Div - curl estimates with critical power weights

Yohei Tsutsui
Shinshu University

1 Introduction

This note is based on [11]. But, the proof of the main result in [11] is not correct, though the result is true. Aim of this note is to give a proof of it, along the talk. It applies Bogovskii formula instead of the Green function for the Neumann problem of Poisson equation, which was used in [11]. I learned this formula from Professor Hideo Kozono, and would like to thank him for teaching me it. It is should be emphasize that thanks to the formula, the result on 3-dimension in [11] is generalized to all dimension.

Div - curl estimate is a inequality of the form, which was firstly studied by Coifman-Linons-Meyer-Semmes [2]: for \( p, q \in (n/(n+1), \infty) \) and \( 1/r = 1/p + 1/q < 1 + 1/n \),

\[
\|(u \cdot \nabla)v\|_{H^r} \lesssim \|u\|_{H^q} \|
abla v\|_{H^p},
\]

(1)

where \( u \) and \( v \) are vector valued functions; \( u = \{u_j\}_{j=1}^n, v = \{v_j\}_{j=1}^n \), and \( \text{div} \ u = \nabla \cdot u = 1 \). Here

\[
(u \cdot \nabla)v = \left( \sum_{j=1}^n u_j \partial_j v_1, \cdots, \sum_{j=1}^n u_j \partial_j v_n \right).
\]

\(H^p; (p \in (0, \infty))\) is the Hardy spaces: for any \( \phi \in C_0^\infty(\mathbb{R}^n) \) with \( \text{supp} \phi \subset \subset B(0, 1) \) and \( \int \varphi dx = 0 \),

\[
\|f\|_{H^p} := \|M_\phi[f]\|_{L^p}, \text{ where } M_\phi[f](x) := \sup_{t>0} |f \ast \phi_t(x)|,
\]

where \( g_t(x) := t^{-n} g(x/t) \). In the case \( p = \infty \), define \( H^\infty := L^\infty \).

As we mentioned in [11], we make use of a real interpolation spaces between weighted Hardy spaces.

**Definition 1.1.** Let \( p, q \in (0, \infty] \) and \( \alpha \in \mathbb{R} \). Define Hardy spaces associated with Herz spaces \( H^{p,q}_\alpha(\mathbb{R}^n) \) as

\[
H^{p,q}_\alpha(\mathbb{R}^n) := \left\{ f \in S'; \|f\|_{H^{p,q}_\alpha} := \|M_\phi f\|_{L^{p,q}_\alpha} < \infty \right\},
\]

where

\[
\|f\|_{L^{p,q}_\alpha} := \left\{ 2^{nk} \|f\|_{L^p(A_k)} \right\}_{k \in \mathbb{Z}}^{1/p_q}.
\]
We explain notations. $S(\mathbb{R}^n)$ and $S'(\mathbb{R}^n)$ denote the Schwartz spaces of rapidly decreasing smooth functions and tempered distributions on $\mathbb{R}^n$, respectively. For a measurable subset $E \subset \mathbb{R}^n$, $|E|$ and $\chi_E$ are the volume and the characteristic function of $E$, respectively. For any integers $j$, $A_j$ denotes a annulus $\{x \in \mathbb{R}^n; 2^{j-1} \leq |x| < 2^j\}$, and $\chi_j$ is the characteristic function of $A_j$. $B(x, r)$ is a ball in $\mathbb{R}^n$, centered at $x$ of radius $r$. $\langle g \rangle_B := |B|^{-1} \int_B f dy$. Also, $A \lesssim B$ means $A \leq cB$ with positive constant $c$, and $A \approx B$ means $A \lesssim B$ and $B \lesssim A$.

To give $(u \cdot \nabla)v$ a definition as a tempered distribution, we define $Y$ by a space of all locally integrable functions $f$ satisfying that there exist $c_f > 0$ and a seminorm $| \cdot |_S$ of $S$ so that $\int |f(x)\varphi(x)| dx \leq c_f |\varphi|_S$, for all $\varphi \in S$.

The main result reads as follows.

**Theorem 1.1.** For $n/(n+1) < p < \infty$, it holds

$$
\|(u \cdot \nabla)v\|_{H^{p,n}_{\alpha(p)}} \leq c \|u\|_{L^\infty} \|\nabla v\|_{H^{p,n/(n+1)}_{\alpha(p)}},
$$

for $u \in L^\infty(\mathbb{R}^n)^n$ with $\text{div} \ u = 0$ and $v \in \left(Y \cap W^{1,r}_{\text{loc}}(\mathbb{R}^n)\right)^n$ for some $r \in (1, \infty)$, where $\alpha(p) := n(1-1/p) + 1$.

**Remark 1.1.** The same argument as the proof of Theorem 1.1, we can also show a weak type estimate:

$$
\|(u \cdot \nabla)v\|_{H^{n/(n+1),\infty}} \lesssim \|u\|_{L^\infty} \|\nabla v\|_{H^{n/(n+1)}},
$$

(2)

because $\alpha(n/(n+1)) = 0$. Here, $f \in H^{(p,q)} \iff M_\phi[f] \in L^{(p,q)}$, where $L^{(p,q)}$ is the Lorentz spaces. Similar estimates were established by Miyakawa [8]. This can be regarded as an endpoint case with $p = n/(n+1)$ and $q = \infty$ of [2]. The ingredient of the proof of (2) is the pointwise estimate:

$$
\|N[v]\|_{L^{n/(n+1),\infty}} \lesssim \|\nabla v\|_{H^{n/(n+1)}},
$$

instead of (4). This is achieved from the pointwise estimate (7) and a Fefferman-Stein’s vector valued inequality (2) of Theorem 1 in [3].

Motivation of this research comes from the optimal $L^2$-energy decay for the incompressible Navier-Stokes equations. Wiegner [12] constructed global weak solutions $u$ having

$$
\|u(t)\|_{L^2} \lesssim t^{-(n+2)/4},
$$

assuming that initial data $a \in L^2$ satisfying $\|e^{t\Delta}a\|_{L^2} \lesssim t^{-(n+2)/4}$. By Miyakawa-Schonbek [9], it is well-known that the decay order $(n+2)/4$ is optimal. $H^p_{\alpha(p)}$ is relevant to this order $(n+2)/4$, because one has that for $p \in (0, 2]$,

$$
\|e^{t\Delta}a\|_{L^2} \lesssim t^{-(n+2)/4}\|a\|_{H^p_{\alpha(p)}},
$$

for
see [10] for the proof. The present author [10] investigated the $L^2$ decay of mild solutions by Kato [5] and constructed solutions whose decay order of $L^2$ energy is $\gamma < (n+2)/4$. One of reasons why the order $\gamma$ in [10] did not reach to the optimal order $(n+2)/4$ is that div-curl estimate in [10] cannot allow us to deal with the critical exponent $\alpha = \alpha(p)$. As mentioned in Remark 7.3 in [7], the bilinear term $(u \cdot \nabla)v$ does not belong to $H^p_{\alpha(p)}$. This observation tells us that if we try to establish div-curl estimate with $\alpha = \alpha(p)$, we has to replace $H^p_{\alpha(p)}$ in the left hand side by some larger spaces. For the purpose, we use Hardy spaces associated to Herz spaces, as in [6] and [7]. Although, a critical div - curl lemma is proved in this article, the author does not know whether or not it is possible to construct global solutions having optimal $L^2$ decay from the similar argument as the previous paper [10].

2  Proof

Before we start the proof of Theorem 1.1, we point out the mistake in [?]. In [?], the Green function for the Neumann problem of Poisson equation:

$$-\Delta h = g \text{ in } B, \quad g = 0 \text{ on } \partial B$$

for $g \in C_0^\infty(B)$ with $\int g dx = 0$, is applied. I treated the solution $h$ as a $C_0^2(B)$ function in [11]. Although $h \in C^2(\mathbb{R}^n)$, this is not true in general. To overcome this difficulty, as we mentioned above, we apply Bogovskiĭ formula. This is a representation, with a kernel function, of solutions to the divergence equation.

2.1  Bogovskiĭ formula

Let $B$ be a ball in $\mathbb{R}^n$ and $g \in C_0^\infty(B)$ with $\int g dx = 0$. We refer Lemma III.3.1 in [4] for next lemma.

Lemma 2.1. There exists a vector function $K = \{K_j\}_{j=1}^n$ on $\mathbb{R}^n \times \mathbb{R}^n \setminus \{(x, y): x = y\}$ so that $G_B(x) := \int K(x, y)g(y)dy \in C_0^\infty(B)^n$ is a solution to the divergence equation $\nabla \cdot G_B = g$ on $B$ satisfying that for $q \in (1, \infty)$

$$\|G_B\|_{L^q} \lesssim |B|^{1/n}\|g\|_{L^q} \quad \text{and} \quad \|\nabla \cdot G_B\|_{BMO} \lesssim \|g\|_{L^\infty}.$$ 

Remark 2.1. The $L^\infty - BMO$ estimate above is deduced from the fact that the operator $g \mapsto G_B$ is a Calderón-Zygmund operator.
2.2 Vector valued restricted weak type inequality

Another ingredient for the proof of Theorem 1.1 is a vector-valued “restricted weak” type inequality for Hardy-Littlewood maximal operator; for $r \in (0, \infty)$

$$M_r f(x) := \sup_{B \ni x} (|f|^r)^{1/r} B,$$

where the supremum is taken over all ball $B$ containing $x$. Define $M f(x) := M_1 f(x)$. The following is a generalization of the result of Fefferman-Stein [3], and the proof is found in [11].

**Proposition 2.1.** For $1 < r < p < \infty$ and $\alpha = n(1 - 1/p)$,

$$\left\| \left( \sum_{l=1}^{\infty} (M f_l)^r \right)^{1/r} \right\|_{L_p^\infty} \lesssim \left\| \left( \sum_{l=1}^{\infty} |f_l|^r \right)^{1/r} \right\|_{L_p^1}.$$

This can be rewritten as the following form.

**Corollary 2.1.** For $0 < r < 1 < p < \infty$ and $\alpha = n(1/r - 1/p)$,

$$\left\| \sum_{l=1}^{\infty} M_r f_l \right\|_{L_p^\infty} \lesssim \left\| \sum_{l=1}^{\infty} |f_l| \right\|_{L_p^r}.$$

2.3 Complete of the proof of Theorem 1.1

The proof is almost same as that in [11] except for applying Lemma 2.1 instead of the Green function for the Neumann problem of Poisson equations.

Because

$$\|(u \cdot \nabla)v\|_{H_\alpha^{p,\infty}(\mathbb{R}^n)} = \sum_{k=1}^{n} \left\| \sum_{j=1}^{n} u_j \partial_j v_k \right\|_{H_\alpha^{p,\infty}(\mathbb{R}^n)} = \sum_{k=1}^{n} \left\| M_\phi \left( \sum_{j=1}^{n} u_j \partial_j v_k \right) \right\|_{L_\alpha^{p,\infty}(\mathbb{R}^n)},$$

it is enough to show the inequality

$$\left\| M_\phi \left( \sum_{j=1}^{n} u_j \partial_j v \right) \right\|_{L_\alpha^{p,\infty}(\mathbb{R}^n)} \lesssim \|u\|_{L^\infty} \|\nabla v\|_{H_\alpha^{p,n/(n+1)}(\mathbb{R}^n)},$$

for all divergence free vector fields $u$ and functions $v \in Y \cap W^{1,r}_{loc}$. Firstly, we give a definition of $\sum_{j=1}^{n} u_j \partial_j v$ as a tempered distribution as follows; for $\varphi \in S$

$$\left\langle \sum_{j=1}^{n} u_j \partial_j v, \varphi \right\rangle := -\sum_{j=1}^{n} \int u_j(y)v(y)\partial_j \varphi(y)dy.$$
Our assumption ensures that the integral in the right hand side absolutely converges. Then, it follows
\[
\sum_{j=1}^{n} u_j \partial_j v \ast \phi_t(x) = -C_\phi \|u\|_{L^\infty} \int v(y) \left[ \sum_{j=1}^{n} \tilde{u}_j(y) \partial_j \phi_t(x-y) \right] dy,
\]
where \( C_\phi \) is a constant depending on \( \phi \), and \( \tilde{u}_j(y) = \frac{u_j(y)}{C_\phi \|u\|_{L^\infty}} \). Owing to the divergence free condition on \( u \), we see that for every \( x \in \mathbb{R}^n \)
\[
\sum_{j=1}^{n} \tilde{u}_j(y) \partial_j \phi_t(x-y) = \sum_{j=1}^{n} \partial_{y_j} (\tilde{u}_j(y) \phi_t(x-y)) \text{ in } S'(\mathbb{R}^n).
\] (3)
Hence, we obtain the pointwise estimate
\[
M_\phi \left( \sum_{j=1}^{n} u_j \partial_j v \right)(x) \leq C_\phi \|u\|_{L^\infty} N[v](x),
\]
where \( N[v](x) := \sup_{t>0} \left| \int v(y) g(y) dy \right| \) and \( g(y) = g(y; x, t) := \sum_{j=1}^{n} \tilde{u}_j(y) \partial_{y_j} \phi_t(x-y) \).

It is enough to prove that
\[
\|N[v]\|_{L^p, \infty} \lesssim \|\nabla v\|_{H^{p, n/(n+1)}},
\] (4)
To show this from a pointwise estimate, we make use of Bogovskiĭ formula Lemma 2.1 and the atomic decomposition in \( H^{p, n/(n+1)}_{\alpha(p)} \) due to Miyachi [7].

Fix \( x \in \mathbb{R}^n \) and \( t \in (0, \infty) \). There exists \( \varepsilon_0 > 0 \) so that for all \( \varepsilon \in (0, \varepsilon_0) \), \( g \ast \phi_\varepsilon \subset \subset B(x, t) \). Take \( \eta_0 \in C^\infty_0(\mathbb{R}^n) \) such that \( \eta(y) = \eta(y; x, t) := \eta_0 \left( \frac{y-x}{t} \right) \) satisfies that \( 0 \leq \eta \leq 1 \) and for \( \varepsilon \in (0, \varepsilon_0) \)
\[
\text{supp } g \ast \phi_\varepsilon \subset \subset \text{supp } \eta \subset \subset B(x, t), \text{ and } \eta \equiv 1 \text{ on } \text{supp } g \ast \phi_\varepsilon
\]
Remark that \( \|\eta\|_{L^p} = ct^{n/p} \) for all \( p \in [1, \infty] \) with \( c \) independent of \( x, t \) and \( \varepsilon \). Next we see that
\[
\sigma_\varepsilon := \|\eta\|_{L^p}^{-1} \int g \ast \phi_\varepsilon dy \to 0 \text{ as } \varepsilon \searrow 0.
\]
In fact, for a test function \( \rho \in C^\infty_0(B(x, 2t)) \) with \( \rho \equiv 1 \) on \( B(x, t) \), we have from (3)
\[
\|\eta\|_{L^1} \sigma_\varepsilon = \langle g, \rho \ast \phi_\varepsilon \rangle \to \langle g, \rho \rangle = 0.
\]
For $\varepsilon \in (0, \varepsilon_0)$, letting $g^\varepsilon := g \ast \phi_{\varepsilon} - \sigma_{\varepsilon} \eta \in C_0^\infty(B(x, t))$, we obtain $\int g^\varepsilon dy = 0$. From Lemma 2.1,

$$G^\varepsilon(y) := \int K(y, z)g^\varepsilon(z)dz \in C_0^\infty(B(x, t))$$

solves the Drichlet problem of Poisson equation:

$$\nabla \cdot G^\varepsilon = g^\varepsilon \text{ in } B(x, t), \quad G^\varepsilon = 0 \text{ on } \partial B(x, t).$$

Further, $G^\varepsilon$ fulfills the following estimates: for all $q \in (1, \infty)$,

$$\|G^\varepsilon\|_{L^q} \lesssim t^{-n+n/q} \quad \text{and} \quad \|\partial_j G^\varepsilon\|_{BMO} \lesssim t^{-(n+1)}.$$  \hfill (5)

Indeed, these follows that $\|g^\varepsilon\|_{L^q} \lesssim t^{-1-n+n/q}$ and $\|g^\varepsilon\|_{L^\infty} \lesssim t^{-(n+1)}$, respectively.

Integration by parts yields

$$\int vgdy = -\lim_{\varepsilon \to 0} \int \nabla v \cdot G^\varepsilon dy.$$

Therefore, one obtains

$$\left| \int vgdy \right| \leq \sum_{k=1}^n \limsup_{0<\varepsilon<\varepsilon_0} \left| \int \partial_k v G^\varepsilon_k dy \right|.$$

Since $\partial_k v \in H^{n,n/(n+1)}_{p,\alpha(p)}$, following Miyachi [7], it can be decomposed as

$$\partial_k v = \sum_{j=1}^\infty a_j^{(k)}$$

where $\text{supp } a_j^{(k)} \subset B_j = B(x_j, r_j)$, $a_j^{(k)} \in L^\infty$ and $\int x^\beta a_j^{(k)}(x)dx = 0$ for $|\beta| \leq 1$, also

$$\left\| \sum_{j=1}^\infty \|a_j^{(k)}\|_{L^\infty} \chi_{B_j} \right\|_{L^{p,n/(n+1)}_{p,\alpha(p)}} \lesssim \|\partial_k v\|_{H^{n,n/(n+1)}_{p,\alpha(p)}}.$$

and one obtains

$$\left| \int vgdy \right| \leq \sum_{k=1}^n \sum_{j=1}^\infty \limsup_{0<\varepsilon<\varepsilon_0} \left| \int a_j^{(k)} G^\varepsilon_k dy \right|.$$

From (5), we immediately see that

$$\left| \int a_j^{(k)} G^\varepsilon_k dy \right| \leq \|a_j^{(k)}\|_{L^\infty} |B(x, t)|^{1-1/q} \|G^\varepsilon_k\|_{L^q} \lesssim \|a_j^{(k)}\|_{L^\infty}. $$
Remark 2.2. In [1], the pointwise estimate (6) with $n + 1 - \varepsilon$ replaced by $n + 1$ was proved.

When $x \notin 4B_j$, if $Ct < |x - x_j|$ with $C > 8/3$, then it holds $B_j \cap B(x, t) = \emptyset$ and
\[
\int a_j^{(k)}G_k^jdy = 0.
\]
On the other hand, if $Ct \geq |x - x_j|$, then we can derive the decay estimate
\[
\limsup_{0 < \varepsilon < \epsilon_0} \left\| \int a_j^{(k)}G_k^jdy \right\| \lesssim \left\| a_j^{(k)} \right\|_{L^\infty} \left( \frac{r_j}{|x - x_j|} \right)^{n+1}.
\]
(6)

We may assume $x \neq x_j$. Using the moment condition on $a_j^{(k)}$ twice, one has
\[
\int a_j^{(k)}G_k^jdy = \int a_j^{(k)}(G_k^j - G_k^j(x_j))dy = \sum_{s=1}^n \int_0^1 \int a_j^{(k)}(y - x_j)(\partial_s G_k^j)(\theta y + (1 - \theta)x_j)d\theta dyd\theta
\]
\[
= \sum_{s=1}^n \int_0^1 \int a_j^{(k)}(y - x_j)s[(\partial_s G_k^j)(\theta y + (1 - \theta)x_j) - \langle \partial_s G_k^j \rangle_{B(x_j, \theta r_j)}]d\theta dy.
\]

From this representation, the decay estimate (6) is derived as follows from (5);
\[
\left| \int a_j^{(k)}G_k^jdy \right| \lesssim r_j^n \left\| a_j^{(k)} \right\|_{L^\infty} \sum_{s=1}^n \int_0^1 \theta^{-n} \int_{B(x_j, \theta r_j)} |\partial_s G_k^j(y) - \langle \partial_s G_k^j \rangle_{B(x_j, \theta r_j)}| d\theta dy
\]
\[
\lesssim r_j^{n+1} \left\| a_j^{(k)} \right\|_{L^\infty} \sum_{s=1}^n \left\| \partial_s G_k^j \right\|_{BMO}
\]
\[
\lesssim \left( \frac{r_j}{t} \right)^{n+1} \left\| a_j^{(k)} \right\|_{L^\infty}
\]
\[
\lesssim \left( \frac{r_j}{|x - x_j|} \right)^{n+1} \left\| a_j^{(k)} \right\|_{L^\infty}.
\]

As mentioned in [7], because
\[
\left( \frac{1}{1 + |x - x_j|/r_j} \right)^{n+1} \approx M_{n/(n+1)}(\chi_{B_j})(x),
\]
then it follows that for all $x \in \mathbb{R}^n$,
\[
N[v](x) = \sup \left| \int v(y)g(y; x, t)dy \right| \lesssim \sum_{k=1}^n \sum_{j=1}^\infty \left\| a_j^{(k)} \right\|_{L^\infty} M_{n/(n+1)}(\chi_{B_j})(x).
\]
(7)

Now, we apply Corollary 2.1 with $r = n/(n + 1)$ and obtain
\[
\|N[v]\|_{L^{p,\infty}} \lesssim \sum_{k=1}^n \sum_{j=1}^\infty \left\| a_j^{(k)} \right\|_{L^\infty} \chi_{B_j} \|\nabla v\|_{H^{\alpha, n/(n+1)}} \lesssim \sum_{k=1}^n \left\| \partial_k v \right\|_{H^{\alpha, n/(n+1)}} = \|\nabla v\|_{H^{\alpha, n/(n+1)}}.
\]

Here we have used $n(1 - 1/p) + n((n + 1)/n - 1) = n(1 - 1/p) + 1 = \alpha(p)$. The proof is completed.
参考文献


Department of Mathematical Sciences
Faculty of science
Shinshu University
Asahi 3-1-1
Matsumoto 390-8621
Japan.
E-mail: tsutsui@shinshu-u.ac.jp