönen and Pérez in [6].
In [9], Muckenhoupt and Wheeden proved that for any $w \in A_\infty$, it holds $BMO(w) = BMO$. Recently, Hytönen and Pérez [6] gave the upper bound of the ratio;

$$\|f\|_{BMO(w)} \le c_n \|w\|_{A_\infty} \|f\|_{BMO}, \tag{1}$$

where $\|w\|_{A_\infty}$ is Wilson's A_∞ constant, see Definition 2.4. Moreover, they [6] proved that the power 1 of $\|w\|_{A_\infty}$ cannot be replaced by any smaller quantity. Main result in this paper is the following lower bound of the ratio.

Theorem 1.1. *There exists $c_n > 0$ such that for any $w \in A_\infty$,*

$$\|f\|_{BMO} \le c_n \log(2[w]_{A_\infty}) \|f\|_{BMO(w)}. \tag{2}$$

Remark 1.1. 1. *We do not know whether the order $\log(2[w]_{A_\infty})$ is optimal or not.*

2. *If the inequality*

$$\|f\|_{BMO} \le c_n \|f\|_{BMO(w)}$$

is true, the exponent 0 of $[w]_{A_\infty}$ is optimal. In fact, for $w(x) = t\chi_E(x) + \chi_{E^c}(x) \in A_1$ with a compact set $E \subset \mathbb{R}^n$ and large t, it follows

$$\|\log w\|_{BMO} = \|\log w\|_{BMO(w)} = \frac{1}{2}\log t.$$

We will give two other proofs of the upper bound (1). To verify (1) in [6], they used the reverse Hölder inequality;

$$\langle w^{r_w}\rangle_Q^{1/r_w} \le 2\langle w\rangle_Q,$$

for a cube $Q \subset \mathbb{R}^n$ and $r_w = 1 + (c_n\|w\|_{A_\infty})^{-1}$. Our proofs of (1) are not based on this type inequality. Our main tools are a dual inequality with the sharp maximal operator $M_\lambda^\sharp$ due to Lerner [7] and another representation of $\|w\|_{A_\infty}$.

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These estimates are related to the sharp weighted inequalities for Calderón-Zygmund operators. The sharp weighted inequality for an operator T means the inequality

$$\|Tf\|_{L^p(w)} \le c_{n,p,T}\Phi([w]_{A_p})\|f\|_{L^p(w)} \tag{3}$$

with the optimal growth function Φ on $[1,\infty)$ in the sense that Φ cannot be replaced by any smaller function. Recently, Hytönen [5] solved so-called A_2 conjecture i.e. for any Calderón-Zygmund operator T (3) holds with $\Phi(t) = t$. By combining this with the extrapolation theorem in [1], we can see that for $p \in (1,\infty)$ (3) with $\Phi(t) = t^{\max(1,1/(p-1))}$ holds and the exponent $\max(1, 1/(p-1))$ is optimal. From the upper bound (1), it immediately follows

$$\|Tf\|_{BMO(w)} \le c_n\|T\|_{L^\infty\to BMO}\|w\|_{A_\infty}\|f\|_{L^\infty(w)}$$

which corresponds to(3) with $p = \infty$. Further, they [6] showed the optimality of the exponent 1 of $\|w\|_{A_\infty}$. On the other hand, our lower bound (2) yields that

$$\|T\|_{BMO(w)\to BMO(w)} \le \\ c_n\|T\|_{BMO\to BMO}\|w\|_{A_\infty}\log(2[w]_{A_\infty}).$$

2 Preliminaries

We say w a *weight* if w is a non-negative and locally integrable function. For a subset $E \subset \mathbb{R}^n$, χ_E means the characteristic function of E and $|E|$ denotes the volume of E. By a " cube " Q we mean a cube in $\mathbb{R}^n$ with sides parallel to the coordinate axes. Throughout this article we use the following notations; $w(Q) = \int_Q w dx$, $\langle f\rangle_Q = \frac{1}{|Q|}\int_Q f dx$ and $\langle f\rangle_{Q;w} = \frac{1}{w(Q)}\int_Q fw dx$.

Firstly, we recall definitions of Muckenhoupt classes A_p and BMO spaces.

Definition 2.1. *A weight w is said to be in the Muckenhoupt class if the following A_p constant $[w]_{A_p}$ is finite;*

$$[w]_{A_1} := \sup_Q \langle w\rangle_Q \|w^{-1}\|_{L^\infty(Q)},$$

$$[w]_{A_p} := \sup_Q \langle w\rangle_Q \langle w^{1-p'}\rangle_Q^{p-1}, \text{ for } p \in (1,\infty)$$

and

$$[w]_{A_\infty} := \sup_Q \langle w\rangle_Q \exp(\langle \log w^{-1}\rangle_Q).$$

Remark 2.1. *1. $[w]_{A_p} \ge 1$ and $p < q \Rightarrow A_p \subset A_q$.*

2. Because $\lim_{r\searrow 0}\langle |f|^r\rangle_Q^{1/r} = \exp\langle \log|f|\rangle_Q$, it follows $\lim_{p\nearrow\infty}[w]_{A_p} = [w]_{A_\infty}$.

Definition 2.2. *With a weight w, one defines $BMO(w)$ as the space of locally integrable functions f with respect to w such that*

$$\|f\|_{BMO(w)} = \sup_Q \langle |f - \langle f\rangle_{Q;w}|\rangle_{Q;w} < \infty.$$

Remark 2.2. *There is another weighted BMO, BMO_w, which is defined by*

$$\|f\|_{BMO_w} = \sup_Q \inf_{c\in\mathbb{C}} \frac{1}{w(Q)} \int_Q |f - c| dx < \infty.$$

It is known that for $w \in A_\infty$, this space is the dual space of the weighted Hardy space $H^1(w)$, i.e. $BMO_w = (H^1(w))^*$, see [3].

The definition of Wilson's constant $\|w\|_{A_\infty}$ uses the restricted Hardy-Littlewood maximal operator.

Definition 2.3. *For any measurable subset $E \subset \mathbb{R}^n$, Hardy-Littlewood maximal operator M_E restricted to E is defined by*

$$M_E f(x) = \sup_{E\supset R\ni x} \langle |f|\rangle_R,$$

where the supremun is taken over all cubes R containing x and included in E. When $E = \mathbb{R}^n$, we write $M = M_E$.

Definition 2.4.

$$\|w\|_{A_\infty} = \sup_Q \frac{1}{w(Q)} \int_Q M_Q w dx.$$

Remark 2.3. *1.* $w \in A_\infty \iff \|w\|_{A_\infty} < \infty$, *and* $\|w\|_{A_\infty} \le c_n [w]_{A_\infty}$.

2. There are several equivalent quantities to $\|w\|_{A_\infty}$;

$$\begin{aligned}\|w\|_{A_\infty} &\approx \sup_Q \frac{1}{w(Q)} \int_Q w \log\left(e + \frac{1}{\langle w\rangle_Q}\right) dx \\ &\approx \sup_Q \frac{1}{\langle w\rangle_Q} \|w\|_{L\log L(Q)} \\ &\approx \sup_Q \frac{1}{w(Q)} \int_{2Q} M(\chi_Q w) dx \\ &\approx \sup_Q \frac{1}{w(Q)} \int_{2Q} |R_j(\chi_Q w)| dx,\end{aligned}$$

where $j = 1, \cdots, n$, $\|f\|_{L\log L(Q)}$ *is defined by*

$$\inf\left\{\lambda > 0; \left\langle \frac{|f|}{\lambda} \log\left(e + \frac{|f|}{\lambda}\right)\right\rangle_Q \le 1\right\}$$

and R_j *is the j-th Riesz transformation. The first and second equivalences are proved by* $L\log L$ *theory due to Stein [10]. The third and fourth ones were proved by Fujii [2]. From the third representation, we obtain an inequality*

$$M(\chi_Q w)(2Q) \le c_n \|w\|_{A_\infty} w(Q),$$

which should be compared with the doubling inequality with $[w]_{A_\infty}$;

$$w(2Q) \le 2^{2^n} [w]_{A_\infty}^{2^n} w(Q),$$

see for example [4].

3 Lower bound

Owing to a version of John-Nirenberg inequality in the context of non-doubling measures in [8], one obtains a variant of the equivalence

$$\|f\|_{BMO} \approx \sup_Q \|f - \langle f\rangle_Q\|_{\exp L(Q)} \tag{4}$$

with constants independent of weights.

Lemma 3.1. *There exist constants* $c_1, c_2 > 0$ *such that for any* $w \in A_\infty$, *it follows*

$$\begin{aligned}c_1 \sup_Q \|f - \langle f\rangle_{Q;w}\|_{\exp L(Q;w)} &\le \|f\|_{BMO(w)} \\ &\le c_2 \sup_Q \|f - \langle f\rangle_{Q;w}\|_{\exp L(Q;w)},\end{aligned}$$

where $\|f\|_{\exp L(Q;w)}$ *is defined by*

$$\inf\left\{\lambda > 0; \left\langle \exp\left(\frac{|f|}{\lambda}\right) - 1\right\rangle_{Q;w} \le 1\right\}.$$

With this lemma, we give a proof of our lower bound, Theorem 1.1.

Proof of Theorem 1.1. From the definition of $\|f\|_{\exp L(Q;w)}$ above, it follows

$$\left\langle \exp\left(\frac{|f|}{\|f\|_{\exp L(Q;w)}}\right)\right\rangle_{Q;w} \le 2.$$

By using the version of Jensen's inequality

$$\exp\langle g\rangle_Q \le [w]_{A_\infty}\langle \exp(g)\rangle_{Q;w}, \tag{5}$$

one obtains

$$\langle |f|\rangle_Q \le \log(2[w]_{A_\infty})\|f\|_{\exp L(Q;w)}.$$

The proof is completed by this inequality and Lemma 3.1 as follows:

$$\begin{aligned}\langle |f-\langle f\rangle_Q|\rangle_Q &\le 2\langle |f-\langle f\rangle_{Q;w}|\rangle_Q\\ &\le 2\log(2[w]_{A_\infty})\|f-\langle f\rangle_{Q;w}\|_{\exp L(Q;w)}\\ &\le c_n\log(2[w]_{A_\infty})\|f\|_{BMO(w)}.\end{aligned}$$

□

Remark 3.1. *The inequality (5) is equivalent to*

$$\exp\langle \log|f|\rangle_Q \le [w]_{A_\infty}\langle |f|\rangle_{Q;w}, \tag{6}$$

which should be compared with (7). (6) can be verified by taking $p \nearrow \infty$ *in*

$$\langle |f|^{1/p}\rangle_Q^p \le [w]_{A_p}\langle |f|\rangle_{Q;w},$$

see 2 in Remark 2.1.

4 Two other proofs of the upper bound

Here, we give two other proofs of the upper bound without reverse Hölder inequality.

4.1 Method based on a dual inequality

The key inequality in this method is the following dual inequality with local sharp maximal operator due to Lerner [7];

Proposition 4.1. *There exists* $c_n > 0$ *so that for any* $\lambda < c_n$

$$\frac{1}{|Q|}\int_Q |f-\langle f\rangle_Q| g\,dx \le c_n\int_Q M^\sharp_\lambda f M_Q g\,dx,$$

where $M^\sharp_\lambda f(x) = \sup_{Q\ni x}\inf_{c\in\mathbb{C}}\left(\chi_Q(f-c)\right)^*(\lambda|Q|)$, $(0<\lambda<1)$ *and* g^* *means the non-increasing rearrangement of* g.

Using this proposition, we can immediately show the optimal upper bound (1) as follows:
Proof of (1).

$$\begin{aligned}\langle |f-\langle f\rangle_{Q;w}|\rangle_{Q;w} &\le 2\langle |f-\langle f\rangle_Q|\rangle_{Q;w}\\ &\le c_n\frac{1}{w(Q)}\int_Q M^\sharp_\lambda f M_Q w\,dx\\ &\le c_n\|f\|_{BMO}\|w\|_{A_\infty}.\end{aligned}$$

□

4.2 Method based on another representation of $\|w\|_{A_\infty}$

Next, we give a proof of (1) by using another representation of $\|w\|_{A_\infty}$.

Proposition 4.2.

$$\|w\|_{A_\infty} \approx \sup_{Q,f} \frac{\langle |f| \rangle_{Q;w}}{\|f\|_{\exp L(Q)}},$$

where $\|f\|_{\exp L(Q)}$ *is defined by*

$$\inf \left\{ \lambda > 0; \left\langle \exp\left(\frac{|f|}{\lambda}\right) - 1 \right\rangle_Q \leq 1 \right\}.$$

Remark 4.1. *This form should be compared with*

$$[w]_{A_\infty} = \sup_{Q,f} \frac{\exp\langle \log |f| \rangle_Q}{\langle |f| \rangle_{Q;w}},$$

see for example [3].

We show this proposition and then give a proof of (1).

Proof. By Hölder inequality in the context of Orlicz spaces, we have

$$\begin{aligned}\langle |f| \rangle_{Q;w} &\leq c_n \frac{|Q|}{w(Q)} \|f\|_{\exp L(Q)} \|w\|_{L \log L(Q)} \\ &\leq c_n \|w\|_{A_\infty} \|f\|_{\exp L(Q)}.\end{aligned}$$

On the other hand, for a cube Q, from the duality, we can find a function $g \in \exp L(Q)$ such that

$$\begin{aligned}\|w\|_{L \log L(Q)} \|g\|_{\exp L(Q)} &\leq c_n \frac{1}{|Q|} | \int_Q wg dx| \\ &\leq c_n \langle w \rangle_Q \langle |g| \rangle_{Q;w},\end{aligned}$$

and then, by using the representation of $\|w\|_{A_\infty}$ in Remark 2.3, one obtains

$$\begin{aligned}\|w\|_{A_\infty} &\leq c_n \sup_Q \frac{1}{\langle w \rangle_Q} \|w\|_{L \log L(Q)} \\ &\leq c_n \sup_Q \frac{\langle |g| \rangle_{Q;w}}{\|g\|_{\exp L(Q)}} \\ &\leq c_n \sup_{Q,f} \frac{\langle |f| \rangle_{Q;w}}{\|f\|_{\exp L(Q)}}.\end{aligned}$$

□

Proof of (1). From Proposition 4.2, it holds

$$\langle |f| \rangle_{Q;w} \leq c_n \|w\|_{A_\infty} \|f\|_{\exp L(Q)}. \tag{7}$$

Therefore,

$$\begin{aligned}\langle |f - \langle f \rangle_{Q;w}| \rangle_{Q;w} &\leq 2\langle |f - \langle f \rangle_Q| \rangle_{Q;w} \\ &\leq c_n \|w\|_{A_\infty} \|f - \langle f \rangle_Q\|_{\exp L(Q)} \\ &\leq c_n \|w\|_{A_\infty} \|f\|_{BMO}.\end{aligned}$$

□

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