部分圏の直行性

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1 Motivation The classical Auslander correspondence gives a bijection between the set of Morita-equivalence classes of representation-finite finite-dimensional algebras Λ and that of finite-dimensional algebras Γ with $\mathrm{gl.dim}\,\Gamma \le 2$ and $\mathrm{dom.dim}\,\Gamma \ge 2$. Our motivation comes from a higher dimensional generalization [I2] of Auslander correspondence below 1.2.

1.1 Definition Let $\mathbf{T}$ be a triangulated category (resp. a full subcategory of abelian category) and $n \ge 0$. For a functorially finite full subcategory $\mathbf{C}$ of $\mathbf{T}$, put

$$\begin{aligned} \mathbf{C}^{\perp_n} &:= \{X \in \mathbf{T} \mid \mathrm{Ext}^i(\mathbf{C}, X) = 0 \text{ for any } i\ (0 < i \le n)\}, \\ {}^{\perp_n}\mathbf{C} &:= \{X \in \mathbf{T} \mid \mathrm{Ext}^i(X, \mathbf{C}) = 0 \text{ for any } i\ (0 < i \le n)\}. \end{aligned}$$

We call $\mathbf{C}$ a *maximal n-orthogonal subcategory* of $\mathbf{T}$ if $\mathbf{C} = \mathbf{C}^{\perp_n} = {}^{\perp_n}\mathbf{C}$ holds [I1]. By definition, $\mathbf{T}$ is a unique maximal 0-orthogonal subcategory of $\mathbf{T}$.

1.2 Theorem *For any $n \ge 1$, there exists a bijection between the set of equivalence classes of maximal $(n-1)$-orthogonal subcategories $\mathbf{C}$ of $\mathrm{mod}\,\Lambda$ with additive generators M and finite-dimensional algebras Λ, and the set of Morita-equivalence classes of finite-dimensional algebras Γ with $\mathrm{gl.dim}\,\Gamma \le n+1$ and $\mathrm{dom.dim}\,\Gamma \ge n+1$. It is given by $\mathbf{C} \mapsto \Gamma := \mathrm{End}_\Lambda(M)$.*

Important examples of maximal orthogonal subcategoires appear in the work of Buan-Marsh-Reineke-Reiten-Todorov on cluster categories [BMRRT], that of Geiss-Leclerc-Schröer on preprojective algebras [GLS], and in consideration of invariant subrings of finite subgroups G of $\mathrm{GL}_d(k)$ [I1]. Let us find some kind of higher dimensional analogy of Auslander-Reiten theory by considering maximal orthogonal subcategories.

2 Triangulated category In this section, let $\mathbf{T}$ be a triangulated category with a Serre functor S, and $\mathbf{C}$ a maximal $(n-1)$-orthogonal subcategory of $\mathbf{T}$.

2.1 Theorem [I3] *(1) $S_n := S \circ [-n]$ gives an autoequivalence of $\mathbf{C}$.*

(2) $\mathbf{C}$ has "Auslander-Reiten $(n+2)$-angles", i.e. any $X \in \mathbf{C}$ has a complex

$$S_n X \xrightarrow{f_n} C_{n-1} \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_1} C_0 \xrightarrow{f_0} X$$

which is obtaied by glueing triangles $X_{i+1} \to C_i \xrightarrow{f_i} X_i \to X_{i+1}[1]$ $(0 \le i < n)$ with $X_0 = X$, $X_n = S_n X$, $C_i \in \mathbf{C}$ and the following sequences are exact.

$$\begin{aligned} \mathbf{C}(\ , S_n X) \xrightarrow{\cdot f_n} \mathbf{C}(\ , C_{n-1}) \xrightarrow{\cdot f_{n-1}} \cdots \xrightarrow{\cdot f_1} \mathbf{C}(\ , C_0) \xrightarrow{\cdot f_0} J_{\mathbf{C}}(\ , X) \to 0 \\ \mathbf{C}(X,\ ) \xrightarrow{f_0 \cdot} \mathbf{C}(C_0,\ ) \xrightarrow{f_1 \cdot} \cdots \xrightarrow{f_{n-1} \cdot} \mathbf{C}(C_{n-1},\ ) \xrightarrow{f_n \cdot} J_{\mathbf{C}}(S_n X\ ) \to 0 \end{aligned}$$

It is quite interesting to study the relationship among all maximal $(n-1)$-orthogonal subcategories of $\mathbf{T}$. In the rest of this section, assume that $\mathbf{T}$ is n-*Calabi-Yau*, i.e. $S_n = 1$. For example, if Λ is a d-dimensional symmetric order, then $\underline{\mathrm{CM}}\Lambda$ is $(d-1)$-Calabi-Yau.

2.2 Definition Assume that $\mathbf{C}$ *satisfies (strict no-loop)*, i.e. for any $X \in \operatorname{ind} \mathbf{C}$, $X \notin \operatorname{add} \bigoplus_{i=1}^{n-1} C_i$ holds in 2.1(2). Define a full subcategory $\mu_{X,i}(\mathbf{C})$ of $\mathbf{T}$ by

$$\operatorname{ind} \mu_{X,i}(\mathbf{C}) := (\operatorname{ind} \mathbf{C} \backslash \{X\}) \cup \{X_i\} \qquad (X \in \operatorname{ind} \mathbf{C},\ i \in \mathbf{Z}/n\mathbf{Z})$$

where X_i is the term of the triangle in 2.1(2). This can be regarded as a higher dimensional generalization of Fomin-Zelevinsky mutation in [BMRRT] and [GLS].

2.3 Theorem [I3] *Assume that* $\mathbf{C}$ *satisfies (strict no-loop). For any* $X \in \operatorname{ind} \mathbf{C}$, $\{\mu_{X,i}(\mathbf{C}) \mid i \in \mathbf{Z}/n\mathbf{Z}\}$ *is the set of all maximal* $(n-1)$*-orthogonal subcategories of* $\mathbf{T}$ *containing* $\operatorname{ind} \mathbf{C} \backslash \{X\}$.

2.4 It is an interesting quetion when (transitivity) holds in $\mathbf{T}$, i.e. the set of all maximal $(n-1)$-orthogonal subcategories of $\mathbf{T}$ is transitive under the action of mutations defined in 2.2. It is known that (transitivity) holds for cluster categories $\mathbf{T}$ [BMRRT], and $\mathbf{T} = \mathrm{CM}\,\Lambda$ for Veronese subring Λ of degree 3 of $k[[x, y, z]]$ [Y].

3 Derived equivalence It is suggestive to relate our question 2.4 to Van den Bergh's generalization [V] of Bondal-Orlov conjecture [BO] in algebraic geometry, which asserts that *all (commutative or non-commutative) crepant resolutions of a normal Gorenstein domain have the same derived category.* Let us generalize the concept of Van den Bergh's non-commutative crepant resolution [V] of commutative normal Gorenstein domains to our situation.

3.1 Let Λ be an R-order which is an isolated singularity. We call $M \in \mathrm{CM}\,\Lambda$ a *NCC resolution* of Λ if $\Lambda \oplus \mathrm{Hom}_R(\Lambda, R) \in \operatorname{add} M$ and $\Gamma := \mathrm{End}_\Lambda(M)$ is an R-order with $\operatorname{gl.dim} \Gamma = d$. We have the remarkable relationship below between NCC resolutions and maximal $(d-2)$-orthogonal subcategories [I2].

Proposition *Let* $d \geq 2$*. Then* $M \in \mathrm{CM}\,\Lambda$ *is a NCC resolution of* Λ *if and only if* $\operatorname{add} M$ *is a maximal* $(d-2)$*-orthogonal subcategory of* $\mathrm{CM}\,\Lambda$.

3.2 We conjecture that *the endomorphism rings* $\mathrm{End}_\Lambda(M)$ *are derived equivalent for all maximal* $(n-1)$*-orthogonal subcategories* $\operatorname{add} M$ *of* $\mathrm{CM}\,\Lambda$. This is an analogy of Bondal-Orlov and Van den Bergh conjecture by 3.1, and true for $n = 2$.

Theorem [I2] *Let* $\mathbf{C}_i = \operatorname{add} M_i$ *be a maximal* 1*-orthogonal subcategory of* $\mathrm{CM}\,\Lambda$ *and* $\Gamma_i := \mathrm{End}_\Lambda(M_i)$ $(i = 1, 2)$*. Then* Γ_1 *and* Γ_2 *are derived equivalent.*

3.3 Corollary [I2,3] *All NCC resolutions of* Λ *are derived equivalent if*
(1) $d \leq 3$*, or (2)* Λ *is a symmetric order and (transitivity) holds in* $\underline{\mathrm{CM}}\Lambda$ *(2.4).*

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