The Riemann – Hilbert Problem

~ Workshop in Shinshu Univ ~

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§0 Review : Orthogonal Polynomials

μ ; positive measure on $\mathbb{R}$ s.t. $\int_{\mathbb{R}} |x|^m d\mu(x) < \infty \quad \forall m = 0, 1, 2, \cdots$

$\rightsquigarrow$ $1, x, x^2, \cdots$ $(\in L^2(\mathbb{R}, \mu))$ $\underset{G-S}{\rightsquigarrow}$ $\exists$ monic polynomials $\pi_n(x)$ degree n

s.t. $\int_{\mathbb{R}} \pi_n(x)\pi_m(x)\, d\mu(x) = \frac{1}{\gamma_n^2}\delta_{n,m} \quad (\gamma_n > 0)$

$\rightsquigarrow$ $p_n(x) := \gamma_n \pi_n(x) \cdots \int_{\mathbb{R}} p_n(x) p_m(x)\, d\mu(x) = \delta_{n,m}$

<u>Prop</u> (The three-term recurrence relation)

$\exists\ (a_n)_{n=0}^{\infty}, \ (b_n)_{n=0}^{\infty}$ s.t. $b_n > 0$

& $x p_n(x) = b_{n-1} p_{n-1}(x) + a_n p_n(x) + b_n p_{n+1}(x) \quad \forall n \geq 0$

where $b_{-1} = 0$

<u>Proof</u> $x p_n(x)$: poly of degree $n+1$

i.e. $x p_n(x) = c_0 p_0(x) + c_1 p_1(x) + \cdots + c_n p_n(x)$

∘ $k = 0, \ldots, n-2$; $c_k = \int_{\mathbb{R}} x p_n(x) \underbrace{p_k(x)}_{\deg = k+1 \leq n-1} d\mu(x) = 0$

∘ $c_{n-1} = \int_{\mathbb{R}} x p_n(x) p_{n-1}(x)\, d\mu(x) = b_{n-1}$

$\rightsquigarrow$ $x p_n(x) = b_{n-1} p_{n-1}(x) + \underset{\underset{a_n}{\parallel}}{c_n} p_n(x) + \underset{\underset{b_{n+1}}{\parallel}}{c_{n+1}} p_{n+1}(x)$

& $b_n = \int_{\mathbb{R}} x p_n(x) p_{n+1}(x)\, d\mu(x)$

$= \int_{\mathbb{R}} (\gamma_n x^{n+1} + \cdots) p_{n+1}(x)\, d\mu(x)$

$= \frac{\gamma_n}{\gamma_{n+1}} \int_{\mathbb{R}} \gamma_{n+1} x^{n+1} \cdot p_{n+1}(x)\, d\mu(x)$

$= \frac{\gamma_n}{\gamma_{n+1}} \int_{\mathbb{R}} p_{n+1}(x)^2\, d\mu(x) = \frac{\gamma_n}{\gamma_{n+1}} > 0$

$\square$

<u>Prop</u> (The Cristoffel-Darboux formula)

$$\sum_{k=0}^{n-1} p_k(x)\, p_k(y) = \frac{\gamma_{n-1}}{\gamma_n} \frac{p_n(x)\,p_{n-1}(y) - p_{n-1}(x)\,p_n(y)}{x-y}$$

<u>Proof</u> By the 3-term rec. rel,

$$(x-y)\sum_{k=0}^{n-1} p_k(x)p_k(y) = \frac{\gamma_{n-1}}{\gamma_n}\left(p_n(x)p_{n-1}(y) - p_{n-1}(x)p_n(y)\right)$$

図

§1 Random matrices & determinantal point processes.

$\mathcal{H}_N :=$ N×N hermitian matrices.

ex. $\mathcal{H}_N \ni M = \begin{pmatrix} x_{11} & & x_{ij}+\sqrt{-1}\,Y_{ij} \\ & \ddots & \\ x_{ij}-\sqrt{-1}\,Y_{ij} & & x_{NN} \end{pmatrix} \mapsto \left((x_{ii})_{i=1}^N,\ (x_{ij}, Y_{ij})_{i<j} \right) \in \mathbb{R}^{N^2}$

<u>Def</u> ⊚ $Q(x) = \alpha_{2j}x^{2j} + \cdots + \alpha_0$, $\alpha_{2j} > 0$

$\rightsquigarrow \mu_Q^{(N)}$; the prob meas on $\mathcal{H}_N$ ← <u>unitary</u> <u>ensemble</u>

s.t. $d\mu_Q^{(N)}(M) := \underbrace{C_Q^{(N)}}_{\text{the normalized const.}}\, e^{-\mathrm{Tr}\,Q(M)}\, \underbrace{dM}_{\text{the Leb meas}}$

⊚ $Q(x) = x^2 \rightsquigarrow \mu_Q^{(N)}$: Gaussian unitary ensemble

⊚ $\lambda : (\mathcal{H}_N, \mu_Q^{(N)}) \ni M = {}^{\exists}U \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_N \end{bmatrix} U^*$

$\mapsto \sum_{i=1}^N \delta_{\lambda_i}$ … "random measure" on $\mathbb{R}$

$\lambda_* \mu_Q^{(N)} := \mu_Q^{(N)}(\lambda^{-1}(\cdot))$ on $\mathbb{R}$

the vague top.

Def

- $\mathcal{M}(\mathbb{R})$: the set of Borel measures on $\mathbb{R}$ s.t. $\sum_i \delta_{x_i}$, where $(x_i)_i$ has no limit points
- A prob meas ν on $\mathcal{M}(\mathbb{R})$ is called a <u>point process</u> on $\mathbb{R}$

e.g. $\lambda_* \mu_Q^{(N)}$, recall, $\lambda : (\mathcal{X}_N, \mu_Q^{(N)}) \to \mathcal{M}(\mathbb{R})$

- $\forall D \in \mathcal{B}(\mathbb{R})$,

$$\begin{aligned} \chi_D : (\mathcal{M}(\mathbb{R}), \nu) &\longrightarrow \mathbb{Z}_{\geq 0} \cup \{\infty\} \\ m &\longmapsto m(D) \\ (=\textstyle\sum_i \delta_{x_i}) &\qquad (=\textstyle\sum_i 1_D(x_i)) \end{aligned}$$

assume $\exists\ \rho_k : \mathbb{R}^k \to \mathbb{R}$

s.t. $\forall D_1, \dots, D_k \in \mathcal{B}(\mathbb{R})$: mutually disjoint,

$$E[\chi_{D_1} \cdots \chi_{D_k}] = \int_{D_1 \times \cdots \times D_k} \rho_k(x_1, \dots, x_k)\, dx_1 \cdots dx_k$$

Then ρ_k is called the <u>k-th correlation function</u>

<u>Q</u> When $\nu = \lambda_* \mu_Q^{(N)}$, $\rho_k = ?$

- A point process ν is called <u>determinantal</u> if $\exists\ \rho_k$ & $\exists\ K : \mathbb{R} \times \mathbb{R} \to \mathbb{C}$ ... <u>determinantal kernel</u>

s.t. $\rho_k(x_1, \dots, x_k) = \det[K(x_i, x_j)]_{i,j=1}^k$

※ $\lambda_* \mu_Q^{(N)}$ is <u>determinantal</u> !!

<u>Thm</u> $\pi_n(x)$; monic orth poly wrt $e^{-Q(x)}dx$ of degree n

& $p_n(x) := \gamma_n \pi_n(x)$ $\left(\int_{\mathbb{R}} \pi_n(x)^2 e^{-Q(x)} dx = 1/\gamma_n^2 \right)$

Then
$$K(x,y) := e^{-\frac{Q(x)+Q(y)}{2}} \frac{\gamma_{N-1}}{\gamma_N} \frac{p_N(x)\, p_{N-1}(y) - p_{N-1}(x)\, p_N(y)}{x - y}$$

is a determinantal kernel of $\lambda_* \mu_Q^{(N)}$

Lem $f : \mathcal{H}_N \to \mathbb{C}$; b'dd, $f(UMU^*) = f(M)$

$\rightsquigarrow \tilde{f} : \mathbb{R}^N \to \mathbb{C}$, $\tilde{f}(x_1, \dots, x_N) := f\left(\begin{bmatrix} x_1 & & \\ & \ddots & \\ & & x_N \end{bmatrix}\right)$

Then $\displaystyle\int_{\mathcal{H}_N} f(M)\, d\mu_Q^{(N)}(M) = \frac{1}{Z_Q^{(N)}} \int_{\mathbb{R}^N} f(x_1, \dots, x_N)\, e^{-\sum_{i=1}^N Q(x_i)} \prod_{1 \le i < j \le N} (x_i - x_j)^2\, dx_1 \cdots dx_N$

where $\displaystyle Z_Q^{(N)} = \int_{\mathbb{R}^N} e^{-\sum_{i=1}^N Q(x_i)} \prod_{1 \le i < j \le N} (x_i - x_j)^2\, dx_1 \cdots dx_N$

Observation

$$Z_Q^{(N)} = \int_{\mathbb{R}^N} e^{-\sum_{i=1}^N Q(x_i)} \det\left[x_j^{i-1}\right]^2 dx_1 \cdots dx_N$$

$$= \int_{\mathbb{R}^N} e^{-\sum_{i=1}^N Q(x_i)} \det\left[\pi_{i-1}(x_j)\right]^2 dx_1 \cdots dx_N$$

$$= \sum_{\sigma, \tau \in S(N)} \mathrm{sgn}(\sigma)\mathrm{sgn}(\tau) \prod_{i=1}^N \int_{\mathbb{R}^N} \pi_{\sigma(i)-1}(x)\, \pi_{\tau(i)-1}(x)\, e^{-Q(x)} dx$$

$$= \sum_{\sigma \in S(N)} \mathrm{sgn}(\sigma)^2 \prod_{i=1}^N \frac{1}{\gamma_{\sigma(i)-1}^2}$$

$$= N! \prod_{i=1}^N \frac{1}{\gamma_i^2}$$

//

Proof (Outline)

Computation : $D_1, \dots, D_K \in \mathcal{B}(\mathbb{R})$; mutually disjoint.

$$E[\chi_{D_1} \cdots \chi_{D_K}] = \int_{\mathcal{M}(\mathbb{R})} \chi_{D_1} \cdots \chi_{D_K}(\nu)\, d\lambda_* \mu_Q^{(N)}(\nu)$$

$$= \int_{\mathcal{H}_N} \chi_{D_1} \cdots \chi_{D_K}(\lambda(M))\, d\mu_Q^{(N)}(M)$$

Lem $\rightarrow$ $\displaystyle = \frac{1}{Z_Q^{(N)}} \int_{\mathbb{R}^N} \tilde{\chi}_{D_1} \cdots \tilde{\chi}_{D_K}(x_1, \dots, x_N)\, e^{-\sum_{i=1}^N Q(x_i)} \prod_{i<j} (x_i - x_j)^2\, dx_1 \cdots dx_N$

$$= \sum_{i_1=1}^{N} \cdots \sum_{i_k=1}^{N} \int_{\mathbb{R}^N} 1_{D_1}(x_{i_1}) \cdots 1_{D_k}(x_{i_k}) \, P_Q^{(N)}(x_1, \dots, x_N) \, dx_1 \cdots dx_N$$

※ $P_Q^{(N)}(x_1, \dots, x_N) = \dfrac{1}{Z_Q^{(N)}} e^{-\sum_{i=1}^N Q(x_i)} \prod_{1 \le i < j \le N} (x_i - x_j)^2$

$$= \sum_{\substack{\sigma : \{1, \dots, k\} \\ \hookrightarrow \{1, \dots, N\}}} \int_{\mathbb{R}^N} 1_{D_1}(x_{\sigma(1)}) \cdots 1_{D_k}(x_{\sigma(k)}) \underbrace{P_Q^{(N)}(x_1, \dots, x_N)}_{S(N)\text{-inv}} dx_1 \cdots dx_N$$

$$= \frac{N!}{(N-k)!} \int_{D_1 \times \cdots \times D_k \times \mathbb{R}^{N-k}} P_Q^{(N)}(x_1, \dots, x_N) \, dx_1 \cdots dx_N .$$

$$\therefore \quad P_k(x_1, \dots, x_k) = \frac{N!}{(N-k)!} \int_{\mathbb{R}^{N-k}} \underline{P_Q^{(N)}(x_1, \dots, x_N)} \, dx_{k+1} \cdots dx_N$$

$$\partial \; P_Q^{(N)}(x_1, \dots, x_N) = \frac{\prod_{n=1}^N \gamma_n^2}{N!} e^{-\sum_{i=1}^N Q(x_i)} \det[\pi_{j-1}(x_i)] \det[\pi_{i-1}(x_j)]$$

$$= \frac{1}{N!} \det\left[e^{-\frac{Q(x_i)}{2}} P_{j-1}(x_i) \right] \det\left[e^{-\frac{Q(x_j)}{2}} P_{i-1}(x_j) \right]$$

$$= \frac{1}{N!} \det\left[e^{-\frac{Q(x_i) + Q(x_j)}{2}} \sum_{k=0}^{N-1} P_{k-1}(x_i) P_{k-1}(x_j) \right]$$

$$= \frac{1}{N!} \det\left[K(x_i, x_j) \right]_{i,j=1}^N$$

$$\rightsquigarrow \quad P_k(x_1, \dots, x_k) = \frac{1}{(N-k)!} \int_{\mathbb{R}^{N-k}} \det\left[K(x_i, x_j)\right]_{i,j=1}^{N} dx_{k+1} \cdots dx_N$$

$$= \det\left[K(x_i, x_j)\right]_{i,j=1}^{k} \qquad //$$

<u>Rem</u>

$$\int_{\mathbb{R}} \det\left[K(x_i, x_j)\right]_{i,j=1}^{k} dx_k$$

$$= \sum_{\sigma \in S(k)} \mathrm{sgn}(\sigma) \int_{\mathbb{R}} \prod_{i=1}^{k} K(x_i, x_{\sigma(i)})\, dx_k$$

$$= \Bigg(\sum_{\substack{\sigma \in S(k) \\ \sigma(k) = k}} + \sum_{\substack{\sigma \in S(k) \\ \sigma(k) \neq k}} \Bigg) \mathrm{sgn}(\sigma) \int_{\mathbb{R}} \prod_{i=1}^{k} K(x_i, x_{\sigma(i)})\, dx_k$$

<u>$\sigma(k) = k$</u>

$$\sum_{\substack{\sigma \in S(k) \\ \sigma(k) = k}} \mathrm{sgn}(\sigma) \int_{\mathbb{R}} \prod_{i=1}^{k} K(x_i, x_{\sigma(i)})\, dx_k$$

$$= \sum_{\substack{\sigma \in S(k) \\ \sigma(k) = k}} \mathrm{sgn}(\sigma) \prod_{i=1}^{k-1} K(x_i, x_{\sigma(i)}) \underbrace{\int_{\mathbb{R}} K(x_k, x_k)\, dx_k}_{= N}$$

$$= N \det\left[K(x_i, x_j)\right]_{i,j=1}^{N-1}$$

<u>$\sigma(k) \neq k$</u>

$$\sum_{l=1}^{k-1} \sum_{\substack{\sigma \in S(k) \\ \sigma(k) = l}} \mathrm{sgn}(\sigma) \prod_{i \neq \sigma^{-1}(k), k} K(x_i, x_{\sigma(i)}) \underbrace{\int_{\mathbb{R}} K(x_{\sigma^{-1}(k)}, x_k) K(x_k, x_l)\, dx_k}_{= K(x_{\sigma^{-1}(k)}, x_l)}$$

$$= \sum_{l=1}^{k-1} \left(- \det\left[K(x_i, x_j)\right]_{i,j=1}^{k-1} \right)$$

$$= -(k-1) \det\left[K(x_i, x_j)\right]_{i,j=1}^{k-1}$$

$$\triangle = (N - k + 1) \det\left[K(x_i, x_j)\right]_{i,j=1}^{k-1} \qquad //$$

§2 Riemann - Hilbert Problem for Orthogonal polynomials

π_n (monic), $p_n = \gamma_n \pi_n$ ‥ orthogonal polyomials wrt μ

& assume $d\mu(x) = w(x)\,dx$

e.g. $w(x) = e^{-x^2}$ $\rightsquigarrow$ p_n ‥‥ hermite polyomial.

Def ◦ $v : \mathbb{R} \ni x \longmapsto \begin{bmatrix} 1 & w(x) \\ 0 & 1 \end{bmatrix} \in Gl_2(\mathbb{R})$

◦ ~~[illegible]~~,

An analytic function $Y : \mathbb{C} \setminus \mathbb{R} \to M_2(\mathbb{C})$

is a solution of RHP $(\mathbb{R}, v)$ if

$\exists\ Y_\pm(z) = \lim_{\substack{z' \to z \\ z' \in \mathbb{C}^\pm}} Y(z')$ for $\forall z \in \mathbb{R}$

& $\tilde{Y}_\pm : z \in \overline{\mathbb{C}^\pm} \longmapsto \begin{cases} Y(z) & z \in \mathbb{C}^\pm \\ Y_\pm(z) & z \in \mathbb{R} \end{cases}$ is continuous

$$Y_+(z) = Y_-(z)\, v(z) \quad \text{for } \forall z \in \mathbb{R}$$

◦ In addition, for $n \geq 0$, we assume

(normalized condition) $\tilde{Y}_\pm(z) \begin{pmatrix} z^{-n} & \\ & z^n \end{pmatrix} \xrightarrow[z \to \infty]{} \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}$ in $\overline{\mathbb{C}^\pm}$

Lemma If a solution Y exists, it is unique.

Proof

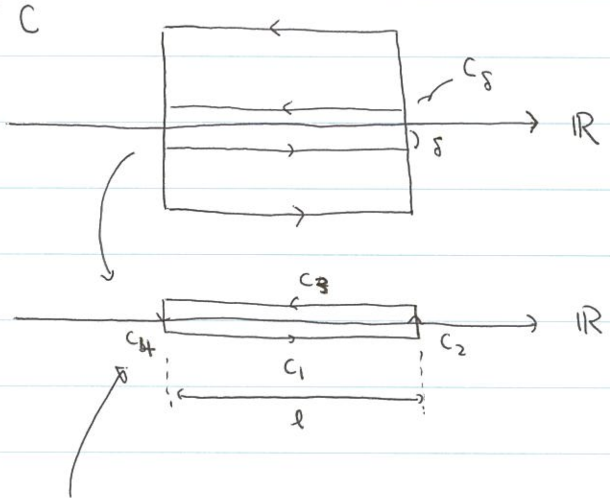
Obs $\det Y$; analytic in $\mathbb{C} \setminus \mathbb{R}$

& $\det Y_+ = \det Y_- v = \det Y_-$

$\rightsquigarrow f(z) := \begin{cases} \det Y(z) & z \in \mathbb{C} \setminus \mathbb{R} \\ \det Y_+(z) & z \in \mathbb{R} \end{cases}$ is continuous

$\leadsto$ f is analytic in $\mathbb{C}$

$\because)$ $\forall C$



$$\int_C f(z)\,dz = \int_{C_\delta} f(z)\,dz$$

cpt i.e. $|f(z)| \leq M$ & f : uniformly continuous

$\forall \varepsilon > 0$, $\exists \delta > 0$

s.t. $M \cdot 2\delta < \varepsilon/4$,

$|z_1 - z_2| < \delta \Rightarrow |f(z_1) - f(z_2)| < \frac{\varepsilon}{2\ell}$

$$\leadsto \left|\int_{C_\delta} f(z)\,dz\right| \leq \underbrace{\left|\int_{C_1 \cup C_3} f(z)\,dz\right|} + \underbrace{\left|\int_{C_2} f(z)\,dz\right|}_{< M \cdot 2\delta < \frac{\varepsilon}{4}} + \underbrace{\left|\int_{C_4} f(z)\,dz\right|}_{< \frac{\varepsilon}{4}}$$

$$\leq \int_{C_1} |f(z) - f(\bar{z})|\,|dz|$$

$$< \frac{\varepsilon}{2\ell} \cdot \ell = \frac{\varepsilon}{2}$$

$$< \varepsilon$$

//

By assumption, $f(z) = \det Y(z)$

$= \det Y(z) \begin{vmatrix} z^{-n} & \\ & z^n \end{vmatrix} \longrightarrow 1$ as $z \to \infty$

$\underset{\text{Liouville}}{\leadsto}$ $\det Y(z) \equiv 1$ i.e. $\exists\, Y(z)^{-1}$

② $U(z)$: another solution

$\rightsquigarrow$ $T(z) := U(z)\,Y(z)^{-1}$; analytic in $\mathbb{C}\setminus\mathbb{R}$
continuous on $\overline{\mathbb{C}^{\pm}}$

& $T_+(z) = U_+(z)\,Y_+(z)^{-1}$
$\forall z \in \mathbb{R}$
$= [U_-(z)\,v(z)]\,[Y_-(z)\,v(z)]^{-1}$
$= U_-(z)\,Y_-(z)^{-1}$
$= T_-(z)$

$$\rightsquigarrow \quad g(z) := \begin{cases} T(z) & z \in \mathbb{C}\setminus\mathbb{R} \\ T_+(z) & z \in \mathbb{R} \end{cases} \quad \text{is analytic in } \mathbb{C}$$

$$\& \quad g(z) = \left[U(z) \begin{bmatrix} z^{-n} & \\ & z^n \end{bmatrix} \right] \left[Y(z) \begin{bmatrix} z^{-n} & \\ & z^n \end{bmatrix} \right]^{-1} \xrightarrow[z\to\infty]{} I$$

$\underset{\text{Liouville}}{\rightsquigarrow}$ $g(z) \equiv I$ i.e. $U(z) = Y(z)$ $\forall z \in \mathbb{C}\setminus\mathbb{R}$ ∎

② Construction of the Solution

- assume $w \in \mathcal{S}(\mathbb{R}) = \{ f \in C^\infty(\mathbb{R}) \mid \sup_{x\in\mathbb{R}} |x^\alpha D^\beta f(x)| < \infty \ \forall \alpha, \beta \}$

- The Cauchy Operator

for $h \in \mathcal{S}(\mathbb{R})$, $Ch(z) := \frac{1}{2\pi i}\int_{\mathbb{R}} \frac{h(s)}{s-z}\,ds \quad (z \in \mathbb{C}\setminus\mathbb{R})$

analytic in $\mathbb{C}\setminus\mathbb{R}$, continuous on $\overline{\mathbb{C}^{\pm}}$,
$Ch(z) \to 0$ as $z\to\infty$ in $\overline{\mathbb{C}^{\pm}}$

<u>Claim</u> $Ch_+(z) - Ch_-(z) = h(z)$ for $z \in \mathbb{R}$

$$\because) \quad Ch_+(z) - Ch_-(z) = \lim_{\varepsilon\searrow 0} \left(Ch(z+i\varepsilon) - Ch(z-i\varepsilon) \right)$$
$$= \lim_{\varepsilon\searrow 0} \int_{\mathbb{R}} \frac{ds}{\pi}\, h(s)\, \frac{\varepsilon}{(s-z)^2+\varepsilon^2}$$
$s - z = \varepsilon \tan x$
$$= \lim_{\varepsilon\searrow 0} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{dx}{\pi}\, h(z+\varepsilon\tan x) = h(z)$$

Y : a solution of $(\mathbb{R}, v)$ $\quad \left(Y = \begin{pmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{pmatrix} \right)$

$\rightsquigarrow$ ① $Y_+ = Y_- v$

$$\begin{pmatrix} Y_{11+} & Y_{12+} \\ Y_{21+} & Y_{22+} \end{pmatrix} = \begin{pmatrix} Y_{11-} & Y_{12-} \\ Y_{21-} & Y_{22-} \end{pmatrix} \begin{pmatrix} 1 & w \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} Y_{11-} & wY_{11-} + Y_{12-} \\ Y_{21-} & wY_{21-} + Y_{22-} \end{pmatrix}$$

② $$\begin{pmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{pmatrix} \begin{pmatrix} z^{-n} & \\ & z^{n} \end{pmatrix} = \begin{pmatrix} z^{-n} Y_{11} & z^{n} Y_{12} \\ z^{-n} Y_{21} & z^{n} Y_{22} \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

① From 1st row

$Y_{11+} = Y_{11-}$ $\rightsquigarrow$ Y_{11} is analytic in $\mathbb{C}$,

$Y_{11}(z) z^{-n} \xrightarrow[z \to \infty]{} 1$ ② $\therefore$ $Y_{11}(z) = z^n + \cdots$ i.e. $Y_{11}|_{\mathbb{R}} \in \mathcal{S}(\mathbb{R})$

& $Y_{12+} - Y_{12-} = wY_{11}$ on $\mathbb{R}$

<u>Claim</u> $Y_{12} = CY_{11}w$

$\because)$ $(Y_{12} - CY_{11}w)_+ - (Y_{12} - CY_{11}w)_-$

$$= (Y_{12+} - Y_{12-}) + (CY_{11}w_+ - CY_{11}w_-) = 0$$

$\rightsquigarrow$ $\underbrace{Y_{12}}_{\to 0} - \underbrace{CY_{11}w}_{\to 0}$ is analytic in $\mathbb{C}$

$\underset{\text{Liouville}}{\rightsquigarrow}$ $Y_{12} = CY_{11}w$ //

& From ... $-Y_{12}(z) = \frac{1}{2\pi i}\int_{\mathbb{R}} \frac{Y_{11}(s)w(s)}{z-s} ds$

$$= \frac{1}{2\pi i}\left(\frac{1}{z}\int_{\mathbb{R}} Y_{11}(s) w(s)\, ds + \frac{1}{z^2}\int_{\mathbb{R}} s Y_{11}(s) w(s)\, ds \right.$$

※ $$\frac{1}{z-s} = \frac{1}{z} - \frac{1}{z} + \frac{1}{z-s}$$

$$= \frac{1}{z} + \frac{s}{z}\frac{1}{z-s}$$

$$= \frac{1}{z} + \frac{s}{z}\left(\frac{1}{z} - \frac{s}{z}\frac{1}{z-s}\right)$$

$$= \frac{1}{z} + \frac{s}{z^2} + \frac{s^2}{z^2}\frac{1}{z-s}$$

$$= \frac{1}{z} + \frac{s}{z^2} + \cdots + \frac{s^{k-1}}{z^k} + \frac{s^k}{z^k}\frac{1}{z-s}$$

$$\rightsquigarrow \ -Y_{12}(z) = \frac{1}{2\pi i}\int_{\mathbb{R}} \frac{Y_{11}(s)w(s)}{z-s}\,ds$$

$$= \frac{1}{2\pi i}\Bigg(\frac{1}{z}\int_{\mathbb{R}} Y_{11}(s)\,w(s)\,ds$$

$$+ \frac{1}{z^2}\int_{\mathbb{R}} s\,Y_{11}(s)\,w(s)\,ds$$

$$+ \cdots$$

$$+ \frac{1}{z^n}\int_{\mathbb{R}} s^{n-1}\,Y_{11}(s)\,w(s)\,ds$$

$$+ \frac{1}{z^n}\int_{\mathbb{R}} s^n \frac{Y_{11}(s)w(s)}{z-s}\,ds \Bigg)$$

$$\rightsquigarrow \ -z^n Y_{12}(z) = \frac{1}{2\pi i}\Bigg(z^{n-1}\int_{\mathbb{R}} Y_{11}(s)\,w(s)\,ds$$

$$\downarrow z \to \infty \qquad + \cdots$$

$$0 \qquad + \int_{\mathbb{R}} s^{n-1}\,Y_{11}(s)\,w(s)\,ds$$

$$+ \int_{\mathbb{R}} s^n \frac{Y_{11}(s)w(s)}{z-s}\,ds \Bigg)$$

$$\rightsquigarrow \ \int_{\mathbb{R}} s^k\,Y_{11}(s)\,w(s)\,ds = 0 \qquad \text{i.e.} \quad Y_{11}(z) = \pi_n(z)$$

$k = 0, \ldots, n-1$

$$\& \qquad Y_{12}(z) = \frac{1}{2\pi i}\int_{\mathbb{R}} \frac{Y_{11}(s)w(s)}{s-z}\,ds$$

∂ From 2nd row

$Y_{21+} = Y_{21-} \rightsquigarrow Y_{21}$ is analytic in $\mathbb{C}$

$Y_{21}(z) z^{-n} \xrightarrow[z \to \infty]{} 0 \quad \therefore \quad Y_{21}(z) = \alpha z^{n-1} + \cdots$ i.e. $Y_{21}|_{\mathbb{R}} \in \mathcal{S}(\mathbb{R})$

& $Y_{22+} - Y_{22-} = w Y_{21}$

$$\rightsquigarrow Y_{22}(z) = C Y_{21} w(z) = \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{Y_{21}(s) w(s)}{s - z} ds$$

Moreover,

$$- z^n Y_{22}(z) = \frac{1}{2\pi i} \Big(z^{n-1} \int_{\mathbb{R}} Y_{21}(s) w(s) ds$$

$$+ z^{n-2} \int_{\mathbb{R}} s Y_{21}(s) w(s) ds$$

$$+ \cdots$$

$$+ z \int_{\mathbb{R}} s^{n-2} Y_{21}(s) w(s) ds$$

$$+ z \int_{\mathbb{R}} s^{n-1} \frac{Y_{21}(s) w(s)}{z - s} ds \Big)$$

$$\therefore \int_{\mathbb{R}} s^k Y_{21}(s) w(s) ds = 0 \quad k = 0, \ldots, n-2$$

i.e. $Y_{21}(z) = \alpha \pi_{n-1}(z)$

& $\frac{1}{2\pi i} \int_{\mathbb{R}} s^{n-1} Y_{21}(s) w(s) ds = -1$

$$\therefore \frac{1}{2\pi i} \frac{\alpha}{\gamma_{n-1}^2} = -1 \quad \therefore \alpha = -2\pi i \gamma_{n-1}^2$$

$$\therefore Y_{22}(z) = -2\pi i \gamma_{n-1}^2 \, C \pi_{n-1} w(z)$$

Conclusion

$$Y(z) = \begin{pmatrix} \pi_n(z) & C\pi_n w(z) \\ -2\pi i\, \gamma_{n-1}^2 \pi_{n-1}(z) & -2\pi i\, \gamma_{n-1}^2\, C\pi_{n-1} w(z) \end{pmatrix}$$

Rem
$$K(x,y) = \frac{e^{-(Q(x)+Q(y))/2}}{2\pi i (x-y)} \begin{pmatrix} 0 & 1 \end{pmatrix} Y(y)^{-1} Y(x) \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

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