Bounded global solutions to a Keller-Segel system with non-diffusive chemical in $\mathbb{R}^n$

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Abstract

We consider a chemotaxis system, on the whole space, without diffusive term for the chemical substance and prove that even if the chemical sensitivity is large, there exist bounded global solutions, when the initial data is sufficiently small.

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1 Introduction

The present article deals with a chemotaxis system

$$\text{(K-S)}\quad \begin{cases} \partial_t u = \Delta u - \chi \nabla \cdot \left(u \dfrac{\nabla v}{v} \right), & t > 0, \quad x \in \mathbb{R}^n \\ \partial_t v = u v^{\lambda}, \quad t > 0, \quad x \in \mathbb{R}^n \\ u(0,x) = u_0(x) \quad v(0,x) = v_0(x), \quad x \in \mathbb{R}^n. \end{cases}$$

Here, u indicates the unknown cell density of the chemotactic species, v the unknown density of non-diffusive chemical substance, which is produced by the species. $\chi > 0$ is the chemotactic sensitivity and λ is the growth rate of chemical. The term $\dfrac{\nabla v}{v} = \nabla \log v$ means that the magnitude of the sensation of cell from the chemical substance follows from the Weber-Fechner law. This system is a particular case of Keller-Segel system [6], [7] and was derived as a model of self-reinforced random walks [13], [11]. Also, the similar systems are applied as mathematical model as haptotaxis and angiogenesis models, for example [3], [4]. One of main features of the system is that ODE does not contain Δv, that is, the absence of diffusion term for the chemical. There are many papers which studied Keller-Segel systems with the diffusion term in the second equation. Levine and Sleeman [9] investigated finite time blow-up phenomena for the system (K-S) with $\lambda = 1$ in one dimensional case. D. Li, K. Li and Zhao [10] treated with the case $\lambda = 1$ in which the system is transformed into a hyperbolic-parabolic system through the transform $V = -\frac{\nabla v}{v}$, and constructed local and global solutions in Sobolev spaces with positive smoothness. In smooth bounded domains in $\mathbb{R}^n$ with $\lambda \leq 1$, Rascle [12] and Yang, Chen and Liu [17] showed the existence of global solutions. Corrias, Perthame and Zaag [3], [4] studied the same topics in a general system with $\lambda \neq 1$. In [14], asymptotic behavior of radial symmetric solutions is studied. When $\lambda \in [0, 1)$ and $n = 1$, Kang, Stevens and Velázquez [5] proved that for some initial data the corresponding solutions u tends to Dirac mass as $t \to \infty$.

For simplicity, we restrict us to the case $\lambda \neq 1$ in this article. Through transformations

$$z = \frac{v^{1-\lambda}}{1-\lambda} \quad \text{with} \quad \theta = \frac{\chi}{1-\lambda} \in \mathbb{R},$$

(K-S). is turned into

$$(E_\theta)\quad \begin{cases} \partial_t u = \Delta u - \theta \nabla \cdot \left(u \dfrac{\nabla z}{z} \right), & x \in \mathbb{R}^n, \quad t > 0 \\ \partial_t z = u, \quad x \in \mathbb{R}^n, \quad t > 0 \\ u(0,x) = u_0(x) \quad z(0,x) = z_0(x) \left(= \dfrac{v_0(x)^{1-\lambda}}{1-\lambda} \right), & x \in \mathbb{R}^n. \end{cases}$$

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In this article, we assume the lower bound

$$|z_0(x)| \geq c_0 > 0 \text{ for all } x \in \mathbb{R}^n, \tag{1}$$

from a technical reason. Owing to the absence of the diffusive term Δz in ODE, the spatial regularity of z and ∇z is not better than that of z_0 and z_0, respectively. Also, z is monotonically increasing for time, thus $L^\infty(\mathbb{R}^n)$ norm of $1/z$ is a bounded function of t, whenever u or t is small as compared with c_0. This $L^\infty((0,T)\times\mathbb{R}^n))$ boundedness of $1/z$ will be repeatedly used in this article. Of course, z is represented by u. The structure of the representation and the boundedness of $1/z$ imply that some Besov norms of $1/z$ are bounded by the same norms of z.

Recently, Ahn and Kang [1] gave a global existence theorem for (E_θ) on smooth, bounded domains of arbitrary dimensions with $\theta \in (0,1]$ and $u_0, z_0 \in L^\infty \cap W^{1,p}$, $(p > n)$. Their solutions (u,z) belong to the class $L^\infty(0,T;L^\infty) \cap L^p(0,T;W^{1,p})$ for all $T < \infty$, which means that the chemotactic collapse does not occur in finite time. They also assumed the lower bound (1) to control the inverse $1/z$ in the non-linear term. Using a Lyapunov function and the boundedness of $1/z$, which is derived from (1), they obtained an inequality:

$$\partial_t z(t,x) \leq c z(t,x)^\theta.$$

Their condition on θ ensures the boundedness of z, and then u. The purpose of the present article is to give an existence theorem for large θ, under the smallness condition on the initial data (u_0, z_0). But we need the restriction on the dimension: $n \geq 3$. For large θ, it seems that their [1] method is failure. Our method differs from them. To achieve the result, we establish two existence theorems. In the former, we construct global-in-time solutions, having uniformly bounds for time, see Proposition 1.1. But, the solutions may be unbounded. The condition $n \geq 3$ is used to show this proposition. The later states the existence of local-in-time "bounded"solutions, see Proposition 1.2. Smallness condition on the initial data yields the consistency between these solutions in short time interval. Combining these propositions, it turns out that our global solutions are spatial bounded in a time interval. We apply estimates in Proposition 1.1 as a priori ones and then the spatial boundedness is extended. In [15], the authors constructed global solutions of (E_θ) for all θ. But properties of the solutions are not enough to apply the argument in this paper.

Solutions we will construct are mild solutions, more precisely, we construct solutions to the integral equations;

$$\text{(I.E.)} \quad u(t) = e^{t\Delta}a - \theta D[u](t),$$

where

$$D[u](t) = \int_0^t e^{(t-\tau)\Delta}\nabla\cdot F(\tau)d\tau, \; F(\tau) = F_u(\tau) = u(\tau)\frac{\nabla z(\tau)}{z(\tau)}$$

and

$$z(t) = z_0 + \int_0^t u(\tau)d\tau.$$

Our existence theorems involve several norms:

$$\|u\|_{X_T} = \sum_{j=1}^5 \|u\|_{X_T^j} \text{ and } \|u\|_{Y_T} = \sum_{j=1}^7 \|u\|_{Y_T^j}$$

where

$$\begin{cases} \|u\|_{X_T^1} = \int_0^T \|u(t)\|_{L^\infty}dt, & \|u\|_{X_T^2} = \int_0^T \|u(t)\|_{\dot{B}^s_{p,\infty}}dt, \\ \|u\|_{X_T^3} = \sup_{t<T}\left\|\nabla\int_0^t u(\tau)d\tau\right\|_{L^q}, & \|u\|_{X_T^4} = \int_0^T \|\nabla u(t)\|_{L^\infty}dt, \\ \|u\|_{X_T^5} = \sup_{t<T}\left\|\nabla\int_0^t u(\tau)d\tau\right\|_{\dot{B}^s_{p,\infty}} \end{cases}$$

and

$$\begin{cases} \|u\|_{Y_T^1} = \sup_{t<T}\|u(t)\|_{L^\infty}, & \|u\|_{Y_T^2} = \sup_{t<T}\|u(t)\|_{L^1}, \\ \|u\|_{Y_T^3} = \sup_{t<T} t^{s/2}\|u(t)\|_{\dot{B}^s_{\infty,\infty}}, & \|u\|_{Y_T^4} = \sup_{t<T} t^{s/2}\|u(t)\|_{\dot{B}^s_{1,\infty}}, \\ \|u\|_{Y_T^5} = \sup_{t<T} t^{1/2}\|\nabla u(t)\|_{L^\infty}, & \|u\|_{Y_T^6} = \sup_{t<T} t^{1/2}\|\nabla u(t)\|_{L^1}, \\ \|u\|_{Y_T^7} = \sup_{t<T} t^{(1+s)/2}\|\nabla u(t)\|_{\dot{B}^s_{p,\infty}}. \end{cases}$$

X_T and Y_T norms are used in Propositions 1.1 and 1.2, respectively. For simplicity, we use the following notations: $\|\cdot\|_{X_T^{a,b,\cdots}} = \|\cdot\|_{X_T^a} + \|\cdot\|_{X_T^b} + \cdots$ and $\|\cdot\|_{Y_T^{a,b,\cdots}} = \|\cdot\|_{Y_T^a} + \|\cdot\|_{Y_T^b} + \cdots$. Further, the following norms are used for the initial data;

$$\|z_0\|_X = \|z_0\|_Y + \|\nabla z_0\|_{L^q}, \ \|z_0\|_Y = \|z_0\|_{\dot{B}^s_{p,\infty}} + \|\nabla z_0\|_{L^\infty \cap \dot{B}^s_{p,\infty}},$$

and

$$\|(u_0, z_0)\|_X = \|u_0\|_{L^1 \cap L^\infty} + \|z_0\|_X, \ \|(u_0, z_0)\|_Y = \|u_0\|_{L^1 \cap L^\infty} + \|z_0\|_Y.$$

Our main result reads as follows and asserts that even if θ is large, when the initial data (u_0, z_0) is sufficiently small, spatial bounded solutions globally exist.

Theorem 1.1. *Let $n \geq 3$, $1 < q < n < p < \infty$, $s \in (n/p, 1)$, $\theta \in \mathbb{R}$ and $c_0 > 0$. There exists small $\delta > 0$ such that for any initial data (u_0, z_0) satisfying $\|(u_0, z_0)\|_X \leq \delta$, $z_0 \in L^1_{loc}$ and (1), (I.E.) admits a global solution u fulfilling $\|u\|_{X_\infty} \leq c\delta$ and $\|u\|_{Y_T} < \infty$ for all $T < \infty$. Especially, $\|u(t)\|_{L^\infty} \leq ce^{ct}$ for all $t > 0$.*

Remark 1.1. *In addition, if $z_0 \in L^\infty$, then we can show the uniform bound for z;*

$$\|z(t)\|_{L^\infty} \leq \|z_0\|_{L^\infty} + \|u\|_{X^1_\infty} \text{ for all } t > 0.$$

The following two theorems are used in the proof of Theorem 1.1. The former is global existence theorem for the our system (E_θ) that does not ensure the spatial boundedness of solutions. One of merits of such solutions is several uniformly bounds for time, which are applied as a priori estimates to extend local solutions in the proof of Theorem 1.1. Proposition 1.1 is a modification of a result in our previous work [15]. The previous estimates are not enough for the present argument.

Proposition 1.1. *Let $n \geq 3$, $1 < q < n < p < \infty$, $s \in (n/p, 1)$, $\theta \in \mathbb{R}$ and $c_0 > 0$. There exists $\delta = \delta(n, p, q, \theta, c_0) > 0$ such that for any $T \in (0, \infty]$, $u_0 \in L^1 \cap L^\infty$ and $z_0 \in L^1_{loc}$ satisfying (1) and $\|(u_0, z_0)\|_X \leq \delta$, then* (I.E.) *admits a unique global solution u in the closed ball*

$$B_\delta(X_T) = \left\{u \in L^1(0, T; L^\infty); \|u\|_{X_T} \leq 2C_X\delta\right\},$$

with a constant $C_X \geq 1$, for which $\|e^{\cdot\Delta}u_0\|_{X_T} \leq C_X\|u_0\|_{L^1 \cap L^\infty}$ holds.

Remark 1.2. *1. The restriction on dimension n steams from the following: if $n = 1, 2$,*

$$\|e^{\cdot\Delta}u_0\|_{X^1_\infty} = \int_0^\infty \|e^{t\Delta}u_0(t)\|_{L^\infty} dt \approx \|u_0\|_{\dot{B}^{-2}_{\infty,1}} = \infty,$$

for $u_0 \in L^1 \cap L^\infty$ in general.

2. $z_0(x) := c(e^{-|x|^2} + 1)$ with small positive c satisfies conditions in Theorem 1.1.

The later is a local existence theorem, in which solutions are local in time but spatial bounded. Proposition 1.2 is also a modification of a result in our previous work [15].

Proposition 1.2. *Let $n \geq 1$, $p \in (n, \infty)$, $s \in (n/p, 1)$, $\theta \in \mathbb{R}$ and $c_0 > 0$. For any (u_0, z_0) satisfying $\|(u_0, z_0)\|_Y < \infty$, $z_0 \in L^1_{loc}$ and (1), there exists a constant $C > 0$ so that* (I.E.) *admits a unique local solution u up to*

$$T = C \min\left(\|(u_0, z_0)\|_Y^{-1}, \ \|(u_0, z_0)\|_Y^{-2/(1-n/p)}, \ \|(u_0, z_0)\|_Y^{-4/(1-n/p+s)}\right),$$

in the closed ball $B(Y_T)$ defined by

$$\left\{u \in BC((0, T) \times \mathbb{R}^n) \cap BC((0, T); L^1 \cap L^\infty); \|u\|_{Y_T} \leq 2C_Y\|(u_0, z_0)\|_Y\right\},$$

with a constant $C_Y \geq 1$, for which $\|e^{\cdot\Delta}u_0\|_{Y_T} \leq C_Y\|u_0\|_{L^1 \cap L^\infty}$ holds. BC means a space of all bounded continuous functions.

This paper is organized as follows. In next section, we recall the definition and equivalence norms of (homogeneous) Besov space and collect our basic estimates. Applying Propositions 1.1 and 1.2, we prove main result in Section 3. The propositions are proved in Sections 4 and 5, respectively.

2 Preliminaries

Throughout this paper we use the following notations. $A \approx B$ means $c_1 B \le A \le c_2 B$ with some $c_1, c_2 > 0$. In what follows, c denotes a constant that is independent of the functions involved, which may differ from line to line.

Let us recall the definition of Besov spaces. We fix $\varphi \in \mathcal{S}(\mathbb{R}^n)$ satisfying supp $\varphi \subset \{1/2 \le |\xi| \le 2\}$ and $\sum_{j\in\mathbb{Z}} \varphi\left(\frac{\xi}{2^j}\right) = 1$ for $\xi \in \mathbb{R}^n \backslash \{0\}$, and then $\varphi_j(D) f = \mathcal{F}^{-1}\left[\varphi\left(\frac{\cdot}{2^j}\right)\hat{f}(\cdot)\right]$, where $\mathcal{F} = \hat{}$ is Fourier transform and $\mathcal{F}^{-1} = \check{}$ is its inversion transform.

Definition 2.1. *Let $0 < p, q \le \infty$ and $s \in \mathbb{R}$. Besov space $\dot{B}^s_{p,q}$ is defined to be the space of $f \in \mathcal{S}'$ modulo polynomials such that*

$$\|f\|_{\dot{B}^s_{p,q}} = \left\| \left\{ 2^{js} \|\varphi_j(D) f\|_{L^p} \right\}_{j\in\mathbb{Z}} \right\|_{l^q} < \infty.$$

We make use of two equivalence norms of Besov spaces established by Triebel [16]. The first one is the following:

$$\|f\|_{\dot{B}^s_{p,\infty}} \approx \sup_{h\in\mathbb{R}^n\backslash\{0\}} \frac{\|f(\cdot + h) - f(\cdot)\|_{L^p}}{h^s}, \tag{2}$$

where $1 \le p < \infty$ and $0 < s < 1$.

Triebel [16] also showed that Besov spaces are also characterized by means of the heat semigroup $e^{t\Delta}$: with a non-negative integer $m > s/2$

$$\begin{aligned} \|f\|_{\dot{B}^s_{p,q}} &\approx \left(\int_0^\infty \left(t^{m-s/2} \|(-\Delta)^m e^{t\Delta} f\|_{L^p} \right)^q \frac{dt}{t} \right)^{1/q}, \quad q < \infty, \\ \|f\|_{\dot{B}^s_{p,\infty}} &\approx \sup_{t<\infty} t^{m-s/2} \|(-\Delta)^m e^{t\Delta} f\|_{L^p}. \end{aligned} \tag{3}$$

Of course, $e^{t\Delta} f$ is defined by

$$e^{t\Delta} f(x) = f * G_{\sqrt{t}},$$

where $g_t(x) = t^{-n} g(x/t)$ and $G(x) = (4\pi)^{-n/2} e^{-|x|^2/4}$.

We handle the nonlinear term by Leibniz's rule. Since Lemma 2.1 is a well-known result, we omit the proof. For example, it can be verified by using the paraproduct formula due to Bony [2].

Lemma 2.1. *For $s > 0$ and $1 \le q \le p \le \infty$,*

$$\|fg\|_{\dot{B}^s_{q,\infty}} \le c\|f\|_{\dot{B}^s_{q,\infty}} \|g\|_{L^\infty} + \|f\|_{L^{p'}} \|g\|_{\dot{B}^s_{p,\infty}},$$

where $1/p' = 1/q - 1/p$.

Decay estimates of the heat semigroup on Besov spaces are basic tools in Sections 4 and 5.

Lemma 2.2 ([8])**.** *If $1 \le q \le p \le \infty$ and $-\infty < \beta < \alpha < \infty$, it follows*

$$\|e^{t\Delta} f\|_{\dot{B}^\alpha_{p,1}} \le c t^{-n(1/p-1/q)/2-(\alpha-\beta)/2} \|f\|_{\dot{B}^\beta_{q,\infty}}.$$

The following estimate is applied in estimates of X^5_T norm.

Lemma 2.3 ([15])**.** *Let $n \ge 1$, $F = (f_1, \cdots, f_n)$ and $r \in (1, \infty)$. For any $T \in (0, \infty]$, it holds*

$$\left\| \int_0^T \nabla \int_0^t e^{(t-\tau)\Delta} \nabla \cdot F(\tau) d\tau dt \right\|_{L^r} \le c \int_0^T \|F(\tau)\|_{L^r} d\tau.$$

To control Y^7_T norm in Proposition 1.2, we make use of next lemma.

Lemma 2.4 ([15])**.** *For any $s \in \mathbb{R}$, $1 \le p \le \infty$ and $0 \le t_0 < t < \infty$, it holds*

$$\left\| \int_{t_0}^t e^{(t-\tau)\Delta} \nabla \cdot F(\tau) d\tau \right\|_{\dot{B}^s_{p,\infty}} \le c \sup_{t_0<\tau<t} \|F(\tau)\|_{\dot{B}^{s-1}_{p,\infty}}.$$

Remark 2.1. *This estimate is a kind of end-point estimates. Taking the Besov norm inside the time integral seems to violate the inequality, because such estimate yields a strong singularity at* $\tau = t$*:*

$$\|e^{(t-\tau)\Delta}\nabla\cdot F(\tau)\|_{\dot{B}^s_{p,\infty}} \le c(t-\tau)^{-1}\|F(\tau)\|_{\dot{B}^{s-1}_{p,\infty}}.$$

To avoid this singularity, the proof applied the theory of real interpolation, see [15] for the detail. This is the reason why we use Besov spaces instead of Sobolev spaces.

3 Proof of Theorem 1.1

Here, we give a proof of Theorem 1.1 by using Propositions 1.1 and 1.2.

From Propositions 1.1 and 1.2, there exist $\delta > 0$ and $T \in (0,\infty)$ for which the statements of both propositions hold. Let $u \in B_\delta(X_\infty)$ and $v \in B(Y_T)$ denote global and local solutions with the initial data (u_0, z_0) constructed in the propositions, respectively. It is not hard to see that the local solution v also belongs to X_T and has the inequalities:

$$\|v\|_{X_T} \le c\max\left(T, T^{(1-s)/2}\right)\|v\|_{Y_T} \le c\max\left(T, T^{(1-s)/2}\right)\|(u_0,z_0)\|_Y.$$

This means that there is small $T_0 \in (0,T)$, depending on δ and $\|(u_0,z_0)\|_Y$, so that $v \in B_\delta(X_{T_0})$. From the uniqueness in $B_\delta(X_{T_0})$ in Proposition 1.1, it holds $u(t,x) = v(t,x)$ a.e. $(t,x) \in (0,T_0)\times\mathbb{R}^n$, which implies $u \in L^\infty((0,T_0)\times\mathbb{R}^n)$. Because the set E defined by

$$\{T \in (0,\infty];\, u \in BC((0,T)\times\mathbb{R}^n)\cap BC((0,T);L^1\cap L^\infty) \\ \text{with } \|u\|_{Y_{T'}} < \infty \text{ for all } T' \in (0,T)\}$$

is not empty, we define $T_{max} := \sup E \in [T_0,\infty]$. Here we assume that $T_{max} < \infty$.

For any $T \in (0,T_{max})$, because $u(T) \in L^1\cap L^\infty$ and $z(T) \in \dot{B}^s_{p,\infty}$ with $\nabla z(T) \in L^\infty\cap\dot{B}^s_{p,\infty}$ and $|z(T,x)| \ge c > 0$ for all $x \in \mathbb{R}^n$, from Proposition 1.2 and the maximality of T_{max}, we can see that

$$c\|(u(T),z(T))\|_Y^{-\alpha} \le T_{max} - T$$

for $\alpha \in \{1,\ 2/(1-n/p),\ 4/(1-n/p+s)\}$, which implies

$$\limsup_{T\nearrow T_{max}}\|(u(T),z(T))\|_Y = \infty. \tag{4}$$

On the other hand, using the uniform bound for time: $\|u\|_{X_\infty} \le 2C_X\delta$, we can deny (4). Indeed, firstly, the mass is conserved: $\|u(T)\|_{L^1} = \|u_0\|_{L^1}$. Since

$$\|u(T)\|_{L^\infty} \le \|u_0\|_{L^\infty} + \int_0^T h(T-t)\|u(t)\|_{L^\infty}dt,$$

where $h(T-t) = c(T-t)^{-1/2}(\|\nabla z_0\|_{L^\infty} + \|u\|_{X^4_\infty})$, applying a version of Gronwall's inequality, one obtains the uniform bound of $\|u(T)\|_{L^\infty}$;

$$\|u(T)\|_{L^\infty} \le c\|u_0\|_{L^\infty}\exp\left(c(\|\nabla z_0\|_{L^\infty} + \|u\|_{X^4_\infty})^2T\right) \text{ for all } T \in (0,T_{max}).$$

Moreover, it holds

$$\begin{cases} \|z(T)\|_{\dot{B}^s_{p,\infty}} \le \|z_0\|_{\dot{B}^s_{p,\infty}} + \|u\|_{X^2_\infty}, \\ \|\nabla z(T)\|_{L^\infty} \le \|\nabla z_0\|_{L^\infty} + \|u\|_{X^4_\infty} \text{ and} \\ \|\nabla z(T)\|_{\dot{B}^s_{p,\infty}} \le \|\nabla z_0\|_{\dot{B}^s_{p,\infty}} + \|u\|_{X^5_\infty}. \end{cases}$$

Hence,

$$\sup_{T<T_{max}}\|(u(T),z(T))\|_Y < \infty$$

is conformed. Then $T_{max} = \infty$ and we have for all $t > 0$, $\|u(t)\|_{L^\infty} \le ce^{ct}$.

4 Proof of Proposition 1.1

Define a map Φ by

$$\Phi[u](t) = e^{t\Delta}u_0 - \theta D[u](t).$$

We show that Φ is a contraction mapping from $B_\delta(X_T)$ to itself with some δ. Fix $u \in B_\delta(B_T)$.

First, take $\delta \in (0, c_0/(4C_X))$. See Lemma 4.1 below for C_X. We are able to see from the triangle inequality:

$$\sup_{t<T} \left\| \frac{1}{z(t)} \right\|_{L^\infty} \le \frac{2}{c_0}. \tag{5}$$

Besov norms of $1/z$ are also controlled:

$$\left\| \frac{1}{z(t)} \right\|_{\dot{B}^s_{p,\infty}} \le c\|z(t)\|_{\dot{B}^s_{p,\infty}}, \tag{6}$$

which follows from (2) and (5).

4.1 Linear estimates

Lemma 4.1. *There exists a constant $C_X > 0$ so that for any $T \in (0,\infty]$, it holds*

$$\|e^{\cdot\Delta}u_0\|_{X_T} \le C_X\|u_0\|_{L^1\cap L^\infty}.$$

Proof. Smoothing estimates for the heat semigroup Lemma 2.2 ensure this inequality. □

4.2 Nonlinear estimates

Smoothing estimates for the heat semigroup also yield

$$\sum_{j=1}^{3} \|D[u]\|_{X^j_T} \le c\int_0^T \|F(t)\|_{L^q\cap L^p}dt.$$

Applying the characterization of Besov norms in terms of the heat semigroup (3), one has

$$\|D[u]\|_{X^4_T} \le \int_0^T \int_0^{T-\tau} \|\nabla e^{t\Delta}\nabla\cdot F(\tau)\|_{L^\infty}dtd\tau \le c\int_0^T \|F(t)\|_{\dot{B}^0_{\infty,1}}dt.$$

We use Lemma 2.3 and see

$$\begin{aligned}\|D[u]\|_{X^5_T} &= \sup_{t<T}\sup_{k\in\mathbb{Z}} 2^{ks}\left\|\nabla\int_0^t\int_0^\tau e^{(\tau-\sigma)\Delta}\nabla\cdot\phi_k(D)F(\sigma)d\sigma d\tau\right\|_{L^p}\\ &\le c\int_0^T \|F(t)\|_{\dot{B}^s_{p,\infty}}dt.\end{aligned}$$

Because the condition $n/p < s$ ensures the embedding $L^q \cap \dot{B}^s_{p,\infty} \hookrightarrow \dot{B}^0_{\infty,1}$, we have to control the three norms; L^q, L^p and $\dot{B}^s_{p,\infty}$. From Hölder inequality, (5) and interpolation inequalities, the first two norms are bounded as follows

$$\begin{cases}\displaystyle\int_0^T \|F(t)\|_{L^q}dt \le c\|u\|_{X^1_T}(\|\nabla z_0\|_{L^q} + \|u\|_{X^3_T})\\ \displaystyle\int_0^T \|F(t)\|_{L^p}dt \le c\|u\|_{X^1_T}(\|\nabla z_0\|_{L^q\cap L^\infty} + \|u\|_{X^{3,4}_T}).\end{cases}$$

Combining Lemma 2.1 and (6) derives the estimation

$$\begin{aligned}\|F(t)\|_{\dot{B}^s_{p,\infty}} \le\ & c\|u(t)\|_{\dot{B}^s_{p,\infty}}(\|\nabla z_0\|_{L^\infty} + \|u\|_{X^4_T})\\ &+ c\|u(t)\|_{L^\infty}(\|\nabla z_0\|_{\dot{B}^s_{p,\infty}} + \|u\|_{X^5_T})\\ &+ c\|u(t)\|_{L^\infty}(\|\nabla z_0\|_{L^\infty} + \|u\|_{X^4_T})(\|z_0\|_{\dot{B}^s_{p,\infty}} + \|u\|_{X^2_T}).\end{aligned}$$

Therefore, we have

$$\int_0^T \|F(t)\|_{L^q\cap L^p\cap \dot{B}^s_{p,\infty}} dt \le c\|u\|_{X_T}(\|z_0\|_X + \|u\|_{X_T})$$
$$+ c\|u\|_{X_T}(\|z_0\|_X + \|u\|_{X_T})^2$$
$$\le cC_X^2\delta^2 + cC_X^3\delta^3.$$

Thus, for small δ, $\Phi[u] \in B_\delta(X_T)$.

Next, we verify the contraction property of Φ. For $u, v \in B_\delta(B_T)$, we decompose

$$\Phi[u](t) - \Phi[v](t) = -\theta \int_0^t \nabla e^{(t-\tau)}(A(\tau) + B(\tau) + C(\tau))d\tau,$$

where

$$\begin{cases} A(\tau) = -\dfrac{u(\tau) - v(\tau)}{w(\tau)} \nabla w(\tau), \\ B(\tau) = \dfrac{u(\tau)(z(\tau) - w(\tau))}{z(\tau)w(\tau)} \nabla w(\tau), \\ C(\tau) = -\dfrac{u(\tau)}{z(\tau)} \nabla(z(\tau) - w(\tau)) \end{cases}$$

and $w(\tau) = z_0 + \int_0^\tau v(\sigma)d\sigma$. Because

$$\|\Phi[u] - \Phi[v]\|_{X_T} \le c|\theta| \int_0^T \|A(t) + B(t) + C(t)\|_{L^q\cap L^p\cap \dot{B}^s_{p,\infty}} dt,$$

it remains to give controls of the three norms. It is not hard to see that

$$\int_0^T \|A(t) + B(t) + C(t)\|_{L^q\cap L^p} dt \le c(C_X\delta + C_X^2\delta^2)\|u - v\|_{X_T}$$

and

$$\int_0^T \|A(t) + C(t)\|_{\dot{B}^s_{p,\infty}} dt \le c(C_X\delta + C_X^2\delta^2)\|u - v\|_{X_T}.$$

We use Lemma 2.1 and obtain

$$\|B(t)\|_{\dot{B}^s_{p,\infty}} \le c\|u(t)\|_{\dot{B}^s_{p,\infty}}(\|\nabla z_0\|_{L^\infty} + \|v\|_{X_T^4})\|u - v\|_{X_T^1}$$
$$+ c\|u(t)\|_{L^\infty}\left(\left\|\frac{\nabla w(t)}{z(t)w(t)}\right\|_{\dot{B}^s_{p,\infty}} + \|\nabla w(t)\|_{L^\infty}\right)\|u - v\|_{X_T^{1,2}}$$
$$\le c\left(C_X\delta\|u(t)\|_{\dot{B}^s_{p,\infty}} + C_X^2\delta^2\|u(t)\|_{L^\infty}\right)\|u - v\|_{X_T}.$$

Here we have used

$$\left\|\frac{1}{z(t)w(t)}\right\|_{\dot{B}^s_{p,\infty}} \le c(\|z(t)\|_{\dot{B}^s_{p,\infty}} + \|w(t)\|_{\dot{B}^s_{p,\infty}}) \le cC_X\delta.$$

Taken together, one gets

$$\|\Phi[u] - \Phi[v]\|_{X_T} \le c(C_X^2\delta^2 + C_X^3\delta^3)\|u - v\|_{X_T},$$

which shows that for small δ, Φ is a contraction mapping from $B_\delta(X_T)$ to itself. The proof is completed.

5 Proof of Theorem 1.2

We show that Φ is also a contraction mapping on $B(Y_T)$ for $T > 0$ as in Proposition 1.2. Fix $u \in B(Y_T)$. Taking $T \le c_0/(4C_Y\|(u_0, z_0)\|_Y)$, one can obtain the bound (5), again.

5.1 Linear estimates

Linear estimates can be derived from Lemma 2.2.

Lemma 5.1. *There is a constant $C_Y \geq 1$, for which*

$$\|e^{\cdot\Delta}u_0\|_{Y_T} \leq C_Y\|u_0\|_{L^1\cap L^\infty} \text{ for all } T \in (0,\infty].$$

5.2 Nonlinear estimates

For simplicity, let $R_Y = 2C_Y\|(u_0,z_0)\|_Y$. Applying Lemmas 2.2 and 2.4, we can see that

$$\begin{cases} \|D[u]\|_{Y_T^1\cap Y_T^2} & \leq c\sup_{t<T}\int_0^t (t-\tau)^{-1/2}\|F(\tau)\|_{L^1\cap L^\infty}d\tau, \\ \|D[u]\|_{Y_T^3\cap Y_T^4} & \leq c\sup_{t<T} t^{s/2}\int_0^t (t-\tau)^{-1/2}\|F(\tau)\|_{L^1\cap L^\infty}d\tau, \\ \|D[u]\|_{Y_T^5} & \leq c\sup_{t<T} t^{1/2}\int_0^t (t-\tau)^{-n/(2p)-(2-s)/2}\|F(\tau)\|_{\dot{B}^s_{p,\infty}}d\tau, \\ \|D[u]\|_{Y_T^6} & \leq c\sup_{t<T} t^{1/2}\int_0^t (t-\tau)^{-(2-s)/2}\|F(\tau)\|_{\dot{B}^s_{1,\infty}}d\tau \text{ and} \\ \|D[u]\|_{Y_T^7} & \leq c\sup_{t<T} t^{(1+s)/2}\Big\{\int_0^{t/2} (t-\tau)^{-(2+s)/2}\|F(\tau)\|_{L^p}d\tau \\ & \quad + \sup_{t/2<\tau<t}\|F(\tau)\|_{\dot{B}^s_{p,\infty}}\Big\}. \end{cases} \tag{7}$$

As is the case with Proposition 1.1, it holds from Lemma 2.1

$$\begin{cases} \|F(\tau)\|_{L^1\cap L^\infty} & \leq c\left(1+\tau^{1/2}\right)\|u\|_{Y_T^{1,2}}(\|\nabla z_0\|_{L^\infty}+\|u\|_{Y_T^5}), \\ \|F(\tau)\|_{\dot{B}^s_{p,\infty}} & \leq c\tau^{-s/2}(1+\tau^{1/2})\|u\|_{Y_T^{1,3,4}}(\|\nabla z_0\|_{L^\infty\cap \dot{B}^s_{p,\infty}}+\|u\|_{Y_T^{5,7}}) \\ +c\left(1+\tau^{(3-s)/2}\right) & \|u\|_{Y_T^1}(\|\nabla z_0\|_{L^\infty}+\|u\|_{Y_T^5})(\|z_0\|_{\dot{B}^s_{p,\infty}}+\|u\|_{Y_T^{3,4}}), \\ \|F(\tau)\|_{\dot{B}^s_{1,\infty}} & \leq c\tau^{-s/2}(1+\tau^{1/2})\|u\|_{Y_T^{1,2,4}}(\|\nabla z_0\|_{L^\infty\cap \dot{B}^s_{p,\infty}}+\|u\|_{Y_T^{5,7}}) \\ +c(1+\tau^{(3-s)/2}) & \|u\|_{Y_T^{1,2}}(\|\nabla z_0\|_{L^\infty}+\|u\|_{Y_T^5})(\|z_0\|_{\dot{B}^s_{p,\infty}}+\|u\|_{Y_T^{3,4}}). \end{cases} \tag{8}$$

Substituting these estimates into (7), one has

$$\begin{aligned} \|D[u]\|_{Y_T} \leq\ & cT^{(1-n/p)/2}\left(1+T^{(1+n/p)/2}\right)R_Y^2 \\ & + cT^{(1+s-n/p)/2}\left(1+T^{(3+n/p-s)/2}\right)R_Y^3. \end{aligned}$$

Here, we need the condition $n/p < s < 1$. Remark that Y_T^6 norm does not appear in the right hand sides of (8). This norm is needed in the proof of Theorem 1.1. Thus, we can find a constant $C > 0$, depending on c_0 and $|\theta|$, so that if

$$T = C\min\left(\|(u_0,z_0)\|_Y^{-1},\ \|(u_0,z_0)\|_Y^{-2/(1-n/p)}\right),$$

then $\|\Phi[u]\|_{Y_T} \leq 2C_Y\|(u_0,z_0)\|_Y$.

Next, we check the continuity $\Phi[u] \in BC((0,T);L^\infty)$. The continuity of the linear term is derived from

$$\|e^{t\Delta}u_0 - e^{s\Delta}u_0\|_{L^\infty} \leq \|G_{\sqrt{t}} - G_{\sqrt{s}}\|_{L^1}\|u_0\|_{L^\infty} \to 0 \text{ as } |t-s| \searrow 0.$$

To verify the right continuity of the Duhamel term, we write for $\varepsilon > 0$

$$\begin{aligned} D[u](t+\varepsilon) - D[u](t) =\ & \int_0^t e^{(t-\tau)\Delta}\left(e^{\varepsilon\Delta} - I_d\right)\nabla\cdot F(\tau)d\tau \\ & - \int_t^{t+\varepsilon} e^{(t+\varepsilon-\tau)\Delta}\nabla\cdot F(\tau)d\tau =: I(\varepsilon) + II(\varepsilon). \end{aligned}$$

It is not hard to see that for any $f \in L^\infty \cap \dot{B}^s_{\infty,\infty}$, $e^{t\Delta} f \to f$ in L^∞ as $t \searrow 0$. Since $F \in L^\infty \cap \dot{B}^s_{\infty,\infty}$, $\|I(\varepsilon)\|_{L^\infty} \to 0$ as $\varepsilon \searrow 0$ from Lebesgue's dominated convergence theorem. On the other hand, it follows

$$\begin{aligned}\|II(\varepsilon)\|_{L^\infty} &\le c \int_t^{t+\varepsilon} (t+\varepsilon-\tau)^{-1/2} \|F(\tau)\|_{L^\infty} d\tau \\ &\le c\left(1+(t+\varepsilon)^{1/2}\right)\varepsilon^{1/2} R_Y^2 \to 0 \text{ as } \varepsilon \searrow 0.\end{aligned}$$

To show the left continuity, we write

$$\begin{aligned}D[u](t) - D[u](t-\varepsilon) = &-\int_0^{t-\varepsilon} \left(e^{(t-\tau)\Delta} - e^{(t-\varepsilon-\tau)\Delta}\right) \nabla\cdot F(\tau) d\tau \\ &- \int_{t-\varepsilon}^t e^{(t-\tau)\Delta} \nabla\cdot F(\tau) d\tau =: I(\varepsilon) + II(\varepsilon).\end{aligned}$$

We decompose the first term as follows:

$$\begin{aligned}&e^{(t-\tau)\Delta}\nabla\cdot F(\tau) - e^{(t-\varepsilon-\tau)\Delta}\nabla\cdot F(\tau) \\ &\quad= (t-\tau)^{-1/2} F * (\nabla G)_{\sqrt{t-\tau}} - (t-\varepsilon-\tau)^{-1/2} F * (\nabla G)_{\sqrt{t-\varepsilon-\tau}} \\ &\quad= (t-\tau)^{-1/2} F * \left[(\nabla G)_{\sqrt{t-\tau}} - (\nabla G)_{\sqrt{t-\varepsilon-\tau}}\right] \\ &\quad+ \left[(t-\tau)^{-1/2} - (t-\varepsilon-\tau)^{-1/2}\right] F * (\nabla G)_{\sqrt{t-\varepsilon-\tau}} \\ &\quad=: P + Q\end{aligned}$$

and can see that both terms go to zero:

$$\begin{aligned}\left\|\int_0^{t-\varepsilon} P d\tau\right\|_{L^\infty} &\le c\int_0^{t-\varepsilon} \tau^{-1/2} \left\|(\nabla G)_{\sqrt{\tau}} - (\nabla G)_{\sqrt{\tau+\varepsilon}}\right\|_{L^1} d\tau \\ &\times \sup_{0<t<T} \|F(t)\|_{L^\infty} \to 0 \text{ as } \varepsilon \searrow 0\end{aligned}$$

and

$$\begin{aligned}\left\|\int_0^{t-\varepsilon} Q d\tau\right\|_{L^\infty} &\le c\int_0^{t-\varepsilon} \left(\tau^{-1/2} - (\tau+\varepsilon)^{-1/2}\right) d\tau \sup_{0<t<T} \|F(t)\|_{L^\infty} \\ &\to 0 \text{ as } \varepsilon \searrow 0.\end{aligned}$$

Finally,

$$\begin{aligned}\|II(\varepsilon)\|_{L^\infty} &\le c\int_{t-\varepsilon}^t (t-\varepsilon)^{-1/2} d\tau \sup_{0<t<T} \|F(t)\|_{L^\infty} \\ &\le c\varepsilon^{1/2} \sup_{0<t<T} \|F(t)\|_{L^\infty} \to 0 \text{ as } \varepsilon \searrow 0,\end{aligned}$$

hence, $\Phi[u] \in BC((0,T);L^\infty)$. The same argument above yields $\Phi[u] \in BC((0,T);L^1)$ and thus $\Phi[u] \in B(Y_T)$.

To end the proof, we verify that Φ is a contraction mapping from $B(Y_T)$ to itself. For simplicity, let $W(t) = A(t) + B(t) + C(t)$. The same argument above yields

$$\begin{cases}\|W(\tau)\|_{L^1\cap L^\infty} \le c\left[(1+\tau^{1/2})R_Y + \tau\left(1+\tau^{1/2}\right)R_Y^2\right]\|u-v\|_{Y_T} \\ \|W(\tau)\|_{\dot{B}^s_{1,\infty}\cap\dot{B}^s_{p,\infty}} \le c\Big[\tau^{-s/2}(1+\tau^{1/2})R_Y + \left(1+\tau^{(3-s)/2}\right)R_Y^2 \\ \qquad\qquad + \tau(1+\tau^{(3-s)/2})R_Y^3\Big]\|u-v\|_{Y_T}.\end{cases}$$

Because (7) with $\Phi[u] - \Phi[v]$ in the place of $D[u]$ holds, it follows

$$\begin{aligned}\|\Phi[u]-\Phi[v]\|_{Y_T} \le c\Big[&T^{(1-n/p)/2}\left(1+T^{(1+n/p)/2}\right)R_Y \\ &+ T^{(1-n/p+s)/2}\left(1+T^{(3+n/p-s)/2}\right)R_Y^2 \\ &+ T^{(3-n/p+s)/2}\left(1+T^{(3+n/p-s)/2}\right)R_Y^3\Big]\|u-v\|_{Y_T}.\end{aligned}$$

Therefore, there is a constant $C > 0$, depending on c_0 and $|\theta|$, so that if

$$T = C \min\left(\|(u_0, z_0)\|_Y^{-1},\ \|(u_0, z_0)\|_Y^{-2/(1-n/p)},\ \|(u_0, z_0)\|_Y^{-4/(1-n/p+s)} \right),$$

then $\|\Phi[u] - \Phi[v]\|_{Y_T} \leq 1/2 \|u - v\|_{Y_T}$. The proof is completed.

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